Sets of end-points and ramification points in dendroids

by

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Abstract. Let X be a dendroid. We consider the set of all points of X which have a given order of ramification. We show that the set of all end-points of X and the set of ordinary points of X are co-analytic and the set of ramification points of X is analytic. We give two constructions of smooth dendroids with closed set of end-points and non-Borel set of ramification points. We also construct a (necessarily non-smooth) dendroid whose set of end-points is not Borel.

All spaces considered in this paper are compact and metric. A *continuum* is a compact, connected, metric space. An *arc* is a homeomorphic copy of the closed unit interval of real numbers. A *dendroid* X is a non-degenerate continuum such that each pair x, y of distinct points of X is contained in a unique continuum $[x, y]$ which is minimal with respect to containing $\{x, y\}$ and $[x, y]$ is an arc.

Let X be a dendroid. For $x \in X$ let $r_X(x) = r(x)$ denote the order of x in X in the classical sense, i.e. $r(x)$ is the cardinality of the set of arc components of $X - \{x\}$. For each cardinal number α let

$$R_\alpha(X) = \{x \in X\colon\ r(x) = \alpha\} \quad \text{and} \quad S_\alpha(X) = \{x \in X\colon\ r(x) \geqslant \alpha\}.$$

By Lemma 1 below, it suffices to consider the sets $R_\alpha(X)$ and $S_\alpha(X)$ for $\alpha \in \{1, 2, \ldots, \aleph_0, c\}$. Note $S_1(X) = X$. We recall that $R_1(X)$ is usually called the set of *end-points* of X, $R_2(X)$ is the set of *ordinary points* of X and $S_3(X)$ is the set of *ramification points* of X. If $S_3(X) = \{p\}$ then X is said to be a *fan* with *top* p. The Cantor fan is a space homeomorphic to the cone over the Cantor set.

LEMMA 1 [N1, Th. 2, p. 105]. *Let X be a dendroid and $x \in X$. If $r_X(x) > \aleph_0$ then X contains a Cantor fan with top x.*

Let X be a planable dendroid (i.e. X is homeomorphic to a subset of the plane). Lelek [Le] proved that $R_1(X)$ is a $G_{\delta\sigma\delta}$ and asked whether $R_1(X)$ is of the second Borel class. He showed that $R_1(X)$ is not in general of the first Borel class. The first-named

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The authors were supported in part by NSERC grant A5616 and by a grant from the University of Saskatchewan President's Fund.

author in [N2] proved that $S_3(X)$ is contained in the union of countably many subarcs of X and for each arc $J \subset X$ the set $S_3(X) \cap J$ is $G_{\delta\sigma}$ [N1, Th. 1]. Methods similar to those in the proof of [N1, Th. 1] show that for each arc $J \subset X$, $S_4(X) \cap J$ is $G_{\delta\sigma}$.

THEOREM 1. *Let X be a planable dendroid. Then*

(1) *$R_1(X)$ is $G_{\delta\sigma\delta}$,*
(2) *$R_2(X)$ and $S_2(X)$ are $F_{\sigma\delta\sigma}$,*
(3) *$R_3(X)$ is both $F_{\sigma\delta\sigma}$ and $G_{\delta\sigma\delta}$ and $S_3(X)$ is $G_{\delta\sigma}$,*
(4) *$R_4(X)$ and $S_4(X)$ are $G_{\delta\sigma}$ and*
(5) *$S_5(X)$ is at most countable.*

In [Le], [N3, p. 104] and [H, Problem 206] questions were raised about the possibility of extending Theorem 1 to cover arbitrary dendroids. The main purpose of this paper is to provide answers to these questions.

THEOREM 2. *Let X be a dendroid. Then*

(1) *$S_\alpha(X)$ is analytic for $\alpha = 2, 3, \ldots, \aleph_0, c$,*
(2) *$R_c(X)$ is analytic and*
(3) *$R_\alpha(X)$ is co-analytic for $\alpha \leqslant \aleph_0$.*

Thus, the set of all ramification points of X is analytic and the set of end-points of X as well as the set of ordinary points of X is co-analytic.

