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Malliavin calculus for stable processes on homogeneous groups

by

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Abstract. Let $\{\mu_t\}_{t>0}$ be a symmetric semigroup of stable measures on a homogeneous group, with smooth Lévy measure. Applying Malliavin calculus for jump processes we prove that the measures μ_t have smooth densities.

§ 0. Introduction. Stable semigroups of measures on homogeneous Lie groups are a natural generalization of the notion of a strictly stable measure on $\mathbf{R}^n$. Nevertheless, stable measures on homogeneous groups revealed first their importance in connection with some problems in harmonic analysis. A 1-stable semigroup of measures on a homogeneous group G was used to construct a commutative approximate identity on G (see [4]), which plays an important role in a characterization of the Hardy spaces $H^p(G)$. This was done in 1986 by Głowacki [8] (for the Heisenberg group cf. [6] and [7]). In particular, Głowacki showed that if $\{\mu_t\}_{t>0}$ is a symmetric stable semigroup with smooth Lévy measure then the μ_t have smooth densities.

The aim of this paper is to prove the smoothness of $\{\mu_t\}_{t>0}$ in a probabilistic way using the methods of Malliavin calculus. Our approach, based on elementary facts concerning Poisson processes and random measures, seems to be simpler than the methods used by Głowacki.

Malliavin calculus, initiated by Malliavin in 1976 ([13]), is a collection of probabilistic methods for showing smoothness of semigroups of measures connected with stochastic processes. The main idea is to integrate by parts on a probability space. The classical Malliavin calculus deals with diffusions. The Malliavin calculus for $\mathbf{R}^n$-valued jump processes was developed by Bismut ([1], [2]), whose approach is here adapted to the case of stable semigroups on homogeneous groups.

§§1 and 2 have a preliminary character. In §1 we present the basic probabilistic facts needed in the sequel. Most of them are well-known in the case

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$G = \mathbf{R}^n$ and the proofs remain the same in the nonabelian case. §2 provides necessary information on homogeneous groups and stable semigroups of measures. In §3 we formulate the main lemma of Malliavin calculus on G and we justify the reduction to a "truncated" stable process.

§4 is the main section of the paper. We integrate by parts on jumps of the process for differential operators of order 1. The difficulties arising in the nonabelian case are treated in points (b) and (c) of that section. In §5 the method presented in §4 is adapted to differential operators of arbitrary order.

The results of this paper are contained in the author's Ph.D. thesis. The author is indebted to Professor T. Byczkowski for suggesting the problem and for many conversations on this topic. He would also like to thank Professor A. Hulanicki for the opportunity of presenting the results of the paper at the conference on harmonic analysis in Tuczno, 1989. The author is also grateful to J. Dziubański, P. Głowacki and S. Kwapień for helpful comments and remarks.

§ 1. Independent increment jump processes on Lie groups. In this section we introduce some notation and collect basic probabilistic facts needed in the sequel.

Let G be a separable Lie group. A family $\{\mu_t\}_{t>0}$ of probability measures on G is called a *continuous semigroup* of measures if

$$\mu_t * \mu_s = \mu_{t+s}, \quad t, s > 0,$$
$$\mu_t \Rightarrow \delta_e \quad \text{as } t \to 0.$$

Let $\{\mu_t\}_{t>0}$ be a continuous semigroup of measures on G. Since G is a complete separable metric space, there exists ([5]) a stochastic process $\{z_t\}_{t>0}$ with values in G such that:

(i) μ_t is the probability law of z_t, $t > 0$;

(ii) the left increments of $\{z_t\}_{t>0}$ are homogeneous, i.e. for any $s < t$ the probability law of $z_s^{-1}z_t$ is μ_{t-s};

(iii) $\{z_t\}_{t>0}$ has independent left increments, i.e. for any $t_1 < \ldots < t_n$ the random elements $z_{t_1}, z_{t_1}^{-1}z_{t_2}, \ldots, z_{t_{n-1}}^{-1}z_{t_n}$ are independent;

(iv) all sample paths of the process $\{z_t\}_{t>0}$ are right-continuous with finite left-hand limits.

We denote by D_G the Skorokhod space of functions defined on $\mathbf{R}^+$ with values in G which are right-continuous with left-hand limits. Under the so-called Skorokhod distance, D_G is a topologically complete separable metric space ([3]). We assume that $\{z_t\}_{t>0}$ is the canonical process on D_G. Similarly, for $T > 0$ we consider the spaces $D_G[0,T]$ with $[0,T]$ in place of $\mathbf{R}^+$.

The infinitesimal generator A of a continuous semigroup $\{\mu_t\}_{t>0}$ on G is described by the Hunt formula ([10]). The generating functional P of $\{\mu_t\}$ is defined by $\langle P, f\rangle = Af(e)$, $f \in C_c^\infty(G)$. P is a distribution on G such that for $f \in C_c^\infty(G)$

$$\langle P, f\rangle = \lim_{t\to 0} \frac{1}{t}\langle \mu_t - \delta_e, f\rangle.$$

We will use the following limit theorem, which is a special case of [3], Thm. 4.2.5.

THEOREM 1.1. *Let $\{z_t^{(n)}\}_{0\le t\le T}$, $\{z_t\}_{0\le t\le T}$ be stochastic processes on G, with homogeneous independent increments and sample paths in $D_G[0,T]$. Let A_n, A be the infinitesimal generators of the corresponding semigroups of measures. If $\lim_{n\to\infty} A_n f = Af$ for every $f \in C_c^\infty(G)$, then*

$$\{z_t^{(n)}\}_{0\le t\le T} \overset{D_G}{\Rightarrow} \{z_t\}_{0\le t\le T} \quad \text{as } n \to \infty,$$

i.e. the probability distributions of $\{z_t^{(n)}\}$ on $D_G[0,T]$ are weakly convergent to the distribution of $\{z_t\}$. ∎

The *Lévy measure* ν of a continuous semigroup $\{\mu_t\}_{t>0}$, which appears in the Hunt formula, is a measure on $G \setminus \{e\}$ such that $\nu(U^c) < \infty$ for any open neighbourhood U of e and (see [10])

(1.1) $$(1/t)\mu_t|_{U^c} \Rightarrow \nu|_{U^c} \quad \text{as } t \to 0 \quad \text{if } \nu(\partial U) = 0.$$

There are various connections between the Lévy measure ν of $\{\mu_t\}_{t>0}$ and the jumps of the corresponding process $\{z_t\}_{t>0}$. We will present some of them now. The jumps of the process are denoted by $\Delta z_s = z_{s-0}^{-1}z_s \neq e$.

The following theorem is well-known for $G = \mathbf{R}^n$ ([5]); the proof for an arbitrary Lie group is similar, using (1.1) and the Poisson limit theorem.

THEOREM 1.2. *Let $B \in \mathcal{B}_G$ with $\nu(B) < \infty$. Then for every $t > 0$*

$$N(t, B) = \operatorname{card}\{\Delta z_s : s \le t,\ \Delta z_s \in B\}$$

is a random variable with the Poisson law and mean value $EN(t,B) = t\nu(B)$. Moreover, $\{N(t,B)\}_{t>0}$ is a Poisson process. ∎

Similarly to the abelian case ([5]) we have

THEOREM 1.3. *If $B_i \in \mathcal{B}_G$, $\nu(B_i) < \infty$, $i = 1, \ldots, n$, and $B_i \cap B_j = \emptyset$ for $i \ne j$ then $N(t, B_1), \ldots, N(t, B_n)$ are independent.* ∎

We will need a stochastic integral with respect to the random measure $N(t, B)$. We use the notation of Bismut ([1], [2]). If $u : G \setminus \{e\} \to \mathbf{R}$ is ν-integrable then we define

$$S_{s\le t}u = \sum_{\Delta z_s \ne e,\ s\le t} u(\Delta z_s), \quad S^c_{s\le t}u = S_{s\le t}u - ES_{s\le t}u.$$

The following theorem shows that $S_{s\le t}u$ and $S^c_{s\le t}u$ are well-defined and gives their properties.

THEOREM 1.4. *Let* $u: G\setminus\{e\}\to \mathbf{R}$ *be measurable and let* $t>0$.

(a) *If* $\int |u|\,d\nu<\infty$, *then* $S_{s\le t}u=\sum_{\Delta z_s\ne e,\ s\le t} u(\Delta z_s)$ *is convergent a.e.*, $S_{s\le t}u\in L^1$ *and* $ES_{s\le t}u=t\int u\,d\nu$.

(b) *If* $\int|u|\,d\nu<\infty$ *and* $\int u^2\,d\nu<\infty$, *then* $E[(S^c_{s\le t}u)^2]=t\int u^2\,d\nu$.

(c) *If* $\int|u|\,d\nu<\infty$ *and* $\int|u|^n\,d\nu<\infty$ *for an* $n\in\mathbf{N}$, *then* $S_{s\le t}u\in L^n$.

