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W R O C Ł A W S K A D R U K A R N I A N A U K O W A

The prime spectrum of an infinite product of copies of Z

by

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Abstract. G. Cherlin, using model-theoretic techniques, has characterized the prime and maximal ideals in the direct product of copies of the ring of integers. In this paper, we obtain a characterization of these ideals using less machinery. As a consequence, we are able to obtain more information about the structure and order type of the prime spectrum of the ring.

§ 0. Introduction. Let I be an infinite index set and let $R = \prod_I \boldsymbol{Z}$, the direct product of copies of $\boldsymbol{Z}$. The goals of this paper are to characterize the prime ideals of R and to discuss the structure of $\mathrm{Spec}(R)$.

It is known that the ultrafilters $\mathscr{U}$ on I are associated in a natural way with the minimal prime ideals of R (denoted by $(\mathscr{U})$). $R/(\mathscr{U})$ is isomorphic to an ultrapower of $\boldsymbol{Z}$, and hence if $\mathscr{U}$ is non-principal, then $R/(\mathscr{U})$ is a so-called non-standard Peano ring (that is, a ring which is elementarily equivalent to $\boldsymbol{Z}$). G. Cherlin [C] studied the prime spectrum of such rings. His characterization is done in model-theoretic terms. Our construction has certain advantages. We use less machinery so our description is simpler. In addition, our elementwise description of the prime and maximal ideals enables us to obtain information about the order structure of the chain of prime ideals.

We use ultrafilters $\mathscr{F}$ on certain Boolean algebras of functions to describe the maximal ideals of R, denoted by $(\mathscr{F})$ (Theorems 3 & 4). If $\mathscr{U}$ is a principal ultrafilter on I, then rather trivially there are no non-maximal prime ideals strictly containing $(\mathscr{U})$. In any case the set of primes between a fixed maximal ideal and a prime ideal is linearly ordered. We show that over a fixed $(\mathscr{U})$ any two maximal chains of prime ideals are order isomorphic. Moreover, we prove that a maximal chain of prime ideals has cardinality either 2 or at least 2^{ω_1}. Finally, we prove that for ω_1-incomplete ultrafilters $\mathscr{U}$ the set of prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ is essentially the Dedekind completion of the lexicographic product of an η_1-set with 2.

We will always work in at least ZFC, that is, Zermelo–Fraenkel set theory with the axiom of choice. We will, in certain cases, use additional axioms.

Following the classical definition, we say that $\mathscr{U}$ is a *filter* on I if it is a subset of the power set of I that satisfies the following conditions:

1) $\varnothing \notin \mathscr{U}$ and $I \in \mathscr{U}$;
2) if J_1 and J_2 are in $\mathscr{U}$, then so is $J_1 \cap J_2$;
3) if $J \in \mathscr{U}$ and $J \subseteq J' \subseteq I$, then $J' \in \mathscr{U}$.

A filter $\mathscr{U}$ on I is an *ultrafilter* if $\mathscr{U}$ is maximal with respect to being a filter, or equivalently, if whenever $J \subseteq I$, then either $J \in \mathscr{U}$ or $I \backslash J \in \mathscr{U}$. Note that any filter can be extended to an ultrafilter. An ultrafilter $\mathscr{U}$ on I is *principal* if there exists an element p of I such that $\mathscr{U}$ consists of all subsets of I containing p. Other ultrafilters are said to be *non-principal.* It can be shown that there are $2^{2^{\operatorname{card}(I)}}$ non-principal ultrafilters on I. (See [CN] or [GJ].)

§ 1. Maximal ideals and minimal primes. Every prime ideal in a ring with identity contains a minimal prime ideal. Therefore we will first state a well known result characterizing the minimal prime ideals of R.

Now for $a = (a_i)_{i \in I} \in R$, we define the *zero set of* a to be $\zeta(a) = \{i \in I\colon\ a_i = 0\}$. For an ultrafilter $\mathscr{U}$ on I, we denote by $(\mathscr{U})$ the set $\{a \in R\colon\ \zeta(a) \in \mathscr{U}\}$. It is easy to show that $(\mathscr{U})$ is a prime ideal. The following proposition is well known and the proof is omitted.

PROPOSITION 1. *There is a bijection between the minimal prime ideals of R and the ultrafilters on I given by $\mathscr{U} \to (\mathscr{U})$.*

If $\mathscr{U}$ is a principal ultrafilter on I, then $R/(\mathscr{U})$ is isomorphic to $\mathbf{Z}$. Therefore, in this case, the only prime ideals of R strictly containing $(\mathscr{U})$ are maximal and correspond to the maximal ideals of $\mathbf{Z}$.

We now turn our attention to the more interesting maximal ideals of R. Let $\mathscr{U}$ be a non-principal ultrafilter on I, and let $p = (p_i)_{i \in I}$ be an element of R such that for each $i \in I$, p_i is a positive prime integer. We define $(\mathscr{U}, p)$ to be the set $\{a = (a_i)_{i \in I} \in R\colon\ \{i \in I\colon\ p_i \text{ divides } a_i\} \in \mathscr{U}\}$.