Proof. Let I_n denote the straight-line segment with end-points $(0, 0)$ and $(1/n, 1/(n+1))$ in the plane R^2, for $n = 1, 2, \ldots$ For each positive integer n, let $T_n = I_1 \cup \ldots \cup I_n$. Let $T_{\aleph_0} = I_1 \cup I_2 \cup \ldots$ and let T_c be the Cantor fan. Set $p_\alpha = (0, 0)$ for $\alpha \leqslant \aleph_0$ and let p_c be the top of T_c.

Let $C(T_\alpha, X)$ denote the space of all continuous maps of T_α into X with the topology of uniform convergence. Then $C(T_\alpha, X)$ is a separable space with a complete metric. Let

$$C_\alpha = \{f \in C(T_\alpha, X):\ f \text{ is an embedding}\}.$$

It is well known (see for example [K, IV.44.VI, Th. 1]) that C_α is a G_δ subset of $C(T_\alpha, X)$. Obviously, the function φ_α: $C_\alpha \to X$ defined by $\varphi_\alpha(f) = f(p_\alpha)$ for $f \in C_\alpha$ is continuous. Hence, $\varphi_\alpha(C_\alpha)$ is an analytic set. It is easy to see that $S_\alpha(X) = \varphi_\alpha(C_\alpha)$ (for $\alpha = c$ apply Lemma 1). Finally, note that $R_c(X) = S_c(X)$, $R_n(X) = S_n(X) \setminus S_{n+1}(X)$ for $n = 1, 2, \ldots$ and $R_{\aleph_0}(X) = S_{\aleph_0}(X) \setminus S_c(X)$.

A dendroid X is said to be *smooth* (at p) if there is a point $p \in X$ such that for each convergent sequence $x_i \to x$ in X we have $[p, x_i] \to [p, x]$. The reader may consult [CE] for the basic properties of smooth dendroids.

THEOREM 3. *If X is a smooth dendroid, then $R_1(X)$ is a G_δ-set.*

Proof. The method of proof is similar to one given by Lelek in [Le]. Let $p \in X$ so that X is smooth at p and let d be a metric on X which is radially convex with respect to p, i.e. d is an isometry on each arc $[p, x]$ for $x \in X$ (see [CE, Cor. 11]). For $n = 1, 2, \ldots$ let

$$F_n = \{x \in X:\ \text{there is } y \in X \text{ with } x \in [p, y] \text{ and } d(x, y) \geqslant 1/n\}.$$

Since X is smooth with respect to p it follows that each F_n is closed. Note that $R_1(X) = X \setminus (F_1 \cup F_2 \cup \ldots)$. This completes the proof.

Note that the Cantor fan T_c is a smooth dendroid. Lelek [Le] constructed a fan $T \subset T_c$ such that $R_1(T) \cup \{p_c\}$ is connected. Hence, $R_1(T)$ is one-dimensional and by Theorem 3, $R_1(T)$ is a G_δ in T since T is smooth at p_c. Recently it has been proved (see [Ch] and [BO]) that T is the unique up to homeomorphism smooth fan with a dense set of end-points.

In Example 5 we shall construct a dendroid Z with $R_1(Z)$ co-analytic and non-Borel. However, there are conditions other than smoothness which imply that the set of end-points of a dendroid is a G_δ-set, and so a Borel set.

THEOREM 4 [Le, (7.4), p. 310]. *If X is a dendroid such that $R_1(X) \cap \mathrm{cl}(S_3(X))$ is a G_δ-set, then $R_1(X)$ is a G_δ-set as well.*

COROLLARY [Le, p. 311]. *If X is a fan, then $R_1(X)$ is a G_δ-set.*

There exist several examples of dendroids with large sets of ramification points (e.g. [C1], [C2], [Be] and [MN]). In particular, in [MN] a dendroid X is constructed such that X is a universal smooth dendroid, $R_1(X)$ is closed and $X = R_1(X) \cup R_c(X)$.