Proof. Statements (a) and (b) are special cases of general results concerning stochastic integrals with respect to point processes ([11]). They may also be easily proved directly, using Theorems 1.2 and 1.3. To prove (c) one uses the Laplace transform of $S_{s\le t}|u|$, which equals

$$\chi_t(r)=\exp\Big\{\int[e^{r|u(z)|}-1]t\nu(dz)\Big\}. \ \blacksquare$$

COROLLARY 1.5. *If* $u: G\setminus\{e\}\to\mathbf{R}$ *is a bounded function such that* $\int|u|\,d\nu<\infty$ *then* $S_{s\le t}u\in L^p$ *for every* $t>0$ *and* $p\ge 1$. ■

We will also be interested in integrability of $(S_{s\le t}u)^{-1}$. The following simple criterion, whose proof uses the Laplace transform of $S_{s\le t}u$, is taken from Bismut ([1]).

LEMMA 1.6. *Let* $u: G\to\mathbf{R}^+$ *be* ν*-integrable. If*

$$\lim_{x\to 0}\frac{\nu\{u^{-1}(x,+\infty)\}}{\log(1/x)}=+\infty,$$

then $(S_{s\le t}u)^{-1}\in L^p$ *for every* $p\ge 1$ *and* $t>0$. ■

Now fix a bounded neighbourhood U of e. Each trajectory $z\in D_G[0,T]$ has only a finite number of jumps such that $\Delta z\notin\overline{U}$. Let $S_1,S_2,\dots$ be the increasing sequence of stopping times at which z has a jump $\Delta z\notin\overline{U}$. The following lemma is well known in the real case and its proof for a Lie group G is identical.

LEMMA 1.7. (a) *For every* $n\in\mathbf{N}$ *the probability law of* Δz_{S_i} *is given by* $\nu|_{\overline{U}^c}/\nu(\overline{U}^c)$.

(b) $S_1, S_2-S_1,\dots,S_n-S_{n-1},\dots,\Delta z_{S_1},\Delta z_{S_2},\dots,\Delta z_{S_n},\dots$ *are independent.* ■

To end this section we present one more result concerning a connection between the Lévy measure of a semigroup and the semigroup itself. It is an easy consequence of a theorem of Hulanicki ([9]).

We say that a function φ on G is *submultiplicative* if

(i) φ is locally bounded and $\varphi\ge 1$;
(ii) $\varphi(x^{-1})=\varphi(x)$, $x\in G$;
(iii) $\varphi(xy)\le\varphi(x)\varphi(y)$, $x,y\in G$.

THEOREM 1.8. *Let* U *be an open neighbourhood of* e *such that* $\overline{U}$ *is compact. Let* ν *be the Lévy measure of a continuous semigroup* $\{\mu_t\}_{t>0}$ *on* G. *If* φ *is a submultiplicative function on* G *and* $\int_{U^c}\varphi\,d\nu<\infty$ *then for every* $T>0$ *there exists a constant* k *such that* $\int\varphi\,d\mu_t<k$ *for all* $t\in[0,T]$. ■

§ 2. Stable semigroups of measures on homogeneous groups. In this section we present basic facts about homogeneous groups (see [4]) and stable measures.

A family of *dilations* on a nilpotent Lie algebra $\mathfrak{g}$ is a one-parameter group $\{\gamma_t\}_{t>0}$ of automorphisms of $\mathfrak{g}$ determined by

$$\gamma_t X_j=t^{d_j}X_j \tag{2.1}$$

where $\{X_1,\dots,X_M\}$ is a linear basis for $\mathfrak{g}$ and $1\le d_1\le\dots\le d_M=\overline{d}$ are real numbers called the *exponents of homogeneity*. A nilpotent Lie group G whose Lie algebra $\mathfrak{g}$ admits a family of dilations $\{\gamma_t\}$ is said to be a *homogeneous group*. The mappings $\exp\circ\gamma_t\circ\exp^{-1}$ are group automorphisms on G and are called *dilations on* G. They are also denoted by $\gamma_t(x)$ or simply tx, $x\in G$. The *homogeneous dimension* of G is the number $Q=d_1+\dots+d_M$. The group identity of G is denoted by 0 and will be referred to as the *origin*.

A function f on $G\setminus\{0\}$ is called *homogeneous* of degree $\lambda\in\mathbf{R}$ if $f\circ\gamma_t=t^\lambda f$ for $t>0$. A distribution τ on G is *homogeneous* of degree λ if $\langle\tau,t^{-Q}f\circ\gamma_{t^{-1}}\rangle=t^\lambda\langle\tau,f\rangle$ for $f\in C_c^\infty(G)$ and $t>0$. A linear differential operator D on G is *homogeneous* of degree λ if $D(f\circ\gamma_t)=t^\lambda(Df)\circ\gamma_t$ for any f in its domain and $t>0$. The vector fields X_i appearing in (2.1) are homogeneous of degree d_i.

We define a Euclidean norm $\|\cdot\|$ on $\mathfrak{g}$ by declaring the X_j's to be orthonormal. One may regard this norm as a function on G in the obvious way: $\|x\|=\|\exp^{-1}x\|$.

We choose and fix once for all a *homogeneous norm* on G, that is, a continuous positive function $x\to|x|$ which is smooth on $G\setminus\{0\}$ and satisfies $|x|=|x^{-1}|$, $|tx|=t|x|$, $|x|=0\Leftrightarrow x=0$ for any $x\in G$ and $t>0$.

LEMMA 2.1. (a) *There exists a constant* $c>0$ *such that*

$$\|x\|\le c|x| \quad \textit{for } |x|\le 1,\qquad \|x\|\le c|x|^{\overline{d}} \quad \textit{for } |x|>1.$$

(b) *There exists a constant* $c>0$ *such that*

$$|xy|\le c(|x|+|y|),\qquad x,y\in G. \ \blacksquare$$

A continuous semigroup $\{\mu_t\}$ of measures on G is said to be *stable* (with exponent α, $0<\alpha<2$) if for every $B\in\mathcal{B}_G$ and $t>0$

$$\mu_t(B)=\mu_1(t^{-1/\alpha}B).$$

This is equivalent to the generating functional of $\{\mu_t\}_{t>0}$ being a homogeneous distribution of degree $-Q-\alpha$. The Lévy measure ν of $\{\mu_t\}_{t>0}$ has the property

$$\nu(t^{-1}B) = t^\alpha \nu(B) \quad \text{for all } B \in \mathcal{B}_G .$$

Homogeneity of the generating functional and of the Lévy measure of $\{\mu_t\}_{t>0}$ applied to the Hunt formula implies that P is a generating functional of an α-stable semigroup of symmetric measures with smooth Lévy measure if and only if P is of the form

$$\langle P, f\rangle = \lim_{\varepsilon\to 0} \int\limits_{|x|\geq\varepsilon} [f(x)-f(0)]\frac{\Omega(|x|^{-1}x)}{|x|^{Q+\alpha}}\,dx \tag{2.2}$$

where $\Omega \not\equiv 0$ is a nonnegative symmetric C^∞ function on the unit sphere $\Sigma = \{x \in G : |x| = 1\}$ and dx is the bi-invariant Haar measure on G such that the measure of the unit ball $\{x : |x| \leq 1\}$ is 1. (For an analytic proof of (2.2) cf. [8].)

We conclude this section with some technical remarks involving the adjoint representation on G (see e.g. [16]). They are true for an arbitrary Lie group G. We denote respectively by L_σ and R_σ the left and right translation by σ.

LEMMA 2.2. *Let $X \in \mathfrak{g}$, $f \in C^\infty(G)$, $\sigma,\tau,z \in G$. Then*

$$Xf(\tau z\sigma) = \mathrm{Ad}_\sigma X(f\circ L_\tau \circ R_\sigma)(z).$$

Proof. By the invariance of X and elementary properties of the adjoint representation ([16])

$$\begin{aligned} Xf(\tau z\sigma) &= \frac{d}{dt}\Big|_0 (f\circ L_{\tau\sigma})(\sigma^{-1}z\sigma\exp tX)\\ &= \frac{d}{dt}\Big|_0 (f\circ L_{\tau\sigma})(\sigma^{-1}z\cdot\exp \mathrm{Ad}_\sigma(tX)\cdot\sigma)\\ &= \mathrm{Ad}_\sigma X(f\circ L_\tau\circ R_\sigma)(z). \ \blacksquare \end{aligned}$$

To simplify notation we use the multiple Lie brackets defined by

$$\begin{gathered}[X_1,[X_2,\ldots[X_{n-1},X_n]\ldots]] = [X_1,\ldots,X_n],\\ [\underbrace{X,[X,\ldots[X,}_{m}\underbrace{[Y,[Y,\ldots[Y,Y]}_{n}\ldots]]]\ldots]] = [X^m,Y^n].\end{gathered}$$

If $n=0$ we set $[X^n, Y] = Y$.