THEOREM 2. *For $\mathscr{U}$ and p as above, $(\mathscr{U}, p)$ is a maximal ideal of R containing $(\mathscr{U})$.*

Proof. Clearly $(\mathscr{U}, p)$ is an ideal of R containing $(\mathscr{U})$. Now let $x = (x_i)_{i \in I} \in R \backslash (\mathscr{U}, p)$. Then $A = \{i \in I\colon\ p_i \text{ does not divide } x_i\}$ is in $\mathscr{U}$. For each $j \in A$, there exists an integer s_j such that $x_j s_j - 1$ is divisible by p_j. Let $y = (y_i)_{i \in I}$ be the element of R defined via

$$y_i = \begin{cases} s_i & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

Then $xy - 1_R \in (\mathscr{U}, p)$. Therefore $R/(\mathscr{U}, p)$ is a field and hence $(\mathscr{U}, p)$ is a maximal ideal of R. ∎

Note. $(\mathscr{U}, p)/(\mathscr{U})$ is the principal ideal in $R/(\mathscr{U})$ generated by the image of p in $R/(\mathscr{U})$. For that reason, we call $(\mathscr{U}, p)$ a *principal maximal ideal of* R. It is natural to ask if every maximal ideal of R is principal. We now give an example which shows that there are non-principal maximal ideals and which motivates their general construction.

EXAMPLE. Let $I = \omega$ and for each sequence $p = (p_i) \in R$ of non-negative primes, let δ_p be the element of R whose nth coordinate is $q(n)/s(n)$, where $q(n)$ is the product of the first n prime integers and $s(n)$ is the largest expression of the form p_n^k which divides $q(n)$. Let J be the ideal generated by $\{\delta_p\colon\ p \text{ is a sequence of non-negative prime numbers}\}$. It is not difficult to see that J does not contain 1_R. Every proper ideal, in a ring with identity, is contained in a maximal ideal. If M is any maximal ideal of R containing J, then M cannot be of the form $(\mathscr{U}, p)$ because p is relatively prime to δ_p. Thus there must be non-principal maximal ideals.

We now investigate the non-principal maximal ideals of R. Let $\mathscr{S}$ be the set of functions $\sigma\colon\ I \to \{F\colon\ F \text{ a finite set of positive prime integers}\}$. One example of such a σ (with $I = \omega$) is given by defining $\sigma(n)$ to be $\{\text{first } n \text{ prime integers}\}$. This function was used implicitly in the preceding example. Now let σ and ϱ be elements of $\mathscr{S}$. We say that $\varrho \leqslant \sigma$ (ϱ is a *subfunction* of σ) if $\varrho(i) \subseteq \sigma(i)$ for each $i \in I$; in this case we define $\sigma \backslash \varrho$ to be the function given by $(\sigma \backslash \varrho)(i) = \sigma(i) \backslash \varrho(i)$ for each $i \in I$. If ϱ_1 and ϱ_2 are elements of $\mathscr{S}$, we define $\varrho_1 \wedge \varrho_2$ [respectively $\varrho_1 \vee \varrho_2$] to be the function defined by $(\varrho_1 \wedge \varrho_2)(i) = \varrho_1(i) \cap \varrho_2(i)$ $[(\varrho_1 \vee \varrho_2)(i) = \varrho_1(i) \cup \varrho_2(i)]$ for each $i \in I$. Finally, the blank function Φ is defined by $\Phi(i) = \varnothing$ for each $i \in I$. Let σ be a fixed element of $\mathscr{S}$. The set of subfunctions of σ forms a Boolean algebra. Therefore it makes sense to talk about ultrafilters on the set of subfunctions of σ. In particular, a subset $\mathscr{F}$ of this Boolean algebra is called an *ultrafilter on* σ if:

1) $\Phi \notin \mathscr{F}$ and $\sigma \in \mathscr{F}$;
2) if ϱ_1 and ϱ_2 are in $\mathscr{F}$, then so is $\varrho_1 \wedge \varrho_2$;
3) if $\varrho \leqslant \sigma$, then either $\varrho \in \mathscr{F}$ or $\sigma \backslash \varrho \in \mathscr{F}$.

We will use these ultrafilters to describe the maximal ideals of R. But first we give some definitions. Let σ be a fixed element of $\mathscr{S}$. For $a = (a_i)_{i \in I} \in R$, we define a subfunction ϱ_a of σ via $\varrho_a(i) = \{p \in \sigma(i)\colon\ p \text{ divides } a_i\}$. For $\varrho \in \mathscr{S}$, we define $\prod \varrho(i)$ to be $\prod_{p \in \varrho(i)} p$ if $\varrho(i)$ is non-empty and 1 otherwise. Now define $a_\varrho = (a_i)_{i \in I} \in R$, where $a_i = \prod \varrho(i)$. Note that $\varrho_{a_\varrho} = \varrho$ and a_{ϱ_a} divides a.

THEOREM 3. *Let $\mathscr{F}$ be an ultrafilter on some $\sigma \in \mathscr{S} \backslash \{\Phi\}$. Then the set $(\mathscr{F}) = \{a \in R\colon\ \varrho_a \in \mathscr{F}\}$ is a maximal ideal of R.*

Proof. Let $a, b \in R$ be such that $\varrho_a, \varrho_b \in \mathscr{F}$. Then $\varrho_{a+b} \geqslant \varrho_a \wedge \varrho_b \in \mathscr{F}$. Also if $r \in R$, then $\varrho_{ar} \geqslant \varrho_a$. Hence $(\mathscr{F})$ is an ideal. Finally, if $x \in R \backslash (\mathscr{F})$, then $\varrho_x \notin \mathscr{F}$, and hence $\sigma \backslash \varrho_x \in \mathscr{F}$. Therefore $a_{\sigma \backslash \varrho_x}$ is in $(\mathscr{F})$ and is relatively prime to x (at each coordinate). Hence there exists $s \in R$ such that $xs - 1_R \in (\mathscr{F})$. So $(\mathscr{F})$ is a maximal ideal of R. ∎

A function $\varrho \in \mathscr{S}$ will be called *principal* if $\varrho(i)$ is either $\varnothing$ or a singleton for each $i \in I$. If $\mathscr{F}$ contains a principal function ϱ, then $(\mathscr{F}) = (\mathscr{U}, p)$, for $\mathscr{U} = \{U_\delta\colon\ \delta \in \mathscr{F}\}$ where $U_\delta = \{i \in I\colon\ \delta(i) \neq \varnothing\}$ and $p = (p_i)_{i \in I}$ where $p_i = \prod \varrho(i)$. Conversely, it is clear that every maximal ideal of the form $(\mathscr{U}, p)$ can be described as an $(\mathscr{F})$ for a suitable choice of $\mathscr{F}$. Thus the maximal ideals described in Theorem 3 include the principal maximal ideals. The next result shows that these are all the maximal ideals.