Now, we give examples to show that the results of Theorem 2 are the best possible. Our examples (1)–(4) are smooth dendroids with complicated sets of ramification points (compare with Theorem 3). Examples (1)–(3) are based on one construction. The construction in Example 4 is different.

For completeness we state here a modification of a well-known theorem (see [MS]):

LEMMA 2. *There exists a continuous function f: $C \to C$ of the Cantor set onto itself and a G_δ subset G of C such that $f(G)$ is not Borel and $\mathrm{card}(f^{-1}(x) \cap G) = c$ for each $x = f(G)$.*

Proof. Let N^N denote the set of irrational numbers which we take to be embedded densely in C. By [KM, p. 434] there is a closed set K in $C \times N^N$ such that $\pi_1(K)$ is not Borel where π_1: $C \times N^N \to C$ is the first coordinate projection. Let

$$G = \{(x, y, z) \in C^3:\ (x, y) \in K\}$$

and let f: $C^3 \to C$ be the first coordinate projection. Then $f(G) = \pi_1(K)$. Now K is a G_δ-set in $C \times C$ since $C \times N^N$ is topologically complete and, hence, G is a G_δ-set in C^3. Of course C^3 is homeomorphic to C.

EXAMPLE 1. There exists a dendroid X such that the sets $S_3(X)$, $R_c(X)$ and $R_2(X)$ are not Borel. By Theorem 2, $S_3(X)$ and $R_c(X)$ are analytic and $R_2(X)$ is co-analytic.

Let C, f and G be as in Lemma 2. Let $F_1, F_2, \ldots$ be closed subsets of C such that $F_1 \subset F_2 \subset \ldots$ and $C \setminus G = F_1 \cup F_2 \cup \ldots$ Let H denote the decomposition of $C \times [-1, 1]$ into points and the following sets:

(a) $(f^{-1}(c) \cap F_n) \times \{t\}$ for $c \in f(F_n)$, $t \in]\frac{1}{n+2}, \frac{1}{n+1}]$ and $n = 1, 2, \ldots$,
(b) $f^{-1}(c) \times \{t\}$ for $c \in C$ and $t \in]-1, 0]$,
(c) $C \times \{-1\}$.

Let h: $C \times [-1, 1] \to X = C \times [-1, 1]/H$ denote the quotient map and quotient space respectively. Observe that H is an upper semi-continuous decomposition of $C \times [-1, 1]$ and X is a dendroid which is smooth at the point $h(C \times \{-1\})$. Define k: $X \to [-1, 1]$ by $k(x) = t$ for $x = h(c, t)$ for $(c, t) \in C \times [-1, 1]$. Then k is a continuous function.

Observe that

$$h(G \times \{0\}) \subset R_c(X) \subset S_3(X) \subset \{h(C \times \{-1\})\} \cup h(G \times \{0\}) \cup \bigcup_{n=1}^{\infty} h(F_n \times \{\tfrac{1}{n+1}\}).$$

Since k is continuous, $h(G \times \{0\}) = S_3(X) \cap k^{-1}(0) = R_c(X) \cap k^{-1}(0)$ is a closed subset of $S_3(X)$ and $R_c(X)$. It is easy to see that $h(G \times \{0\})$ is homeomorphic to $f(G)$. Thus, $h(G \times \{0\})$ is not Borel. Moreover, $R_2(X) \cap k^{-1}(0) = k^{-1}(0) \setminus h(G \times \{0\})$ is not Borel so $R_2(X)$ is also not Borel. Observe, also, that the set $R_1(X) = k^{-1}(1)$ is closed.

EXAMPLE 2. For each cardinal number $\alpha \in \{3, 4, \ldots, \aleph_0\}$ there exists a smooth dendroid X_α such that the set $R_\alpha(X_\alpha)$ is not Borel.