LEMMA 2.3. *Let G be nilpotent of degree d and let $S_1,\ldots,S_n,X \in \mathfrak{g}$ $(n\in\mathbf{N})$. Then*

$$\mathrm{Ad}_{\exp S_1\ldots\exp S_n} X = \sum_{\substack{0\leq a_1+\ldots+a_n<d\\ a_i\in\mathbf{N}}} \frac{1}{a_1!\ldots a_n!}[S_1^{a_1},\ldots,S_n^{a_n},X]. \tag{2.3}$$

Proof. For $n=1$, (2.3) is an immediate consequence of elementary facts concerning the adjoint representation Ad and its differential ad ([16]). The formula for all n follows by induction. ■

§ 3. A probabilistic approach to the smoothness problem for stable semigroups.

Głowacki ([8]) proved

THEOREM 3.1. *Let $\{\mu_t\}$ be a stable semigroup of symmetric measures on a homogeneous group G. If the Lévy measure of $\{\mu_t\}$ is smooth, then the μ_t have smooth densities on G.*

The aim of this paper is to prove Theorem 3.1 using a version of Malliavin calculus for jump processes, based on methods of Bismut ([1], [2]). In this way we obtain a probabilistic proof of Theorem 3.1.

First we formulate the basic lemma of Malliavin calculus on a nilpotent Lie group G. Fix a linear basis $X_1,\ldots,X_M$ of the Lie algebra $\mathfrak{g}$ and write $X^I = X_{i_1}\ldots X_{i_n}$ for a multiindex $I = (i_1,\ldots,i_n)$, $i_1,\ldots,i_n \leq M$.

LEMMA 3.2. *Let G be a connected and simply connected nilpotent Lie group and let μ be a finite Borel measure on G. If for every multiindex I there exists a constant c_I such that*

$$|\langle\mu, X^I f\rangle| \leq c_I\|f\|_\infty \tag{3.1}$$

for every $f\in C_c^\infty(G)$, then μ has a smooth density with respect to the Haar measure on G.

Proof. The exponential map is a diffeomorphism of $\mathfrak{g}$ and G, and the image of the Lebesgue measure on $\mathfrak{g}$ is the bi-invariant Haar measure on G ([4]). Since the derivatives in a global coordinate system on G are linear combinations of the X_i with smooth coefficients, it is enough to prove the lemma for $G=\mathbf{R}^M$. The proof in this case may be found e.g. in [15]. ■

Remark. Lemma 3.2 is true for an arbitrary Lie group G and for any distribution on G.

We are going to apply Lemma 3.2 to prove Theorem 3.1.

From now on, G will always denote a homogeneous group and $\{\mu_t\}_{t>0}$ a symmetric α-stable semigroup on G with a smooth Lévy measure ν. All constants are denoted by c.

Now we show that the question of smoothness of $\{\mu_t\}$ may be reduced to the problem of smoothness of a truncated semigroup $\{\widetilde{\mu}_t\}$.

PROPOSITION 3.3. *Let h be a smooth function on G such that $0 \le h \le 1$, $h(x) = 1$ for $|x| < 1/2$ and $h(x) = 0$ for $|x| > 1$. Let $\{\widetilde{\mu}_t\}_{t>0}$ be a semigroup of measures on G with the Lévy measure $\widetilde{\nu} = h\nu$ and the generating functional*

$$\langle \widetilde{P}, f\rangle = \lim_{\varepsilon\to 0} \int_{|x|\ge\varepsilon} [f(x) - f(0)]\, d\widetilde{\nu}(x).$$

Assume that the measures $\widetilde{\mu}_t$ are smooth. Then so are the μ_t.

Proof. Set $k = \nu - \widetilde{\nu}$. Then k is a smooth function on G. By (2.2), $k(x) = \Omega(|x|^{-1}x)/|x|^{Q+\alpha}$ for $|x| > 1$ where $\Omega \in C^\infty(\Sigma)$. Hence k and $X^I k$ are bounded for any multiindex I. Using (2.2) and the perturbation formula ([14]) we get

$$f * \mu_t = f * \widetilde{\widetilde{\mu}}_t + \int_0^t f * \mu_s * k * \widetilde{\widetilde{\mu}}_{t-s}\, ds$$

where $\widetilde{\widetilde{\mu}}_t = e^{-\|k\|_1 t}\widetilde{\mu}_t$ and $f \in C_c^\infty(G)$. In particular, for any multiindex I

$$\langle X^I\mu_t, f\rangle = \langle X^I\widetilde{\widetilde{\mu}}_t, f\rangle + \int_0^t \langle X^I(\mu_s * k * \widetilde{\widetilde{\mu}}_{t-s}), f\rangle\, ds. \tag{3.2}$$

Observe that

$$\mu_s * k * \widetilde{\widetilde{\mu}}_{t-s}(x) = \int k(y^{-1}xz^{-1})\, d\widetilde{\widetilde{\mu}}_{t-s}(z)\, d\mu_s(y). \tag{3.3}$$

By Lemma 2.2, for any vector field $X \in \mathfrak{g}$

$$X(k \circ L_{y^{-1}} \circ R_{z^{-1}})(x) = \mathrm{Ad}_z X(k)(y^{-1}xz^{-1}).$$

Define $S = \exp^{-1}(z)$. If $S = \sum_{i=1}^M a_i(z)X_i$, then using Lemma 2.3 we have

$$\begin{aligned}\mathrm{Ad}_z X &= X + [S, X] + \ldots + \frac{1}{(d-1)!}[S^{d-1}, X]\\ &= X + \sum_{l=1}^{d-1}\frac{1}{l!}\sum_{j_1,\ldots,j_l} a_{j_1}(z)\ldots a_{j_l}(z)[X_{j_1},\ldots,X_{j_l},X]\end{aligned}$$

where d is the degree of nilpotency of G and $j_1,\ldots,j_l \in \{1,\ldots,M\}$.

For any $j_1,\ldots,j_l$ the function $[X_{j_1},\ldots,X_{j_l},X]k$ is bounded. Lemma 2.1(a) implies that

$$|a_{j_1}(z)\ldots a_{j_l}(z)| \le \|S\|^l \le c(1+|z|^{ld}).$$

Thus $|\mathrm{Ad}_z X(k)(\sigma)| \le c(1+|z|^{(d-1)d})$ for every $\sigma \in G$. From Theorem 1.8 applied to the function $|z|^{(d-1)d} + K$, which is submultiplicative for K sufficiently large, it follows that $X_x k(y^{-1}xz^{-1})$ is dominated by a function integrable with respect to $d\widetilde{\widetilde{\mu}}_{t-s}(z)\, d\mu_s(y)$ and independent of x. Therefore we may differentiate in (3.3) under the integral sign.

Similarly, for $X^I = X_{i_1}\ldots X_{i_n}$

$$X^I(k \circ L_{y^{-1}} \circ R_{z^{-1}})(x) = (\mathrm{Ad}_z X_{i_1})\ldots(\mathrm{Ad}_z X_{i_n})k(y^{-1}xz^{-1}).$$

The same argument as for X shows that such a function is estimated by $c(1+|z|^{n(d-1)d})$ and one may differentiate in (3.3) writing X^I under the integral sign. Coming back to (3.2) and using Theorem 1.8 we have

$$\begin{aligned}\Big|\int_0^t \langle X^I(\mu_s * k * \widetilde{\widetilde{\mu}}_{t-s}), f\rangle\, ds\Big| &\le c\|f\|_\infty \int_0^t \int_G (1+|z|^{n(d-1)d})\, d\widetilde{\mu}_{t-s}(z)\, ds\\ &\le c\|f\|_\infty.\end{aligned}$$

The distribution $\int_0^t \mu_s * k * \widetilde{\widetilde{\mu}}_{t-s}\, ds$ satisfies the assumptions of Lemma 3.2 so it is a smooth function. Since, by assumption, the $\widetilde{\widetilde{\mu}}_t$ are smooth, the same is true for the μ_t. ■

§ 4. Integration by parts for differential operators of order 1.

As in the preceding section, let G be a homogeneous group and $\{\mu_t\}_{t>0}$ a symmetric α-stable semigroup on G with a smooth Lévy measure ν. Let $\{\widetilde{\mu}_t\}_{t>0}$ be the truncated semigroup defined in Proposition 3.3. The purpose of this section is to show that (3.1) holds for the measures $\widetilde{\mu}_t$ and for $X \in \mathfrak{g}$. This is certainly of little value in itself because we merely obtain in this way the absolute continuity of $\{\widetilde{\mu}_t\}_{t>0}$, already stated by a theorem of Janssen ([12]). However, the approach for all X^I is similar and we want to present first the most transparent situation when $|I| = 1$.