THEOREM 4. *There is a surjection, given by $\mathscr{F} \to (\mathscr{F})$, between the set of ultrafilters on elements of $\mathscr{S}$ and the maximal ideals of R.*

Proof. First we introduce some notation. For $x=(x_i)_{i\in I}\in R$, let $\sqrt{x}=(y_i)_{i\in I}\in R$ where y_i is the product of all the distinct positive prime integers dividing x_i if $x_i\neq 0, 1$. If $x_i=0$ or 1, then $y_i=x_i$.

Now let M be any maximal ideal of R. If $x\in M$, then so is $\sqrt{x}$; for otherwise, $\sqrt{x}$ is relatively prime to an element $z\in M$ at each coordinate, and hence x is relatively prime to z, which is a contradiction.

Let $x\in M$ be such that $\sqrt{x}=x$. Define $\sigma\in\mathcal{S}$ such that for $i\in I$, $\sigma(i)=\{$positive prime integers that divide $x_i\}$. Let $\mathcal{F}=\{\varrho\leqslant\sigma\colon a_\varrho\in M\}$.

Claim 1. *$\mathcal{F}$ is an ultrafilter on σ.*

Clearly $\Phi\notin\mathcal{F}$ (since $1_R\notin M$) and $\sigma\in\mathcal{F}$ (since $a_\sigma=\sqrt{x}=x$). Also, if ϱ_1 and ϱ_2 are elements of $\mathcal{F}$ (i.e., a_{ϱ_1} and a_{ϱ_2} are in M), then $a_{\varrho_1\wedge\varrho_2}$ is also in M, since $a_{\varrho_1\wedge\varrho_2}$ is the greatest common divisor of a_{ϱ_1} and a_{ϱ_2}. Finally, let $\varrho\leqslant\sigma$. Then $a_\varrho a_{\sigma\backslash\varrho}=\sqrt{x}=x\in M$ and hence $\varrho\in\mathcal{F}$ or $\sigma\backslash\varrho\in\mathcal{F}$.

Claim 2. $(\mathcal{F})\subseteq M$.

Let $a\in(\mathcal{F})$. Then $\varrho_a\in\mathcal{F}$ and thus $a_{\varrho_a}\in M$. But a_{ϱ_a} divides a, so $a\in M$ proving the claim. By Theorem 3, $(\mathcal{F})$ is a maximal ideal and hence $(\mathcal{F})=M$. ∎

§ 2. The prime ideals of R. *From now on, we fix $\sigma\in\mathcal{S}\backslash\{\Phi\}$ and $\mathcal{F}$, an ultrafilter on σ.* As pointed out previously, every prime ideal contains a minimal prime ideal. So in particular $(\mathcal{F})$ contains a prime ideal of the form $(\mathcal{U})$ for some ultrafilter $\mathcal{U}$ on I. Note that this ultrafilter is again the ultrafilter $\mathcal{U}$ generated by $\{U_\varrho\colon\varrho\in\mathcal{F}\}$, where $U_\varrho=\{i\colon\varrho(i)\neq\varnothing\}$.

We now turn our attention to the prime ideals of R. Let $\varrho\in\mathcal{F}$ and let $x=(x_i)_{i\in I}\in R$. We define $T_\varrho(x_i)$ to be the greatest non-negative integer k such that $[\prod\varrho(i)]^k$ divides x_i if $x_i\neq 0$; we define $T_\varrho(0)$ to be ∞. If $h\colon I\to\boldsymbol{R}$ is any function, we say that h is *bounded on* $\mathcal{U}$ if for some $U\in\mathcal{U}$, the set $\{h(i)\colon i\in U\}$ is a bounded subset of $\boldsymbol{R}$. Otherwise h is *unbounded on* $\mathcal{U}$. Now let $g\colon I\to[1,\infty)$ be any function. We define $(\mathcal{F})_g$ by $(\mathcal{F})_g=\{x\in R\colon$ there exists a $\varrho\in\mathcal{F}$ such that the function $T_\varrho(x)/g$ is unbounded on $\mathcal{U}\}$, where $T_\varrho(x)/g$ is defined by $(T_\varrho(x)/g)(i)=T_\varrho(x_i)/g(i)$.