We keep all of the notation of Example 1. Moreover, we let T_β and p_β for $\beta \in \{1, 2, \ldots, \aleph_0\}$ be as in the proof of Theorem 2. Let $\gamma = \alpha - 2$ if $\alpha \in \{3, 4, \ldots\}$ and let $\gamma = \aleph_0$ if $\alpha = \aleph_0$. Let

$$X_\alpha = (X \times \{p_\gamma\}) \cup (k^{-1}(0) \times T_\gamma).$$

Then X_α is a smooth dendroid and

$$(k^{-1}(0) \setminus h(G \times \{0\})) \times \{p_\gamma\} = R_\alpha(X_\alpha) \cap (k^{-1}(0) \times \{p_\gamma\})$$

is a closed subset of $R_\alpha(X_\alpha)$ which is not Borel. Hence, $R_\alpha(X_\alpha)$ is not Borel.

EXAMPLE 3. There exists a smooth dendroid Y such that none of the sets $R_2(Y)$ and $R_\beta(Y)$ and $S_\beta(Y)$ for $\beta \in \{3, 4, \ldots, \aleph_0, c\}$ is Borel. The space Y is a wedge of the spaces X and X_α of Examples 1 and 2.

In the dendroid X of Example 1 let q denote the unique point in $h(C \times \{-1\})$. Moreover, for each dendroid X_α in Example 2, let $q_\alpha = (q, p_\gamma) \in X_\alpha$ denote the point of smoothness of X_α.

There exist embeddings e: $X \to R^4$ and e_α: $X_\alpha \to R^4$ for $\alpha \in \{3, 4, \ldots, \aleph_0\}$ such that

(a) $e(q) = e_{\aleph_0}(q_{\aleph_0}) = e_3(q_3) = e_4(q_4) = \ldots$,
(b) $e(X) \cap e_\alpha(X_\alpha) = \{e(q)\}$ and $e_\alpha(X_\alpha) \cap e_\delta(X_\delta) = \{e(q)\}$ for $\alpha \neq \delta$,
(c) $\operatorname{diam} e_n(X_n) < 1/n$ for $n = 3, 4, \ldots$

Let $Y = e(X) \cup e_{\aleph_0}(X_{\aleph_0}) \cup e_3(X_3) \cup e_4(X_4) \cup \ldots \subset R^4$. Then Y is a dendroid which is smooth at the point $e(q)$. Note that $e(R_2(X))$ is a closed subset of $R_2(Y)$. Since e is an embedding and $R_2(X)$ is not Borel, it follows that $R_2(Y)$ is not Borel. A similar argument shows that $R_\beta(Y)$ and $S_\beta(Y)$ are not Borel for each $\beta > 2$. Observe that

$$R_1(Y) = e(R_1(X)) \cup e_{\aleph_0}(R_1(X_{\aleph_0})) \cup e_3(R_1(X_3)) \cup \ldots$$

and $\operatorname{cl}(R_1(Y)) = R_1(Y) \cup \{e(q)\}$. Hence, the sets $R_1(Y)$ and $S_2(Y)$ are both G_δ and F_σ.

In [N1] it was proved that if X is a planable dendroid and J is an arc in X then $S_3(X) \cap J$ is $G_{\delta\sigma}$ (see also Theorem 1(3–5)). In non-planar dendroids the situation is more complicated as the next example shows.

Recall that a function f: $X \to [0, 1]$ is said to be *upper semi-continuous* if $f^{-1}([t, 1])$ is closed in X, for each $t \in [0, 1]$. We are indebted to the referee for a simplification of our proof of the following:

LEMMA 3. *There exists an upper semi-continuous function f: $C \to [0, 1]$ from the Cantor set into $[0, 1]$ such that*

(a) *$f(C)$ is not a Borel set,*
(b) *$\{x \in C$: $f^{-1}(f(x))$ is countable$\}$ is a countable subset of C, and*
(c) *$f(C)$ is a nowhere dense set in $[0, 1]$.*

Proof. Let A be a countable dense subset of C. Then $C \setminus A$ is homeomorphic to the set of irrational numbers in the real line. Hence, as in Lemma 2 there exists a continuous function k: $C \setminus A \to [0, 1]$ such that

(a′) $k(C \setminus A)$ is not a Borel set,
(b′) $k^{-1}(k(x))$ is uncountable for each $x \in C \setminus A$, and
(c′) $k(C \setminus A)$ is a nowhere dense set in $[0, 1]$.