Denote by g the density of the Lévy measure $\widetilde{\nu}$ of $\{\widetilde{\mu}_t\}$. By (2.2) and the definition of $\{\widetilde{\mu}_t\}$

$$g(x) = \Omega(|x|^{-1}x)/|x|^{Q+\alpha} \quad \text{for } |x| < 1/2. \tag{4.1}$$

Let $\{z_t\}_{t>0}$ be an independent homogeneous increment process with sample paths in D_G, corresponding to the semigroup $\{\widetilde{\mu}_t\}$. Observe that the estimate (3.1) for $\widetilde{\mu}_t$ and $X \in \mathfrak{g}$ may be expressed in terms of the process $\{z_t\}$ as

$$|E[Xf(z_T)]| \le c\|f\|_\infty. \tag{4.2}$$

We will show (4.2) following the technique of Bismut ([2]). For the sake of clarity we divide our procedure into four parts. Part (a) presents the basic integration by parts formula. In part (b) we deal with random coefficients appearing in this formula in the nonabelian case. In (c) and (d) we remove additional assumptions made in (a) and finally derive (4.2).

(a) *Integration by parts on jumps of the process.* Fix $\varepsilon > 0$ and $T > 0$. Let $S_1, S_2, \ldots$ be the sequence of consecutive stopping times at which the process $\{z_t\}$ has jumps $|\Delta z| > \varepsilon$. By Lemma 1.7 and Theorem 1.2 the independent increment process $\{z_t^{(\varepsilon)}\}$ defined by

$$z_t^{(\varepsilon)} = \Delta z_{S_1} \ldots \Delta z_{S_{N(t,\{|x|>\varepsilon\})}}$$

corresponds to the semigroup of measures $\{\exp t(\widetilde{\nu}_\varepsilon - \widetilde{\nu}_\varepsilon(G))\}_{t>0}$ where $\widetilde{\nu}_\varepsilon = \widetilde{\nu}|_{\{|x|>\varepsilon\}}$. The generating functional of this semigroup is

$$\langle P_\varepsilon, f\rangle = \int_{\{|x|>\varepsilon\}} [f(x) - f(0)]\, d\widetilde{\nu}(x), \quad f \in C_c^\infty(G), \tag{4.3}$$

and its Lévy measure is $\widetilde{\nu}_\varepsilon$.

We shall integrate by parts on the process $\{z_t^{(\varepsilon)}\}$. To integrate by parts in $E[Xf(z_T^{(\varepsilon)})]$ we must have a term to transfer X on from the function f.

Consider a smooth function ϱ on $\mathbf{R}$ such that $0 \le \varrho \le 1$ and $\varrho(s) = 1$ when $|s| \ge 1$, $\varrho(s) = 0$ when $|s| \le 1/2$. For $\eta > 0$ we write $\varrho_\eta(s) = \varrho(s/\eta)$. Let $u \in C^1(G)$ be a nonnegative function with support in $\{|x| \ge \varepsilon\}$. We will integrate by parts in integrals of the form

$$E[\varrho_\eta(S_{s\le T}u)Xf(z_T^{(\varepsilon)})], \quad f \in C_c^\infty(G).$$

Write $N = N(T, \{|x| > \varepsilon\})$. By Lemma 1.7(b), N is independent of the jumps $\Delta z_{S_1}, \Delta z_{S_2}, \ldots$ and we have

$$E[\varrho_\eta(S_{s\le T}u)Xf(z_T^{(\varepsilon)})] = \sum_{n=1}^{\infty} P\{N = n\} E\Big[\varrho_\eta\Big(\sum_{i=1}^{n} u(\Delta z_{S_i})\Big) Xf(\Delta z_{S_1} \ldots \Delta z_{S_n})\Big].$$

Putting $\varrho(s)/s = 0$ for $s = 0$ and using again Lemma 1.7 we get

$$\begin{aligned} E\Big[\varrho_\eta\Big(\sum_{i=1}^{n} u(\Delta z_{S_i})\Big) Xf(\Delta z_{S_1} \ldots \Delta z_{S_n})\Big] &= \sum_{j=1}^{n} E\left[\frac{\varrho_\eta(\sum_{i=1}^n u(\Delta z_{S_i}))}{\sum_{i=1}^n u(\Delta z_{S_i})} u(\Delta z_{S_j}) Xf(\Delta z_{S_1} \ldots \Delta z_{S_n})\right] \\ &= \sum_{j=1}^{n} E\Big[\int_G \frac{\varrho_\eta(K_j + u(z))}{K_j + u(z)} u(z) Xf(\Delta z_{S_1} \ldots z \ldots \Delta z_{S_n}) \frac{g(z)}{\widetilde{\nu}_\varepsilon(G)}\, dz\Big] \end{aligned}$$

where $K_j = \sum_{i\ne j} u(\Delta z_{S_i})$. Since f has a compact support one may integrate by parts in the variable z in each integral over G in the last sum. By using Lemma 2.2 and setting $\sigma_j = \Delta z_{S_{j+1}} \ldots \Delta z_{S_n}$, these integrals are equal

to

$$\begin{aligned} &-\int_G \left[\frac{\varrho'_\eta(K_j + u(z))}{K_j + u(z)} - \frac{\varrho_\eta(K_j + u(z))}{(K_j + u(z))^2}\right] (u(z)(\mathrm{Ad}_{\sigma_j} X)u(z)) \\ &\qquad\qquad \times f(\Delta z_{S_1} \ldots z \ldots \Delta z_{S_n}) \frac{g(z)}{\widetilde{\nu}_\varepsilon(G)}\, dz \\ &-\int_G \frac{\varrho_\eta(K_j + u(z))}{K_j + u(z)} \frac{(\mathrm{Ad}_{\sigma_j} X)(ug)(z)}{g(z)} f(\Delta z_{S_1} \ldots z \ldots \Delta z_{S_n}) \frac{g(z)}{\widetilde{\nu}_\varepsilon(G)}\, dz \end{aligned}$$

(one may divide by $g(z)$ because $g(\Delta z_s) \ne 0$ almost everywhere so that $\widetilde{\nu}\{z : g(z) = 0\} = 0$).

Now we reverse our procedure. Defining

$$\Phi_0 = \frac{\varrho_\eta(S_{s\le T}u)}{S_{s\le T}u}, \quad \Phi_1 = \frac{\varrho'_\eta(S_{s\le T}u)}{S_{s\le T}u} - \frac{\varrho_\eta(S_{s\le T}u)}{S_{s\le T}^2 u}, \tag{4.4}$$

we obtain the following integration by parts formula.

LEMMA 4.1.

$$\begin{aligned} E[\varrho_\eta(S_{s\le T}u)Xf(z_T^{(\varepsilon)})] &= -E\Big[\Phi_1 \sum_{j=1}^{N} u(\Delta z_{S_j})(\mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}} X)u(\Delta z_{S_j}) f(z_T^{(\varepsilon)})\Big] \\ &\quad -E\Big[\Phi_0 \sum_{j=1}^{N} \Big(\frac{1}{g} \mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}} X(ug)\Big)(\Delta z_{S_j}) f(z_T^{(\varepsilon)})\Big]. \quad \blacksquare \end{aligned} \tag{4.5}$$

Now we analyse the form of the operators $\mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}}$ occurring in (4.5). Observe that when G is nilpotent of degree 2 then by Lemma 2.3

$$\mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}} X = X + \sum_{i=j+1}^{N} [\exp^{-1}(\Delta z_{S_i}), X].$$

If

$$\exp^{-1}(\Delta z_{S_i}) = \sum_{k=1}^{M} a_k(\Delta z_{S_i}) X_k \tag{4.6}$$

for $X_1, \ldots, X_M$ as in (2.1) then

$$\mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}} X = X + \sum_{k=1}^{M} d_k^{(j)} [X_k, X]$$

where

$$d_k^{(j)} = \sum_{i=j+1}^{N} a_k(\Delta z_{S_i}).$$

For an arbitrary degree of nilpotency of G the formula for $\mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}}$ is more complicated. Using Lemma 2.3, in the same way as for $d = 2$ we obtain

$$(4.7)\quad \mathrm{Ad}_{\Delta z_{S_{j+1}} \ldots \Delta z_{S_N}} X = X + \sum_{\substack{1 \le l < d \\ k_1, \ldots, k_l = 1, \ldots, M}} d_{k_1 \ldots k_l}^{(j)} [X_{k_1}, \ldots, X_{k_l}, X]$$

where

$$(4.8)\quad d_{k_1 \ldots k_l}^{(j)} = \sum_{q=1}^{l} \sum_{\substack{j_1 + \ldots + j_q = l \\ j_1, \ldots, j_q \ne 0}} \frac{1}{j_1! \ldots j_q!} \sum_{j < i_1 < \ldots < i_q} (a_{k_1} \ldots a_{k_{j_1}})(\Delta z_{S_{i_1}}) \ldots \ldots (a_{k_{l-j_q+1}} \ldots a_{k_l})(\Delta z_{S_{i_q}})$$

and the functions a_k are given by (4.6). For instance, the coefficient of the vector field $[X_{k_1}, X_{k_2}, X]$ equals

$$d_{k_1 k_2}^{(j)} = \frac{1}{2} \sum_{i=j+1}^{N} a_{k_1}(\Delta z_{S_i}) a_{k_2}(\Delta z_{S_i}) + \sum_{j < i_1 < i_2}^{N} a_{k_1}(\Delta z_{S_{i_1}}) a_{k_2}(\Delta z_{S_{i_2}}).$$