Theorem 5. *For any function $g\colon I\to[1,\infty)$, the set $(\mathcal{F})_g$ is a prime ideal of R satisfying $(\mathcal{U})\subset(\mathcal{F})_g\subset(\mathcal{F})$.*

Proof. Let $x=(x_i)_{i\in I}$, $y=(y_i)_{i\in I}\in(\mathcal{F})_g$. Then, since $\mathcal{F}$ is a filter, there exists $\varrho\in\mathcal{F}$ such that $T_\varrho(x)/g$ and $T_\varrho(y)/g$ are unbounded on $\mathcal{U}$. Now $T_\varrho(x_i+y_i)\geqslant\min\{T_\varrho(x_i), T_\varrho(y_i)\}$. Fix $U\in\mathcal{U}$, then let $W=\{i\in U\colon T_\varrho(x_i+y_i)\geqslant T_\varrho(x_i)\}$ and let $V=U\backslash W$ (so for $i\in V$, $T_\varrho(x_i+y_i)\geqslant T_\varrho(y_i)$). Either $W\in\mathcal{U}$ or $V\in\mathcal{U}$. In the first case, since $T_\varrho(x)/g$ is unbounded on W, so is $T_\varrho(x+y)/g$, and hence $T_\varrho(x+y)/g$ is unbounded on U. On the other hand, if $V\in\mathcal{U}$, the same argument applies to $T_\varrho(y)$. Hence $x+y\in(\mathcal{F})_g$. Also, if $r=(r_i)\in R$ then $T_\varrho(r_ix_i)\geqslant T_\varrho(x_i)$, so that $T_\varrho(rx)/g$ is unbounded on $\mathcal{U}$. Hence $rx\in(\mathcal{F})_g$ and $(\mathcal{F})_g$ is an ideal of R.

We now show that $(\mathcal{F})_g$ is prime. Suppose that x and y are elements of R such that $xy\in(\mathcal{F})_g$. Thus there exists $\varrho\in\mathcal{F}$ such that $T_\varrho(xy)/g$ is unbounded on all U in $\mathcal{U}$. Define $\tau\leqslant\varrho$ as follows:

$$\tau(i)=\{p\in\varrho(i)\colon \max\{k\colon p^k|x_i\}\geqslant\max\{k\colon p^k|y_i\}\}.$$

Observe that $T_\tau(x_iy_i)\geqslant T_\varrho(x_iy_i)$ and $T_\tau(x_iy_i)\leqslant 2T_\tau(x_i)$ for all i, by our definition of τ. Furthermore, either $\tau\in\mathcal{F}$ or $\varrho\backslash\tau\in\mathcal{F}$. In the first case it follows that, since $T_\tau(x)/g$ is unbounded on $\mathcal{U}$, x is in $(\mathcal{F})_g$. While if $\varrho\backslash\tau$ is in $\mathcal{F}$, then a similar argument shows that y is in $(\mathcal{F})_g$. Therefore $(\mathcal{F})_g$ is a prime ideal of R.

Now let $x=(x_i)\in(\mathcal{U})$; then $U_x=\{i\in I\colon x_i=0\}\in\mathcal{U}$ and hence for any $\varrho\in\mathcal{F}$, for all $i\in U_x$, $T_\varrho(x_i)=\infty$. Since $U\cap U_x\neq\varnothing$ for each $U\in\mathcal{U}$, it follows that $T_\varrho(x)/g$ is unbounded on $\mathcal{U}$, i.e., $x\in(\mathcal{F})_g$. Hence $(\mathcal{U})\subseteq(\mathcal{F})_g$.

Finally, let $x=(x_i)_{i\in I}\in(\mathcal{F})_g$. Then there exists $\varrho\in\mathcal{F}$ such that $T_\varrho(x)/g$ is unbounded on $\mathcal{U}$. Hence the set $\{i\in I\colon T_\varrho(x_i)\geqslant 1\}$ is in $\mathcal{U}$. Clearly, the set $\{i\colon\prod\varrho(i)$ divides $x_i\}$ is also in $\mathcal{U}$, and so $x\in(\mathcal{F})$. ∎

We next want to investigate the relationship between $(\mathcal{F})_f$ and $(\mathcal{F})_g$ for different functions f and g. But before we do that we give a result which is Corollary 4.10 in [C]. We include a proof (which is different from Cherlin's) for completeness.

Proposition 6. *Let $\mathcal{U}$ be the ultrafilter on I determined by $\mathcal{F}$. Then any two prime ideals P and Q between $(\mathcal{U})$ and $(\mathcal{F})$ are comparable.*

Proof. Suppose to the contrary that there exist $x\in P\backslash Q$ and $y\in Q\backslash P$. Since $x, y\in(\mathcal{F})$, there exists $\varrho\in\mathcal{F}$ such that a_ϱ divides both x and y. Now define a subfunction τ of ϱ by $\tau(i)=\{p\in\varrho(i)\colon T_p(x_i)\geqslant T_p(y_i)\}$. Let $\delta=\varrho\backslash\tau$. Then either τ or δ is in $\mathcal{F}$. Assume that $\tau\in\mathcal{F}$. Factor x into $x'\cdot s$, where for each $i\in I$ the only primes dividing x_i' are in $\tau(i)$, while s_i is relatively prime to $\prod\tau(i)$. Notice that $s\notin(\mathcal{F})$, forcing x' into $P\backslash Q$. So replace x with x' and do the same for y. Now it is clear that y divides x, so $x\in Q$, which is a contradiction. Otherwise δ is in $\mathcal{F}$ so, with suitable substitutions, x divides y. Hence P and Q are comparable. ∎

We introduce some notation. Let $g\colon I\to[1,\infty)$ and let $x\in R$. Then $x^g=(x_i^{[g(i)]})_{i\in I}$, where [ ] denotes the greatest integer function.