Let $\mathcal{U}_1, \mathcal{U}_2, \ldots$ be coverings of C such that, for each positive integer n, $\mathcal{U}_n = \{U_1^n, \ldots, U_{m_n}^n\}$ consists of pairwise disjoint closed-open sets of diameters $\leqslant 1/n$ and $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$. For each positive integer n define f_n: $C \to [0, 1]$ by

$$f_n(x) = \sup(k(U_i^n \setminus A)) \quad \text{provided } x \in U_i^n,\ 1 \leqslant i \leqslant m_n.$$

Obviously, all the functions f_n are continuous. Let f: $C \to [0, 1]$ be defined by $f(x) = \inf\{f_n(x)$: $n = 1, 2, \ldots\}$ for each $x \in C$. By [D, Corollary 10.4, p. 85], f is upper semi-continuous. It is easy to see that $f(x) = k(x)$ provided $x \in C \setminus A$. It follows that f has the properties (a), (b) and (c). The proof is complete.

EXAMPLE 4. There exists a smooth dendroid X such that

(a) $S_3(X)$ is contained in a subarc J of X,
(b) $R_1(X)$ is closed,
(c) $S_3(X) = R_c(X)$,
(d) $R_c(X)$ is an analytic, non-Borel set,
(e) $R_2(X) \cap J$ is a co-analytic, non-Borel set.

Let $T \subset [0, 1]$ be the Cantor ternary set and let $\tilde{f}$: $T \to [0, 1]$ be an upper semi-continuous function such that $\tilde{f}(T)$ is not a Borel set and

$$D = \{x \in T\colon \tilde{f}^{-1}(\tilde{f}(x)) \text{ is countable}\}$$

is a countable set. Let $D = \{x_1, x_2, \ldots\}$. For each $i = 1, 2, \ldots$ let $T_i = T \cap [0, 3^{-i}]$. Let

$$C = T \times \{0\} \cup \bigcup_{i=1}^{\infty} \{x_i\} \times T_i.$$

Define $f\colon C \to [0, 1]$ by $f(x, t) = \tilde{f}(x)$. Then C is a Cantor set and f is an upper semi-continuous function such that $f(C) = \tilde{f}(T)$. We may reimbed C into $[0, 1]$ and we denote (by abusing notation) by C this copy of the Cantor set in $[0, 1]$.

Since f is upper semi-continuous there exist continuous functions $g_n\colon C \to [0, 1]$ for $n = 1, 2, \ldots$ such that $g_n(x) \geqslant g_{n+1}(x)$ and $\lim g_k(x) = f(x)$ for $n = 1, 2, \ldots$ and $x \in C$ (see for example [En, p. 88]).

Define for each positive integer n a function $f_n\colon C \to [0, 2]$ as follows: Let $f_1(x) = 2$ for $x \in C$ and $f_k(x) = g_k(x) + 1/k$ for $x \in C$ and $k = 2, 3, \ldots$ Then $f_k(x) > f_{k+1}(x)$ and $\lim f_n(x) = f(x)$ for each $k = 1, 2, \ldots$ and $x \in C$.

For each positive integer k and $x \in C$ let J_k^x denote the straight-line segment in R^3 with end-points $(1/k, x/k, f_k(x))$ and $(1/(k+1), x/(k+1), f_{k+1}(x))$.