Now we substitute (4.7) into (4.5). Using the Schwarz inequality and setting for $Y \in \mathfrak{g}$

$$(4.9)\quad \Psi_1^{(Y)} = uYu, \quad \Psi_2^{(Y)} = Y(ug)/g,$$

we obtain

LEMMA 4.2. *If G is nilpotent of degree d then*

$$(4.10)\quad |E[\varrho_\eta(S_{s \le T} u) X f(z_T^{(\varepsilon)})]| \le \|f\|_\infty \Big\{ E[|\Phi_1 S_{s \le T} \Psi_1^{(X)}| + |\Phi_0 S_{s \le T} \Psi_2^{(X)}|] + \sum_{\substack{1 \le l < d \\ k_1, \ldots, k_l = 1, \ldots, M}} (E \sup_{1 \le j < N} |d_{k_1 \ldots k_l}^{(j)}|^2)^{1/2} \times [(E[\Phi_1^2 S_{s \le T}^2 |\Psi_1^{([X_{k_1}, \ldots, X_{k_l}, X])}|])^{1/2} + (E[\Phi_0^2 S_{s \le T}^2 |\Psi_2^{([X_{k_1}, \ldots, X_{k_l}, X])}|])^{1/2}] \Big\}$$

where Φ_0, Φ_1 are defined in (4.4), $\Psi_1^{(Y)}$, $\Psi_2^{(Y)}$ *in* (4.9) *and the* $d_{k_1 \ldots k_l}^{(j)}$ *are given by* (4.8). ■

Observe that in the abelian case the right-hand side of (4.10) contains only the first term.

(b) *Estimation of random coefficients*. We now estimate the coefficients

$$E \sup_{1 \le j < N} |d_{k_1 \ldots k_l}^{(j)}|^2$$

occurring in (4.10). The estimation will not depend on ε. Consider first the case $l = 1$ (when G is nilpotent of degree 2 this is the only possibility). Since

$$E \sup_{1 \le j < N} |d_k^{(j)}|^2 = \sum_{n=0}^{\infty} P\{N = n\} E \sup_{1 \le j < n} \Big| \sum_{i=j+1}^{n} a_k(\Delta z_{S_i}) \Big|^2$$

it suffices to estimate $E \sup_{1 \le j < n} |\sum_{i=j+1}^{n} a_k(\Delta z_{S_i})|^2$ for a fixed $n \in \mathbb{N}$. Observe that $a_k(\Delta z_{S_1}), \ldots, a_k(\Delta z_{S_n})$ are independent symmetric bounded random variables. By the Lévy inequality and (4.6)

$$E \sup_{1 \le j < n} \Big| \sum_{i=j+1}^{n} a_k(\Delta z_{S_i}) \Big|^2 \le 2E \Big| \sum_{i=1}^{n} a_k(\Delta z_{S_i}) \Big|^2 = 2E \sum_{i=1}^{n} a_k^2(\Delta z_{S_i}) \le 2E \sum_{i=1}^{n} \|\Delta z_{S_i}\|^2.$$

Since $|\Delta z_{S_i}| < 1$, by Lemma 2.1(a) we have $\|\Delta z_{S_i}\|^2 \le c|\Delta z_{S_i}|^2$ and finally, writing $r(z) = |z|$,

$$E \sup_{1 \le j < N} |d_k^{(j)}|^2 \le cE \sum_{|\Delta z_s| > \varepsilon,\ s \le T} |\Delta z_s|^2 \le cE S_{s \le T} r^2.$$

This estimate does not depend on ε and by Theorem 1.4(a) it is finite. We generalize it in the following lemma.

LEMMA 4.3. *For every $m \in \mathbb{N}$ there exists a constant $c = c(m)$ such that*

$$E \sup_{1 \le j < N} |d_k^{(j)}|^{2m} \le cE S_{s \le T}^m r^2 < \infty.$$

Proof. As in case $m = 1$ the Lévy inequality implies

$$E \sup_{1 \le j < N} |d_k^{(j)}|^{2m} \le 2E \Big[\sum_{i=1}^{N} a_k(\Delta z_{S_i}) \Big]^{2m}.$$

Set $b_i = a_k(\Delta z_{S_i})$. Then the b_i are independent symmetric random variables with mean value 0. For a fixed $n \in \mathbb{N}$, by the multinomial formula

$$(4.11)\qquad \Big(\sum_{i=1}^{n} b_i\Big)^{2m} = \sum_{q=1}^{2m} \sum_{\substack{j_1+\ldots+j_q=2m\\ j_1\ldots j_q\neq 0}} \frac{(2m)!}{j_1!\ldots j_q!} \sum_{\substack{i_1<\ldots<i_q\\ i_s\neq i_t}} b_{i_1}^{j_1}\ldots b_{i_q}^{j_q}.$$

We take the expected value of both sides of (4.11). Then only the terms corresponding to even exponents $j_1,\ldots,j_q$ will remain on the right-hand side. Since for each such sequence $j_1,\ldots,j_q$

$$\sum b_{i_1}^{j_1}\ldots b_{i_q}^{j_q} \le \Big(\sum_{i=1}^{n} b_i^2\Big)^m$$

we get

$$E\Big(\sum_{i=1}^{n} b_i\Big)^{2m} \le cE\Big(\sum_{i=1}^{n} b_i^2\Big)^m .$$

Arguing as for $m=1$ and using Corollary 1.5 we finally obtain

$$E\sup_{1\le j<N} |d_k^{(j)}|^{2m} \le cES_{s\le T}^m r^2 < \infty .$$

PROPOSITION 4.4. *For every $m\in\mathbb{N}$ there exists a constant $c=c(m)$ independent of ε such that*

$$E\sup_{1\le j<N} |d_{k_1\ldots k_l}^{(j)}|^{2m} < c .$$

Proof. For $l=1$ the proposition reduces to Lemma 4.3. For $l>1$ we need a stronger tool than the Lévy inequality. Consider the case $l=2$, $m=1$. Just as for $l=1$ we may treat N as a constant. By (4.8), $d_{k_1k_2}^{(j)} = w_j^{(1)} + w_j^{(2)}$, where

$$w_j^{(1)} = \frac{1}{2}\sum_{i=j+1}^{N} (a_{k_1}a_{k_2})(\Delta z_{S_i}),\qquad w_j^{(2)} = \sum_{j<i_1<i_2}^{N} a_{k_1}(\Delta z_{S_{i_1}})a_{k_2}(\Delta z_{S_{i_2}}) .$$

We estimate these terms separately. Since

$$|w_j^{(1)}| \le \sum_{i=j+1}^{N} |a_{k_1}a_{k_2}(\Delta z_{S_i})| \le \sum_{i=1}^{N} \|\Delta z_{S_i}\|^2 \le cS_{s\le T}r^2$$

we have

$$E\sup_{1\le j<N} |w_j^{(1)}|^2 \le cES_{s\le T}^2 r^2 < \infty .$$

In order to estimate $E\sup_{1\le j<N}|w_j^{(2)}|^2$ we notice that $\{w_{N-2}^{(2)},\ldots\ldots,w_1^{(2)},w_0^{(2)}\}$ is a martingale with respect to the increasing family of σ-fields $\{\sigma(\Delta z_{S_{N-1}},\Delta z_{S_N}),\ldots,\sigma(\Delta z_{S_1},\ldots,\Delta z_{S_N})\}$. Indeed, by the independence and symmetry of $\{\Delta z_{S_i}\}$

$$\begin{aligned}
&E\{w_{j-1}^{(2)} \mid \sigma(\Delta z_{S_{j+1}},\ldots,\Delta z_{S_N})\}\\
&= E\{a_{k_1}(\Delta z_{S_j})[a_{k_2}(\Delta z_{S_{j+1}})+\ldots+a_{k_2}(\Delta z_{S_N})]+w_j^{(2)} \mid \sigma(\Delta z_{S_{j+1}},\ldots,\Delta z_{S_N})\}\\
&= w_j^{(2)}+[a_{k_2}(\Delta z_{S_{j+1}})+\ldots+a_{k_2}(\Delta z_{S_N})]E\{a_{k_1}(\Delta z_{S_j}) \mid \sigma(\Delta z_{S_{j+1}},\ldots,\Delta z_{S_N})\}\\
&= w_j^{(2)} .
\end{aligned}$$

By the Doob–Kolmogorov inequality for martingales ([11])

$$\begin{aligned}
E\sup_{1\le j<N} |w_j^{(2)}|^2 &\le 4E|w_0^{(2)}|^2 = 4E\sum_{1\le i_1<i_2}^{N} a_{k_1}^2(\Delta z_{S_{i_1}})a_{k_2}^2(\Delta z_{S_{i_2}})\\
&\le 4E\Big(\sum_{i_1=1}^{N}\|\Delta z_{S_{i_1}}\|^2\cdot\sum_{i_2=1}^{N}\|\Delta z_{S_{i_2}}\|^2\Big) \le cES_{s\le T}^2 r^2 < \infty .
\end{aligned}$$