Proposition 7. *Let $f, g\colon I\to[1,\infty)$. If f/g is unbounded on $\mathcal{U}$, then $(\mathcal{F})_f\subsetneq(\mathcal{F})_g$.*

Proof. Consider the element $y=a_\sigma^f$. Then $y\notin(\mathcal{F})_f$, yet clearly y is an element of $(\mathcal{F})_g$. So, by Proposition 6, $(\mathcal{F})_f\subsetneq(\mathcal{F})_g$. ∎

We note that $(\mathcal{F})_f=(\mathcal{F})_g$ if and only if both f/g and g/f are bounded on $\mathcal{U}$. One direction follows from Proposition 7, the other from the definition of $(\mathcal{F})_g$. As a consequence, $(\mathcal{F})_g=(\mathcal{F})_{[g]}$. So it follows that when considering prime ideals $(\mathcal{F})_g$ we can restrict our attention to integer-valued functions g. We will assume that the range of g is either ω or $[1,\infty)$ depending on which is more convenient.

We now define another prime ideal using functions $g\colon I\to[1,\infty)$. Let $(\mathcal{F})^g=\{x\in R\colon$ there exists $\varrho\in\mathcal{F}$ such that $g/T_\varrho(x)$ is bounded on $\mathcal{U}\}$. We make the convention that if $\{i\colon T_\varrho(x_i)=0\}\in\mathcal{U}$, then $g/T_\varrho(x)$ is unbounded on $\mathcal{U}$.

THEOREM 8. *The set* $(\mathscr{F})^g$ *is a prime ideal of R which contains* $(\mathscr{F})_g$. *Furthermore,* $(\mathscr{F})^g$ *is the smallest prime ideal below* $(\mathscr{F})$ *containing the element* $t = a^g_\sigma$.

Proof. The proof that $(\mathscr{F})^g$ is an ideal is analogous to the proof that $(\mathscr{F})_g$ is an ideal as given in Theorem 5. To show that the ideal is prime let x, y be elements of R such that $xy \in (\mathscr{F})^g$. Then there exists $\varrho \in \mathscr{F}$ and $U \in \mathscr{U}$ such that $g/T_\varrho(xy)$ is bounded on U. As in Theorem 5, there exists $\tau \leqslant \varrho$ such that $T_\tau(x_i y_i) \leqslant 2T_\tau(x_i)$ for all i. Clearly then $g/T_\tau(x)$ is bounded on U. If $\tau \in \mathscr{F}$, then $x \in (\mathscr{F})^g$. Otherwise $\varrho\backslash\tau$ is in $\mathscr{F}$, which implies that y is in $(\mathscr{F})^g$. So $(\mathscr{F})^g$ is prime.

Finally, let Q be a prime ideal of R between $(\mathscr{U})$ and $(\mathscr{F})$ containing the element t. Pick $x \in (\mathscr{F})^g$. Then there exists a positive integer n, an element ϱ in $\mathscr{F}$ and $U \in \mathscr{U}$ such that $g(i)/n \leqslant T_\varrho(x_i)$ for all $i \in U$. Now let $\delta = \sigma\backslash\varrho$ and let $z = a_\delta$. It follows that t divides $(xz)^n$. Thus either $x \in Q$ or $z \in Q$. But, since $\delta \notin \mathscr{F}$, $z \notin (\mathscr{F})$. Therefore $x \in Q$, proving that $(\mathscr{F})^g \subseteq Q$. ∎

Note. If g is constant or even bounded on some $U \in \mathscr{U}$, then $(\mathscr{F})^g = (\mathscr{F})$.

If $\varrho \in \mathscr{F}$, then $\mathscr{F}$ defines an ultrafilter $\mathscr{F}'$ on ϱ in a natural fashion: $\tau \leqslant \varrho$ is in $\mathscr{F}'$ if and only if τ is in $\mathscr{F}$. It is not difficult to see that these two ultrafilters determine the same maximal ideal, namely $(\mathscr{F}) = (\mathscr{F}')$. In particular, if $x = a^h_\varrho$ for some function $h\colon I \to \omega$, then $(\mathscr{F})^h$ is the smallest prime ideal in $(\mathscr{F})$ which contains x.

PROPOSITION 9. $(\mathscr{F})_g$ *is the largest prime ideal of R below* $(\mathscr{F})$ *not containing the element* $y = a^g_\sigma$.

Proof. In Theorem 8, we showed that $y \notin (\mathscr{F})_g$. Now let Q be a prime ideal of R between $(\mathscr{U})$ and $(\mathscr{F})$ not containing y. Assume that Q is not contained in $(\mathscr{F})_g$ so there exists $x \in Q\backslash(\mathscr{F})_g$. Since $x \in (\mathscr{F})$, we can, as in the proof of Theorem 8, assume that x equals a^h_ϱ for some function $h\colon I \to \omega$ and some ϱ in $\mathscr{F}$. Since x is not in $(\mathscr{F})_g$, $h(i)/g(i)$ is bounded on some $U \in \mathscr{U}$; i.e. $h(i)/g(i) < N$ on U for some positive integer N. Hence y is in $(\mathscr{F})^h$. Since $(\mathscr{F})^h$ is the smallest prime ideal containing x, $(\mathscr{F})^h \subseteq Q$. Thus $y \in Q$, which is a contradiction. ∎

The following corollaries are now immediate using Theorem 8 and Proposition 9.

COROLLARY 10. *There are no prime ideals between* $(\mathscr{F})_g$ *and* $(\mathscr{F})^g$.

Note that we then have $(\mathscr{F})_g \subseteq (\mathscr{F})^f$ if and only if $(\mathscr{F})_g \subseteq (\mathscr{F})_f$. Furthermore, $(\mathscr{F})^g = (\mathscr{F})^f$ if and only if $(\mathscr{F})_g = (\mathscr{F})_f$. We will show in the next section that not every prime ideal is of the form $(\mathscr{F})_g$ or $(\mathscr{F})^g$. However, all the prime ideals can be described in terms of these primes.