For each $x \in C$ let $J^x = J_1^x \cup J_2^x \cup \ldots$ Then

$$\mathrm{cl}(J^x) = J^x \cup \{(0, 0, f(x))\}$$

is an arc in R^3. Moreover, J^x is contained in the plane $\{(t_1, t_2, t_3)\colon t_2 = x \cdot t_1\}$. It follows that $J^x \cap J^y = \emptyset$ for $x \neq y$.

For each positive integer k let $F_k = \bigcup_{x \in C} J_1^x \cup \ldots \cup J_k^x$. By construction and the continuity of the functions $\{f_n\}$ it follows that the sets F_k are compact for $k = 1, 2, \ldots$

Let J be the straight-line segment in R^3 with end-points $(0, 0, 0)$ and $(0, 0, 2)$. Let $X = J \cup \bigcup_{k=1}^{\infty} F_k$. An easy argument shows that X is a dendroid which is smooth with respect to the point $(0, 0, 0)$. Moreover,

$$R_c(X) = S_3(X) = \{(0, 0, f(x))\colon x \in C\} \subset J.$$

An alternative construction of X may be obtained as follows: Let $Z = (\{2\} \cup C) \times [0, 2]$ and let G be the decomposition of Z into points and the sets $(\{2\} \cup f^{-1}([t, 2])) \times \{t\}$, $t \in [0, 2]$. It is easy to see that the quotient space Z/G is homeomorphic to the dendroid X above.

EXAMPLE 5. We construct a dendroid Z such that $R_1(Z)$ is co-analytic and not Borel.

Let C be a Cantor set. Let $f\colon C \to C$ be a continuous, onto function such that there is a G_δ-subset G of C such that $f(G)$ is not Borel. Let $F_1 \subset F_2 \subset \ldots$ be compact sets in C such that $C \backslash G = F_1 \cup F_2 \cup \ldots$

Let $\sim$ be the equivalence relation on $C \times [-1, \frac{1}{2}]$ such that $(x, t) \sim (y, s)$ if and only if either

(1) $t = s = -1$ or
(2) $(x, t) = (y, s)$ or
(3) $x = y \in F_n$ and $|s| = |t| \leqslant 1/n$.

Then clearly $\sim$ is upper semi-continuous. Let $Y = C \times [-1, \frac{1}{2}]/\sim$ be the quotient space and $h\colon C \times [-1, \frac{1}{2}] \to Y$ the quotient map. Then Y is a dendroid.

Define an equivalence relation $\sim_2$ on Y by setting $a \sim_2 b$ if $a, b \in Y$ and either $a = b$ or there exist $x, y \in C$ and $t \in [-1, 0]$ such that $f(x) = f(y)$, $a = h(x, t)$ and $b = h(y, t)$. Then $\sim_2$ is upper semi-continuous. Let $Z = Y/\sim_2$ be the quotient space and let $k\colon Y \to Z$ be the quotient map. Then Z is a dendroid.

Define an embedding $m\colon C \to C \times [-1, \frac{1}{2}]$ by $m(c) = (c, 0)$. Let $A = k \circ h \circ m(C)$. Then A is a closed subset of Z. Define $n\colon A \to C$ by $n(k \circ h \circ m(c)) = f(c)$. Then n is a well-defined function. In fact n is a homeomorphism of A onto C. If $c \in C$ then $k \circ h \circ m(c) \in R_1(Z)$ if and only if $f(c) \notin f(G)$. Hence, $A \cap R_1(Z) = A \backslash n^{-1}(f(G))$. Since n is a homeomorphism of A onto C and $f(G)$ is analytic and not Borel it follows that $R_1(Z) \cap A$ is co-analytic and not Borel. Since A is closed in Z it follows by Theorem 2 that $R_1(Z)$ is co-analytic and not Borel.

If in Example 5 for each $x \in G$ there exists $y \in G$ with $y \neq x$ and with $f(y) = f(x)$ then $A \subset R_1(Z) \cup S_3(Z)$. So in this case neither $R_1(Z)$ nor $S_3(Z)$ is a Borel set.

The following observation solves a problem posed in [N3, Problem 9.5].