For $l>2$ the idea of the proof is the same as for $l=2$. One combines the methods of estimation of $w_j^{(1)}$ and $w_j^{(2)}$ and uses martingale inequalities for moments of higher order ([11]). For $m>1$ one follows the proof of Lemma 4.3. We omit the details. ■

By Proposition 4.4 and Lemma 4.2 we get

COROLLARY 4.5.

$$\begin{aligned}
(4.12)\qquad &|E[\varrho_\eta(S_{s\le T}u)Xf(z_T^{(\varepsilon)})]|\\
&\le \|f\|_\infty\Big\{E|\Phi_1S_{s\le T}\Psi_1^{(X)}| + E|\Phi_0S_{s\le T}\Psi_2^{(X)}|\\
&\quad + c\sum_{\substack{k_1,\ldots,k_l=1,\ldots,M\\ 1\le l<d}} [(E[\Phi_1^2S_{s\le T}^2|\Psi_1^{([X_{k_1},\ldots,X_{k_l},X])}|])^{1/2}\\
&\quad + (E[\Phi_0^2S_{s\le T}^2|\Psi_2^{([X_{k_1},\ldots,X_{k_l},X])}|])^{1/2}]\Big\}
\end{aligned}$$

where the constant c depends only on the process $\{z_t\}_{t\le T}$ and Φ_0, Φ_1, Ψ_1, Ψ_2 are defined in (4.4) *and* (4.9). ■

Remark. Observe that the right-hand side of (4.12) does not depend on ε; all the stochastic integrals are taken with respect to the process $\{z_t\}_{t\le T}$.

(c) *The estimate for a specific function u.* We first let $\varepsilon\to 0$ in (4.12) under the assumption that $u\in C^1(G)$ and $\operatorname{supp} u \subseteq \{|x|\ge\varepsilon_0\}$ for an $\varepsilon_0>0$. Next, for a specific function u such that $0\in\operatorname{supp} u$, we will get rid of the assumption on the support of u.

PROPOSITION 4.6. *Let $u\in C^1(G)$ with $\operatorname{supp} u\subseteq\{|x|\ge\varepsilon_0\}$ for an $\varepsilon_0>0$. Then* (4.12) *holds for $E[\varrho_\eta(S_{s\le T}u)Xf(z_T)]$.*

Proof. According to the remark after Corollary 4.5 we only have to prove that

$$E[\varrho_\eta(S_{s\le T}u)Xf(z_T^{(\varepsilon)})] \to E[\varrho_\eta(S_{s\le T}u)Xf(z_T)] \quad \text{as } \varepsilon \to 0. \tag{4.13}$$

By (4.3) and the Taylor formula, the generators of $\{z_t^{(\varepsilon)}\}$ converge on $C_c^\infty(G)$ to the generator of the process $\{z_t\}$. By Theorem 1.1 this means that the processes $\{z_t^{(\varepsilon)}\}_{t\le T}$ converge as $\varepsilon \to 0$ to $\{z_t\}_{t\le T}$ on the Skorokhod space D_G. Using the Skorokhod metric on D_G ([3]) it is not difficult to see that the mapping $D_G \ni z \to \sum_{s\le t} u(\Delta z_s)$ is continuous on D_G. Since the projection $\{z_t\} \to z_T$ is continuous a.e. with respect to the probability law of $\{z_t\}$ on $D_G[0,T]$ and the functions ϱ_η and Xf are continuous and bounded it follows that (4.13) holds. ∎

THEOREM 4.7. *Let $u(x) = |x|^m \Omega(x)h(x)$, where $m = \overline{d} + 2$ ($\overline{d}$ the maximal exponent of homogeneity), $\Omega(x) = \Omega(|x|^{-1}x)$ is as in* (4.1) *and h was defined in Proposition* 3.3. *Then*

$$\begin{aligned}
&|E[\varrho_\eta(S_{s\le T}u)Xf(z_T)]| \\
&\quad\le \|f\|_\infty \Big\{ E|\Phi_1 S_{s\le T}\Psi_1^{(X)}| + E|\Phi_0 S_{s\le T}\Psi_2^{(X)}| \\
&\qquad + c \sum_{\substack{k_1,\dots,k_l=1,\dots,M\\ 1\le l<\overline{d}}} [(E[\Phi_1^2 S^2_{s\le T}|\Psi_1^{([X_{k_1},\dots,X_{k_l},X])}|])^{1/2} \\
&\qquad + (E[\Phi_0^2 S^2_{s\le T}|\Psi_2^{([X_{k_1},\dots,X_{k_l},X])}|])^{1/2}]\Big\}.
\end{aligned} \tag{4.14}$$

Proof. For $\varepsilon > 0$ we consider $u_\varepsilon = \widetilde{\varrho}_\varepsilon u$, where $\widetilde{\varrho}(x) = \varrho(|x|)$, ϱ defined at the beginning of point (a). Then $\operatorname{supp} u_\varepsilon \subseteq \{x : |x| \ge \varepsilon/2\}$ and $\lim_{\varepsilon\to 0} u_\varepsilon = u$. By Proposition 4.6 the estimate (4.14) holds for u_ε.

Since $S_{s\le T}u_\varepsilon \to S_{s\le T}u$ for every ω, the dominated convergence theorem implies that the left-hand side of (4.14) for u_ε tends as $\varepsilon \to 0$ to $|E[\varrho_\eta(S_{s\le T}u)Xf(z_T)]|$. It is easy to see that the random variables Φ_0, Φ_1 given by (4.4) are bounded by a constant independent of u. Since $\Phi_0(u_\varepsilon) \to \Phi_0(u)$ and $\Phi_1(u_\varepsilon) \to \Phi_1(u)$ everywhere, the convergence of the right-hand side of (4.14) for $u_\varepsilon \to u$ will be proved when we show that for $Z = X_1, \dots, X_M$

$$S_{s\le T}|\Psi_i^{(Z)}(u_\varepsilon) - \Psi_i^{(Z)}(u)| \to 0 \quad \text{in } L^2,\ i = 1, 2.$$

To prove this, it is sufficient to show that, as $\varepsilon \to 0$,

$$\begin{aligned}
&S_{s\le T}|\Psi_i^{(Z)}(u_\varepsilon) - \Psi_i^{(Z)}(u)| \to 0 \quad \text{in } L^1, \\
&S^c_{s\le T}|\Psi_i^{(Z)}(u_\varepsilon) - \Psi_i^{(Z)}(u)| \to 0 \quad \text{in } L^2.
\end{aligned} \tag{4.15}$$

First observe that $S_{s\le T}|\Psi_i^{(Z)}(u)| \in L^p$ for every $p \ge 1$. Indeed, by (4.9), $|\Psi_1^{(Z)}(u)| = |uZu| \le c|x|^{2m-d_Z}$ for $|x| < 1/2$ and $\Psi_1^{(Z)}(u)$ is bounded for $|x| > 1/2$ (d_Z denotes the degree of homogeneity of Z). Therefore $\Psi_1^{(Z)}$ is $\widetilde{\nu}$-integrable and bounded and we apply Corollary 1.5. Similarly $|\Psi_2^{(Z)}(u)| \le c|x|^{m-d_Z} \le c|x|^2$ for $|x| < 1/2$ and $\Psi_2^{(Z)}(u)$ is bounded for $|x| > 1/2$ so $S_{s\le T}|\Psi_2^{(Z)}(u)| \in L^p$ for all $p \ge 1$.

To prove (4.15) we use Theorem 1.4. For $i = 1$ we have

$$\begin{aligned}
E[S_{s\le T}|\Psi_1^{(Z)}(u_\varepsilon) - \Psi_1^{(Z)}(u)|] &\le \int_{\{|x|\le\varepsilon\}} |u_\varepsilon Z u_\varepsilon - uZu| g(x)\,dx \\
&\le 2 \int_{\{|x|\le\varepsilon\}} |\Psi_1^{(Z)}| g(x)\,dx + \int_{\{|x|\le\varepsilon\}} u^2(x)\varepsilon^{-d_Z}|Z\widetilde{\varrho}(x/\varepsilon)| g(x)\,dx.
\end{aligned}$$

Since $\Psi_1^{(Z)}$ is $\widetilde{\nu}$-integrable, the first integral tends to 0 as $\varepsilon \to 0$. The same is true for the second integral because $Z\widetilde{\varrho}$ is bounded and $u(x) \le c\varepsilon^{d_Z}$ for $|x| \le \varepsilon$. Next,

$$\begin{aligned}
E[S^c_{s\le T}|\Psi_1^{(Z)}(u_\varepsilon) - \Psi_1^{(Z)}(u)|] &\le \int_{\{|x|\le\varepsilon\}} |u_\varepsilon Z u_\varepsilon - uZu|^2 g(x)\,dx \\
&\le 6 \int_{\{|x|\le\varepsilon\}} |\Psi_1^{(Z)}|^2 g(x)\,dx + 4 \int_{\{|x|\le\varepsilon\}} u^4\varepsilon^{-2d_Z}|Z\widetilde{\varrho}(x/\varepsilon)|^2 g(x)\,dx.
\end{aligned}$$

Arguing as before we see that both integrals tend to 0 as $\varepsilon \to 0$. The proof in case $i = 2$ is similar. ∎

(d) *The final estimate*. The final step in our procedure is to let $\eta \to 0$. To do this we need the following fact.