COROLLARY 11. *Every prime ideal below* $(\mathscr{F})$ *can be written as the intersection of a family of* $(\mathscr{F})_g$*'s.*

COROLLARY 12. *Every prime ideal below* $(\mathscr{F})$ *can be written as the union of a family of* $(\mathscr{F})^g$*'s.*

Observe that the prime ideals between a given $(\mathscr{U})$ and $(\mathscr{F})$ are completely determined by functions g from I to $[1, \infty)$ modulo $\mathscr{U}$. In particular, if $(\mathscr{F})$ and $(\mathscr{F}')$ are two maximal ideals containing $(\mathscr{U})$, then the chain of primes contained in $(\mathscr{F})$ is order isomorphic to the chain of primes contained in $(\mathscr{F}')$.

In view of the note after Theorem 8, the question arises as to whether functions g, from I to ω, which are unbounded on every $U \in \mathscr{U}$ exist. The next few paragraphs deal with this question.

For f and g functions from I to ω, we say that $f \sim g$ (modulo $\mathscr{U}$) if f/g is bounded away from 0 and ∞ on some $U \in \mathscr{U}$. Note that $\sim$ is an equivalence relation. The equivalence classes are linearly ordered and in order reversing bijection with the set of prime ideals of the form $(\mathscr{F})_g$ $[(\mathscr{F})^g]$ for a fixed $\mathscr{F}$.

An ultrafilter $\mathscr{U}$ on I is called ω_1*-complete* if every countable partition of I contains an element of $\mathscr{U}$ (see [CN]). An ultrafilter is called ω_1*-incomplete* if it is not ω_1-complete.

Let $\mathscr{U}$ be an ω_1-complete ultrafilter on I, and let g be any function from I to ω. Then $\{g^{-1}(n)\colon n \in \omega\}$ forms a countable partition of I, and hence g is bounded on a filter element. Therefore, there are no prime ideals of R strictly contained between $(\mathscr{U})$ and $(\mathscr{F})$ by Corollary 11. Since every principal ultrafilter on I is ω_1-complete, there are no prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ as noted after Proposition 1.

Let $\mathscr{U}$ be an ω_1-incomplete ultrafilter on I. Then there exists a countable partition $\{A_n\colon n \in \omega\}$ of I such that $A_n \notin \mathscr{U}$ for every $n \in \omega$. Define $g\colon I \to [1, \infty)$ via $g(A_n) = n$; then g is unbounded on every $U \in \mathscr{U}$. Now for $r \in [1, \infty)$, let $g_r\colon I \to [1, \infty)$ be defined by $g_r(i) = g(i)^r$. Then g_r is unbounded on every $U \in \mathscr{U}$, and if $r < s$, then g_s/g_r is unbounded on every $U \in \mathscr{U}$. Thus there are at least $\mathfrak{c} = |\boldsymbol{R}|$ equivalence classes (modulo $\mathscr{U}$) of functions g from I to $[1, \infty)$ (or to ω).

If I has non-measurable cardinal (e.g., if I is countable), then every non-principal ultrafilter on I is ω_1-incomplete. A set with a measurable cardinal (if such a set exists) has a non-principal ω_1-complete ultrafilter on it. (See [CN].)

The following example, which is due to John Kulesza, shows that, assuming the Continuum Hypothesis (CH), there exists an ultrafilter $\mathscr{U}$ on a set I of cardinality $\mathfrak{c}$ for which there are more than $\mathfrak{c}$ many equivalence classes (modulo $\mathscr{U}$) of functions g from I to ω.

EXAMPLE. Let $\mathscr{V}$ be a non-principal ultrafilter on ω and let $\mathscr{W}$ be a uniform ultrafilter on $\boldsymbol{R}$, that is, an ultrafilter each element of which has cardinality $\mathfrak{c}$. Define an ultrafilter $\mathscr{U}$ on $I = \omega \times \boldsymbol{R}$ by $U \in \mathscr{U}$ if and only if

$$\{x\colon \{y\colon (x, y) \in U\} \in \mathscr{V}\} \in \mathscr{W}.$$

Assuming CH we well order $\boldsymbol{R}$ as $\{y_\lambda\colon \lambda < \omega_1\}$. Let $\{f_\alpha\colon \alpha < \omega_1\}$ be a collection of $\mathfrak{c}$ many functions from I to ω. For $\alpha < \omega_1$, let $g_\alpha\colon \omega \times \{y_\alpha\} \to \omega$ be such that for all $\lambda \leqslant \alpha$, $\{n \in \omega\colon g_\alpha(n, y_\alpha) < f_\lambda(n, y_\alpha)\}$ is finite. Define $g\colon I \to \omega$ by $g|_{\omega \times \{y_\alpha\}} = g_\alpha$.

CLAIM. *For each* $\lambda < \omega_1$, $\{(x, y)\colon f_\lambda(x, y) \leqslant g(x, y)\} \in \mathscr{U}$.

Assume, to the contrary, that $\{(x, y)\colon f_\lambda(x, y) > g(x, y)\} \in \mathscr{U}$. Since $\mathscr{W}$ is uniform, there is a $y = y_\alpha$ such that $\alpha > \lambda$ and $\{x\colon f_\lambda(x, y_\alpha) > g(x, y_\alpha)\} \in \mathscr{V}$. But $f_\lambda|_{\omega \times \{y_\alpha\}} \leqslant g_\alpha = g|_{\omega \times \{y_\alpha\}}$. This proves the claim.