THEOREM 5. *If X is a dendroid and $x \in X$, then there exists a function $f\colon X \to [0, 1]$ such that $f(x) = 0$ and $f|_{[x,y]}\colon [x, y] \to [0, f(y)]$ is a homeomorphism for each $y \in X$.*

Proof. Let $C(X)$ denote the hyperspace of all subcontinua of X and let $\mu\colon C(X) \to [0, 1]$ be a Whitney map (see. e.g. [Na]), i.e. μ is continuous, $\mu(X) = 1$, $\mu(\{y\}) = 0$ for $y \in X$, and $\mu(Y) < \mu(Z)$ if $Y, Z \in C(X)$, $Y \subset Z$ and $Y \neq Z$. Let $g\colon X \to C(X)$ be defined by $g(y) = [x, y]$. Then $g|_{[x,y]}\colon [x, y] \to g([x, y])$ is a homeomorphism, for each $y \in X$. Now, it suffices to let $f = \mu \circ g$.

We remark that if the map f given in Theorem 5 is continuous, then the dendroid X is smooth with respect to the point x. Conversely, if X is smooth with respect to x, then f can be chosen to be continuous.

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Received 6 February 1990;
in revised form 29 May 1990

Strong cellularity and global asymptotic stability

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Abstract. For (semi)dynamical systems on infinite-dimensional Banach spaces, a topological characterization of nonempty compact invariant globally asymptotically stable sets is given. The proofs are based on a paper by McCoy [11] and on other results of infinite-dimensional topology.

I. Introduction and the finite-dimensional case. Let $(X, \|\cdot\|)$ be a Banach space. The origin of X is denoted by 0_X. The closed ball and sphere of radius r centered at 0_X are denoted by $B(r)$ and $\partial B(r)$, respectively. In general, ∂ denotes the boundary of sets in X. The distance between a point $x \in X$ and a nonempty set $Y \subset X$ is defined as $d(x, Y) = \inf\{\|x-y\| \mid y \in Y\}$.

A closed subset C of X is called a *cell* in X if there exists a homeomorphism from the pair $(B(1), \partial B(1))$ onto the pair $(C, \partial C)$. A subset A of X is called *cellular* if there is a cellular sequence for A, i.e. a sequence $\{C_n\}$ of cells in X such that $\bigcap\{C_n \mid n \in N\} = A$ and $C_{n+1} \subset \operatorname{int}(C_n)$ for each $n \in N$. A subset A of X is called *strongly cellular* if there is a strongly cellular sequence for A, i.e. a cellular sequence $\{C_n\}$ with the additional property that for each open set U in X containing A, there is an integer n such that $C_n \subset U$.

A subset A of X is called *point-like* if $X \setminus A$ is homeomorphic to $X \setminus \{0_X\}$. A compact connected subset A of X is cellular if and only if it is point-like. Strongly cellular subsets are compact and connected. Compact subsets of infinite-dimensional Banach spaces are point-like and cellular. In the finite-dimensional case, cellularity is equivalent to strong cellularity. For these and other properties of cellularity resp. strong cellularity, see [11], [10].

The continuous mapping $\pi\colon \boldsymbol{R} \times X \to X$ ($\pi\colon \boldsymbol{R}^+ \times X \to X$) is called a *dynamical (semidynamical) system* if $\pi(0, x) = x$ for all $x \in X$ and $\pi(t+\tau, x) = \pi(t, \pi(\tau, x))$ for all $t, \tau \in \boldsymbol{R}$, $x \in X$ (for all $t, \tau \in \boldsymbol{R}^+$, $x \in X$). In most applications, dynamical systems are induced (both on finite- and infinite-dimensional Banach spaces) by ordinary differential equations. Similarly, in most applications, semidynamical systems are induced (on infinite-dimensional Banach spaces) by retarded or partial differential equations.

This paper was written while the author was an Alexander-von-Humboldt Research Fellow at the Mathematical Institute, University of Augsburg, Germany.