PROPOSITION 4.8. *Let $u(x) = |x|^m \Omega^k(x)h^l(x)$ for $m, k, l \in \mathbb{N}$. Then $(S_{s\le T}u)^{-1} \in L^p$ for all $1 \le p < \infty$.*

Proof. We will use Lemma 1.6. Integrating in polar coordinates on G ([4]) we have

$$\begin{aligned}
\widetilde{\nu}\{u^{-1}(t, +\infty)\} &= \int 1_{(t,+\infty)}(u(x))g(x)\,dx \\
&\ge \int_{\{|x|\le 1/2\}} 1_{(t,+\infty)}(u(x))\frac{\Omega(|x|^{-1}x)}{|x|^{Q+\alpha}}\,dx \\
&= \int d\sigma(y) \int_0^{1/2} 1_{(t,+\infty)}(r^m\Omega^k(y))\frac{\Omega(y)}{r^{Q+\alpha}}r^{Q-1}\,dr \\
&= \frac{1}{\alpha}\int_{\{\Omega^k > 2^m t\}} \Omega(y)\left[\frac{\Omega(y)^{k\alpha/m}}{t^{\alpha/m}} - 2^\alpha\right] d\sigma(y).
\end{aligned}$$

Hence

$$\varliminf_{t\to 0+}\frac{\widetilde{\nu}\{u^{-1}(t,+\infty)\}}{\log(1/t)} \ge \frac{1}{\alpha}\varliminf_{t\to 0+}\left\{\frac{\int_{\{\Omega^k>2^m t\}}\Omega(y)^{1+k\alpha/m}\,d\sigma(y)}{t^{\alpha/m}\log(1/t)}-\frac{2^\alpha\int_{\{\Omega^k>2^m t\}}\Omega(y)\,d\sigma(y)}{\log(1/t)}\right\}.$$

The integrals on the right-hand side converge as $t\to 0$ to the corresponding integrals over the set $\{\Omega>0\}$, which are positive for $\Omega\not\equiv 0$. Since $\lim_{t\to 0+}t^{\alpha/m}\log t=0$ we get

$$\varliminf_{t\to 0+}\frac{\widetilde{\nu}\{u^{-1}(t,+\infty)\}}{\log(1/t)}=+\infty\,. \ \blacksquare$$

THEOREM 4.9. *Let $u(x)=|x|^m\Omega(x)h(x)$, $m=\overline{d}+2$. Then*

$$\begin{aligned}|E[Xf(z_T)]| \le \|f\|_\infty\Bigg\{ & E\frac{S_{s\le T}|\Psi_1^{(X)}|}{S^2_{s\le T}u}+E\frac{S_{s\le T}|\Psi_2^{(X)}|}{S_{s\le T}u}\\ &+c\sum_{\substack{k_1,\dots,k_l=1,\dots,M\\ 1\le l<d}}\Bigg[\bigg(E\frac{S^2_{s\le T}|\Psi_1^{([X_{k_1},\dots,X_{k_l},X])}|}{S^4_{s\le T}u}\bigg)^{1/2}\\ &+\bigg(E\frac{S^2_{s\le T}|\Psi_2^{([X_{k_1},\dots,X_{k_l},X])}|}{S^2_{s\le T}u}\bigg)^{1/2}\Bigg]\Bigg\}.\end{aligned}$$

Proof. By Proposition 4.8, $(S_{s\le T}u)^{-1}\in L^p$ for $p\ge 1$. In particular, $S_{s\le T}u>0$ a.e. The dominated convergence theorem implies that the left-hand side of (4.14) tends to $|E[Xf(z_T)]|$ as $\eta\to 0$. Observe that for all $\eta>0$

$$|\Phi_0|\le 1/S_{s\le T}u\,,\quad |\Phi_1|\le c/S^2_{s\le T}u\,.$$

As we saw in the proof of Theorem 4.7, $S_{s\le T}|\Psi_i^{(Z)}|\in L^p$, $i=1,2$, for all $1\le p<\infty$. It follows that functions of the form $|\Phi_i S_{s\le T}\Psi_j^{(X)}|$ and $\Phi_i^2 S^2_{s\le T}|\Psi_j^{(Z)}|$ are dominated by an L^1 function independent of η. Applying the Lebesgue theorem to the right-hand side of (4.14) completes the proof. ∎

COROLLARY 4.10. *Let $X\in\mathfrak{g}$. Then there exists a constant c such that for $f\in C_c^\infty(G)$*

$$\langle\widetilde{\mu}_T,Xf\rangle\le c\|f\|_\infty\,. \ \blacksquare$$

§ 5. Integration by parts for differential operators of order n.

In this section we generalize the procedure of integration by parts and the

estimation presented in §4 to the case of operators X^I where $|I|=n$. Our aim is to derive estimates of the type (4.2):

$$|E[Vf(z_T)]|\le c\|f\|_\infty$$

for an operator $V=X^I=X_{i_1}\dots X_{i_n}$ where $X_1,\dots,X_M$ form a homogeneous linear basis of $\mathfrak{g}$.

All the notation as well as the scheme of the procedure are the same as in §4.

(a) *Integration by parts on jumps of the process.* Fix $\varepsilon>0$ and $T>0$. Let $u\in C^n(G)$ be a nonnegative function such that $\operatorname{supp}u\subseteq\{|x|\ge\varepsilon\}$. Similarly to §4(a) we have

$$\begin{aligned}&E[\varrho_\eta(S_{s\le T}u)Vf(z_T^{(\varepsilon)})]\\ &=\sum_{m=1}^\infty P\{N=m\}\sum_{j=1}^m E\left[\frac{\varrho_\eta(K_j+u(\Delta z_{S_j}))}{K_j+u(\Delta z_{S_j})}u(\Delta z_{S_j})Vf(\Delta z_{S_1}\dots\Delta z_{S_m})\right].\end{aligned}$$

In the integrals in the last sum we integrate with respect to the probability law of Δz_{S_j} and next we integrate by parts on G. The jth integral is then equal to

$$\text{(5.1)}\qquad E\left[\frac{(-1)^n}{\widetilde{\nu}_\varepsilon(G)}\int \mathrm{Ad}_{\sigma_j}X^I\left(\frac{\varrho_\eta(K_j+u)}{K_j+u}ug\right)(z)\times f(\Delta z_{S_1}\dots z\dots\Delta z_{S_m})\,dz\right]$$

where $\mathrm{Ad}_\sigma X^I=\mathrm{Ad}_\sigma X_{i_n}\dots\mathrm{Ad}_\sigma X_{i_1}$. By (4.7),

$$\mathrm{Ad}_{\Delta z_{S_{j+1}}\dots\Delta z_{S_N}}X^I=\sum_W d_W^{(j)}W$$

where the sum is taken over all W of the form

$$\text{(5.2)}\qquad W=[X_{k_1^{(n)}},\dots,X_{k_{l_n}^{(n)}},X_{i_n}]\dots[X_{k_1^{(1)}},\dots,X_{k_{l_1}^{(1)}},X_{i_1}]$$

for $0\le l_i\le d-1$, with

$$\text{(5.3)}\qquad d_W^{(j)}=\prod_{i=1}^n d^{(j)}_{k_1^{(i)}\dots k_{l_i}^{(i)}}\qquad(\text{for } l_i=0 \text{ we put } d_0^{(j)}=1)\,.$$

For a fixed W of the form (5.2) we will calculate the derivative

$$W\left(\frac{\varrho_\eta(K_j+u)}{K_j+u}ug\right).$$

Write $W=Y_n\dots Y_1=Y^J$ where $Y_s=[X_{k_1^{(s)}},\dots,X_{k_{l_s}^{(s)}},X_{i_s}]$.

If $F \in C^n(\mathbf{R})$ and $v \in C^n(G)$, then by induction

$$(5.4)\qquad Y^J(F\circ v)(z) = \sum_{l=1}^{n} F^{(l)}(v(z)) \sum_{\substack{J_1^{(l)}+\ldots+J_l^{(l)}=J \\ J_i^{(l)}\neq 0}} Y^{J_1^{(l)}}v\ldots Y^{J_l^{(l)}}v\,.$$

In our case F has the form $F(t)=\varrho(t)/t$, $\varrho\in C^\infty(\mathbf{R})$, and by induction we obtain for $k\geq 1$

$$(5.5)\qquad F^{(k)}(t) = \frac{\varrho^{(k)}(t)}{t} + c_{k-1,k}\frac{\varrho^{(k-1)}(t)}{t^2} + \ldots + c_{0k}\frac{\varrho(t)}{t^{k+1}}$$

for some constants $c_{k-1,k},\ldots,c_{0k}$.