Now let $h(x, y) = g(x, y)\cdot(1+x+y)$. Then for each $U \in \mathscr{U}$, h/g is unbounded on U, so $h > f_\lambda$ for each $\lambda < \omega_1$. Therefore there are more than continuum many equivalence classes (modulo U) of functions f from I to ω.

§ 3. The order structure of prime ideals. We want to investigate the order type of the chain of prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$. As we have seen, if $\mathscr{U}$ is ω_1-incomplete, then there are at least $\mathfrak{c}$ such prime ideals. Now if $|I| = \lambda$, then the ring R has 2^λ elements and hence the chain of prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ contains at most 2^{2^λ} elements.

We can actually get more information on the structure of the order of the prime ideals of the form $(\mathscr{F})_g$ in the case where $I = \omega$. A linearly ordered set S is called an *η_1-set* if given any countable subsets F and G of S such that every element of F is less than every element of G, then there exists an element $h \in S$ such that $f < h < g$ for every $f \in F$ and every $g \in G$.

THEOREM 13. *Let $I = \omega$ and let $\mathscr{F}$ be an ultrafilter on some $\sigma \in \mathscr{S}$. Assume that the associated ultrafilter $\mathscr{U}$ on I is non-principal. Then the set of prime ideals of the form $(\mathscr{F})_g$ between $(\mathscr{U})$ and $(\mathscr{F})$, where g is unbounded on $\mathscr{U}$, is an η_1-set.*

Proof. Recall that the prime ideals of the form $(\mathscr{F})_g$ between $(\mathscr{U})$ and $(\mathscr{F})$ are in order reversing bijection with the equivalence classes of functions f from I to ω, where $f \sim g$ if f/g is bounded away from 0 and ∞ on some $U \in \mathscr{U}$. The order is given by $f < g$ if g/f is unbounded on $\mathscr{U}$. It is therefore sufficient to show that the set of equivalence classes with this order is an η_1-set.

Let $F = \{f_0, f_1, f_2, \ldots\}$ and $G = \{g_0, g_1, g_2, \ldots\}$ be countable sets of equivalence classes of functions f_n and g_n from I to ω such that $f_n < g_m$ for all n and m. Clearly, we can assume that

$$f_0 \leqslant f_1 \leqslant f_2 \leqslant \ldots \leqslant g_2 \leqslant g_1 \leqslant g_0.$$

We will assume that F and G have infinitely many equivalence classes. The other cases are similar and easier. Therefore, we can assume that the above inequalities are strict.

CLAIM. *We can assume that $f_n(i) \leqslant f_{n+1}(i)$ and $g_{n+1}(i) \leqslant g_n(i)$ for all $i \in I$.*

Let $\hat{f}_n$ and $\hat{g}_n$ be defined via $\hat{f}_n(i) = \max_{k \leqslant n} f_k(i)$ and $\hat{g}_n(i) = \inf_{k \leqslant n} g_k(i)$. Then $f_n \sim \hat{f}_n$ and $g_n \sim \hat{g}_n$. So we have the claim.

It is not too difficult to see that it is possible to define a countable collection $\{U_k\colon k = 0, 1, 2, \ldots\}$ of elements of $\mathscr{U}$ satisfying the following:

1) $U_0 = I = \omega$;
2) $U_k \supseteq U_{k+1}$ for all k;
3) $\bigcap U_k = \varnothing$ (e.g., delete k from U_k for each k);
4) $g_k(i)/f_k(i) > k$ for all $i \in U_k$ and for all k.

To accomplish 4) we use the fact that, since g_k/f_k is unbounded on $\mathscr{U}$, the set $\{i\colon g_k(i)/f_k(i) > k\}$ is in $\mathscr{U}$.

Let h be the function from I to $[1, \infty)$ defined via $h(i) = \sqrt{f_k(i)\, g_k(i)}$ for $i \in U_k \setminus U_{k+1}$. Note that conditions 1), 2) and 3) assure that h is well defined.

CLAIM. *$h > f_k$ for all k.*

Suppose to the contrary that for a fixed k, $h(i)/f_k(i) < B$ on some some $U \in \mathscr{U}$ and some $B \in \omega$. Let m be a positive integer larger than B^2 and k, and let $i \in U \cap U_m$. Then $h(i) = \sqrt{f_r(i)\, g_r(i)}$ for some $r \geqslant m$. Hence

$$\frac{h(i)}{f_k(i)} = \sqrt{\frac{f_r(i)\, g_r(i)}{f_k(i)^2}} \geqslant \sqrt{\frac{f_r(i)\, g_r(i)}{f_r(i)^2}} = \sqrt{\frac{g_r(i)}{f_r(i)}}.$$

By condition 4) and the assumptions we have on r and m, we also know that

$$\sqrt{g_r(i)/f_r(i)} > \sqrt{r} \geqslant \sqrt{m} \geqslant B.$$

The two inequalities combined give a contradiction, so the claim is proved.

The proof that $h < g_k$ for all k is done in a similar fashion. ∎

Remarks. (i) The proof of Theorem 13 can be generalized to the case where $\mathscr{U}$ is an ω_1-incomplete ultrafilter on an arbitrary index set I. The only potential difficulty is in condition 3) on the sets (U_k). So let $(I_k)_{k=1,2,\ldots}$ be a countable partition of I such that no I_k is in $\mathscr{U}$. Then we can replace U_k with $U_k \setminus \bigcup_{i=1}^{k} I_i$ if need be, which satisfies condition 3).

(ii) It follows from Theorem 13 that, if $\mathscr{U}$ is ω_1-incomplete, then the set of prime ideals of the form $(\mathscr{F})_g$ and $(\mathscr{F})^g$, where g is unbounded, is the lexicographic product of an η_1-set with $\{0, 1\}$.

(iii) It follows from (ii) and Corollaries 11 & 12 that, if $\mathscr{U}$ is ω_1-incomplete, then the primes between $(\mathscr{F})$ and $(\mathscr{U})$ can be obtained from the Dedekind completion of an η_1-set by splitting each element of the η_1-set into two consecutive elements and then adjoining a first element, a last element and a next to last element (respectively $(\mathscr{U})$, $(\mathscr{F})$ and $(\mathscr{F})_g$ where g is any constant). Furthermore, since any Dedekind complete space containing an η_1-set has cardinality at least $2^{\aleph_1}$ (see [GJ]), there are at least $2^{\aleph_1}$ primes between $(\mathscr{U})$ and $(\mathscr{F})$. In particular, if $I = \omega$, then, assuming CH, there are exactly $2^{\aleph_1}$ prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$.

COROLLARY 14. *Let $\mathscr{U}$ be an ω_1-incomplete ultrafilter on I. Then no prime between $(\mathscr{U})$ and $(\mathscr{F})$ is both the union of countably many $(\mathscr{F})^g$'s and the intersection of countably many $(\mathscr{F})_g$'s.*

Proof. In the Dedekind completion of an η_1-set, no singleton is the intersection of countably many non-degenerate intervals. ∎

COROLLARY 15. *Let $I = \omega$ and assume that $2^{\aleph_0} < 2^{\aleph_1}$. (This set-theoretic assumption will hold if, for example, the continuum hypothesis is assumed.) Then there exist prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ which are neither a union of countably many $(\mathscr{F})^g$'s nor an intersection of countably many $(\mathscr{F})_g$'s.*

Proof. There are only continuum many ideals of the form $(\mathscr{F})^g$ or $(\mathscr{F})_g$ because there are only continuum many choices for the functions g. Hence, there are only $\mathfrak{c}^{\aleph_0} = \mathfrak{c}$ intersections or unions of countably many sets of this form. But as noted in Remark (iii) above, there are $2^{\aleph_1}$ prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$. ∎

We close with some questions:

1. Does Corollary 15 hold with no set-theoretic assumptions?
2. Does Corollary 15 hold under MA+ $\neg$CH?
3. Is there a generalization of Corollary 15 to higher cardinals?
4. Given an index set I and an ultrafilter $\mathscr{U}$ on I, is it possible to determine the cardinality of the set of prime ideals between $(\mathscr{U})$ and $(\mathscr{F})$ (perhaps assuming some additional set-theoretic hypothesis)?

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On some subclasses of Darboux functions

by

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Abstract. The maximal additive, multiplicative and lattice-like classes for some classes of real functions are computed.

1. Introduction. We shall consider mainly real functions of a real variable, however, some of the considered functions will be defined on different sets, and sometimes the range sets will be different. Let us settle some of the notations to be used in the article.

$\mathscr{C}onst$ – the class of constant functions,
$\mathscr{C}on$ – the class of connected functions,
$\mathscr{C}$ – the class of continuous functions,
$\mathscr{A}$ – the class of almost continuous functions,
$\mathscr{D}$ – the class of Darboux functions,
$\mathscr{D}\cap\mathscr{B}_1$ – the class of Darboux functions of the first class of Baire,
$\mathscr{F}$ – the class of functionally connected functions ([5]),
$lsc\,(usc)$ – the class of lower (upper) semicontinuous functions,
$\mathscr{M}$ – the class of Darboux functions f with the following property: if x_0 is a right-hand (left-hand) point of discontinuity of f, then $f(x_0)=0$ and there is a sequence (x_n) converging to x_0 such that $x_n > x_0$ $(x_n < x_0)$ and $f(x_n)=0$,
$\mathscr{D}_0$ – the class of all functions f such that for each x from the domain $f(x)\in L^-(f,x)\cap L^+(f,x)$ and the sets $L^-(f,x)$, $L^+(f,x)$ are closed intervals.

The symbols $L^-(f,x)$, $L^+(f,x)$ denote the cluster sets from the left and right, respectively, of the function f at the point x.

Notice that if $f\in\mathscr{M}$, then the set E of all points of discontinuity of f is nowhere dense and $f(x)=0$ for each x in $\bar{E}$. Consequently, f is a function of the first class of Baire, hence $\mathscr{M}\subseteq\mathscr{D}\cap\mathscr{B}_1$. Since $\mathscr{A}\cap\mathscr{B}_1=\mathscr{C}on\cap\mathscr{B}_1=\mathscr{D}\cap\mathscr{B}_1$ ([1]), we have $\mathscr{M}\subseteq\mathscr{A}$. Thus for the classes of real functions defined on an interval we have

$$\mathscr{C}\subseteq\mathscr{M}\subseteq\mathscr{A}\subseteq\mathscr{C}on\subseteq\mathscr{F}\subseteq\mathscr{D}\subseteq\mathscr{D}_0.$$

Let $\mathscr{X}$ be a class of real functions. The *maximal additive* (*multiplicative*, *lattice-like*, respectively) *class* for $\mathscr{X}$ is defined to be the class of all $f\in\mathscr{X}$ for which $f+g\in\mathscr{X}$ $(fg\in\mathscr{X}$, $\max(f,g)$ and $\min(f,g)\in\mathscr{X}$, respectively) whenever $g\in\mathscr{X}$. The respective classes are denoted by $\mathscr{M}_a(\mathscr{X})$, $\mathscr{M}_m(\mathscr{X})$, $\mathscr{M}_l(\mathscr{X})$.