Using the Leibniz formula, (5.4) and (5.5) we find that the term corresponding to the operator W in the integral (5.1) is

$$(-1)^n E\Big[d_W^{(j)} \sum_{K\leq J} c_{KJ} \sum_{0\leq l\leq |K|} \Phi_l \Psi_{Kl}^{(W)}(\Delta z_{S_j})\cdot f(z_T^{(\varepsilon)})\Big]$$

where the c_{KJ} are constants from the Leibniz formula,

$$(5.6)\qquad \Phi_l = F^{(l)}(S_{s\leq T}u)$$

for $F^{(l)}$ given by (5.5) and

$$\Psi_{Kl}^{(W)} = \frac{Y^{J-K}(ug)}{g} \sum_{\substack{K_1^{(l)}+\ldots+K_l^{(l)}=K \\ K_i^{(l)}\neq 0}} Y^{K_1^{(l)}}u\ldots Y^{K_l^{(l)}}u$$

(if $l=0$ then $K=0$). Defining

$$(5.7)\qquad \Psi_l^{(W)} = \sum_{K\leq J,\ |K|\geq l} c_{KJ}\Psi_{Kl}^{(W)}$$

we obtain the following integration by parts formula.

LEMMA 5.1.

$$E[\varrho_\eta(S_{s\leq T}u)Vf(z_T^{(\varepsilon)})] = (-1)^n \sum_W \sum_{l=0}^{n} E\Big[\Phi_l \sum_{j=1}^{N} d_W^{(j)}\Psi_l^{(W)}(\Delta z_{S_j})f(z_T^{(\varepsilon)})\Big]$$

where the W are of the form (5.2), $d_W^{(j)}$ *is given by* (5.3) *and* Φ_l, $\Psi_l^{(W)}$ *are defined in* (5.6) *and* (5.7). ∎

Arguing as in the proof of Lemma 4.2 we get an analogous estimate:

LEMMA 5.2.

$$(5.8)\qquad |E[\varrho_\eta(S_{s\leq T}u)Vf(z_T^{(\varepsilon)})]| \leq \|f\|_\infty \sum_{l=0}^{n}\sum_W (E\sup_{1\leq j\leq N}|d_W^{(j)}|^2)^{1/2}(E[\Phi_l^2 S_{s\leq T}^2|\Psi_l^{(W)}|])^{1/2}\,. \ ∎$$

(b) *Estimation of random coefficients.* By (5.3) the coefficients $d_W^{(j)}$ are products of $d_{k_1\ldots k_l}^{(j)}$ investigated in §4(b). Since

$$(E|\xi_1\ldots\xi_n|)^n \leq E|\xi_1|^n\ldots E|\xi_n|^n$$

in order to estimate $E\sup_{1\leq j\leq N}|d_W^{(j)}|^2$ it is sufficient to estimate $E(\sup_{1\leq j\leq N}|d_{k_1\ldots k_l}^{(j)}|^{2n})$. By Proposition 4.4 such integrals are bounded by a constant $c<\infty$ independent of ε.

Thus by Lemma 5.2 we obtain

COROLLARY 5.3.

$$(5.9)\qquad |E[\varrho_\eta(S_{s\leq T}u)Vf(z_T^{(\varepsilon)})]| \leq c\|f\|_\infty \sum_{l=0}^{n}\sum_W (E[\Phi_l^2 S_{s\leq T}^2|\Psi_l^{(W)}|])^{1/2}$$

where the constant c depends only on n and on the process $\{z_t\}$, the functions Φ_l, $\Psi_l^{(W)}$ are given by (5.6) *and* (5.7) *and the W are of the form* (5.2). ∎

(c) *The estimate for a specific function u*

PROPOSITION 5.4. *Let $u\in C^n(G)$ with* $\operatorname{supp} u\subseteq\{|x|\geq\varepsilon_0\}$ *for an* $\varepsilon_0>0$. *Then*

$$(5.10)\qquad |E[\varrho_\eta(S_{s\leq T}u)Vf(z_T)]| \leq c\|f\|_\infty \sum_{l=0}^{n}\sum_W (E[\Phi_l^2 S_{s\leq T}^2|\Psi_l^{(W)}|])^{1/2}\,.$$

Proof. We make $\varepsilon\to 0$ in (5.9) and argue as in the proof of Proposition 4.6. ∎

THEOREM 5.5. *Let $u(x)=|x|^m\Omega^n(x)h^n(x)$. Then* (5.10) *holds for u with m sufficiently large.*

Proof. The proof is analogous to that of Theorem 4.7. We consider the functions

$$u_\varepsilon(x) = \varrho(|x|/\varepsilon)u(x)$$

for which (5.10) holds by Proposition 5.4. Since ϱ_η is bounded, $\lim_{\varepsilon\to 0}E[\varrho_\eta(S_{s\leq T}u_\varepsilon)Vf(z_T)] = E[\varrho_\eta(S_{s\leq T}u)Vf(z_T)]$. To prove the convergence of the right-hand side of (5.10) for $u_\varepsilon\to u$, observe first that the Φ_l are bounded for fixed $\eta>0$ and converge everywhere when $u_\varepsilon\to u$. Thus it is sufficient to show that

$$S_{s\leq T}|\Psi_l^{(W)}(u_\varepsilon)-\Psi_l^{(W)}(u)|\to 0 \quad \text{in } L^2\,.$$

To show this we prove that

$$(5.11)\qquad \begin{aligned} S_{s\leq T}|\Psi_{Kl}^{(W)}(u_\varepsilon)-\Psi_{Kl}^{(W)}(u)|\to 0 &\quad \text{in } L^1\,,\\ S_{s\leq T}^c|\Psi_{Kl}^{(W)}(u_\varepsilon)-\Psi_{Kl}^{(W)}(u)|\to 0 &\quad \text{in } L^2\,,\end{aligned}$$

for all K, l and W.

The proof of (5.11) is similar to that of (4.15) in Theorem 4.7. First we show that for m sufficiently large $|\Psi_{Kl}^{(W)}| \le c|x|^2$ for $|x| < 1/2$. Since the $\Psi_{Kl}^{(W)}$ are bounded, Corollary 1.5 implies that $S_{s\le T}|\Psi_{Kl}^{(W)}| \in L^p$ for all $1 \le p < \infty$. By (5.7) the same is true for $S_{s\le T}|\Psi_l^{(W)}(u)|$. Next we show (5.11) using Theorem 1.4. The derivatives of the function $\widetilde{\varrho}_\varepsilon$ appearing in $\Psi_{Kl}^{(W)}(u_\varepsilon)$ yield an $O(\varepsilon^{-nd})$ factor. Similarly to the proof of Theorem 4.7 this factor disappears if the exponent m in u is sufficiently large. ∎

(d) *The final estimate*

THEOREM 5.6. *Let* $u(x) = |x|^m \Omega^n(x) h^n(x)$, *where* m *is as in the previous theorem. Then* (5.10) *holds with* ϱ_η *replaced by* 1.

Proof. By Proposition 4.8, $(S_{s\le T}u)^{-1} \in L^p$ for $p \ge 1$, so $S_{s\le T}u > 0$ a.e. and the left-hand side of (5.10) converges to $|E[Vf(z_T)]|$ as $\eta \to 0$.

Now (5.5) implies that $|\Phi_l| \le c/S_{s\le T}^{l+1}u \in L^p$ $(p \ge 1,\ l = 0,\dots,n)$ and $\Phi_l(\varrho_\eta) \to c_{0l}/S_{s\le T}^{l+1}u$ as $\eta \to 0$. Since $S_{s\le T}|\Psi_l^{(W)}| \in L^p$ for all $p \ge 1$ and the $\Psi_l^{(W)}$ do not depend on η we deduce that $\Phi_l^2(\varrho_\eta) S_{s\le T}^2 |\Psi_l^{(W)}|$ is bounded by an integrable function independent of η. By the dominated convergence theorem, the right-hand side of (5.10) is convergent as $\eta \to 0$. In particular, we obtain the estimate

$$|E[Vf(z_T)]| \le c\|f\|_\infty \sum_{l=0}^{n} \sum_{W} \left(E \frac{S_{s\le T}^2 |\Psi_l^{(W)}|}{(S_{s\le T}u)^{2l+2}} \right)^{1/2}.$$

COROLLARY 5.7. *There exists a constant* $c < \infty$ *such that for* $f \in C_c^\infty(G)$

$$|\langle \widetilde{\mu}_t, Vf \rangle| \le c\|f\|_\infty. \ ∎$$

The proof of Theorem 3.1 now follows by Corollary 5.7, Lemma 3.2 and Proposition 3.3. ∎

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