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Every Radon–Nikodym Corson compact space is Eberlein compact

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Abstract. We prove the result announced in the title. The Banach space version of this topological result reads as follows: A Banach space E whose dual unit ball is a weak* Corson compact and which is GSG (i.e. there is an Asplund space X and a continuous linear operator from X into E with dense range) is weakly compactly generated. We also analyze a relevant example of M. Talagrand and obtain solutions to three problems posed by I. Namioka.

1. Introduction. A compact topological space is called an *Eberlein compact* if it is homeomorphic to a weakly compact subset of some Banach space and is called *Radon–Nikodym compact* if it is homeomorphic to a weak* compact subset of the dual of an Asplund space. By the factorization result of [DFJP] every Eberlein compact space is homeomorphic to a weakly compact subset of a reflexive Banach space, therefore an Eberlein compact space is a Radon–Nikodym compact space. (For definitions and unexplained notation we refer to the end of the introduction.)

To see that these two notions are different, observe that for a compact scattered space K the Banach space $C(K)$ is Asplund as the dual $C(K)^*$ equals $l^1(K)$ and therefore has RNP. Hence K is a Radon–Nikodym compact space. For example the ordinal interval $[0, \omega_1]$ is a Radon–Nikodym compact but fails to be Eberlein (by Eberlein's theorem). Using the idea of dentability I. Namioka gave a topological characterization of Radon–Nikodym compacta [N] as those compact spaces which admit a lower semicontinuous fragmenting metric (see below). This gives rise to the notion of fragmented compacta ([JR], [N] where the lower semicontinuity assumption is dropped and we obtain the following chain of implications:

Eberlein compact $\Rightarrow$ Radon–Nikodym compact $\Rightarrow$ fragmented compact.

We shall prove in this paper that the second implication above also fails to be an equivalence.

Another direction of generalizing the notion of Eberlein compacta was done in the work of M. Talagrand [T] and S. Gul'ko [Gu3] which we summarize (as in [Ne]) by the following chain of implications, which all are known not to be equivalences (for definitions see below):

$$\text{Eberlein} \Rightarrow \text{Talagrand} \Rightarrow \text{Gul'ko} \Rightarrow \text{Corson}.$$

As regards the relationship between these two chains of implications it was shown in [G], [R] that

$$\text{Gul'ko} \Rightarrow \text{fragmented space}.$$

This implication summarizes the work done around the problem, posed by Talagrand, of whether every Gul'ko compact space has a dense G_δ metrizable subset. In [G] this question is positively answered and in [R] the spaces used by Gruenhage are seen to be fragmented compact spaces. Indeed, every fragmented compact space has a dense G_δ subset where the induced topology coincides with the topology induced by the fragmenting metric.

In [To], an example of a Corson compact space without dense metrizable subsets is given, so we know that

$$\text{Corson} \not\Rightarrow \text{fragmented space}.$$

On the other hand, every scattered Corson compact space is an Eberlein compact space ([A], see also [V]).

The following questions related to the concepts above are posed in the paper [N] of Namioka:

PROBLEM 1. Find a compact Hausdorff space that is fragmented by a metric but is not Radon–Nikodym compact.

PROBLEM 2. Do the implications Talagrand compact $\Rightarrow$ Radon–Nikodym compact or Gul'ko compact $\Rightarrow$ Radon–Nikodym compact hold true?

PROBLEM 3. Find conditions for a Radon–Nikodym compact space to be Eberlein compact.

E. A. Reznichenko [K] has given an example of a Talagrand compact space which is not Radon–Nikodym compact, thus answering the questions posed in Problems 1 and 2. We show that Talagrand's initial example of a Talagrand compact space which is not Eberlein compact also fails to be Radon–Nikodym compact, thus also solving these two questions. We then proceed to prove the main result of this paper which is announced in the title, i.e. a Radon–Nikodym compact which is also Corson is already Eberlein. This gives in particular a complete solution to the first two questions and a rather general answer to Problem 3.

We shall use some embedding results which we summarize as follows: every compact Hausdorff space K can be embedded into a cube I^Γ, where I denotes the unit interval $[-1, +1]$, and Γ is some set of continuous functions on K. K is an Eberlein compact if and only if this embedding can be chosen to take values in $c_0(\Gamma)$ [AL]. By definition K is a Corson compact if this embedding can be chosen to take values in the Σ-product space

$$\Sigma(\Gamma) = \{x \in I^\Gamma : \{\gamma \in I : x(\gamma) \text{ is not zero}\} \text{ is countable}\}.$$

K is a Radon–Nikodym compact space if and only if this embedding can be done in such a way that the metric d_Γ of uniform convergence on Γ:

$$d_\Gamma(x, y) = \sup\{|x(\gamma) - y(\gamma)| : \gamma \in \Gamma\},$$

x and y in I^Γ, fragments K [N]. Note that d_Γ fragments K if, and only if, for every countable subset A of Γ the pseudometric

$$d_A(x, y) = \sup\{|x(\gamma) - y(\gamma)| : \gamma \in A\}$$

is separable on K, which is parallel to Stegall's characterization of dual Banach spaces with the Radon–Nikodym property [S1].

Let us now give an outline of the organization of this paper:

In Section 2 we study in detail Talagrand's example of a Talagrand compact space K which fails to be Eberlein. We show that it also fails to be Radon–Nikodym. Let us point out that the facts proved in Section 2 for the particular space K will also follow from the general results obtained in the subsequent sections (which are logically independent of Section 2). Yet we recommend the reader to. first follow the arguments in the special case of Talagrand's example as many of the ideas used in the general case appear already here. Hence we make a point of giving detailed proofs for this special case; for example we explicitly define a fragmenting (not lower semicontinuous) metric on K.

In the subsequent sections we work on the proof of the main result of this paper, in its topological and Banach space versions:

THEOREM A. *A compact Hausdorff space is Eberlein if (and only if) it is Radon–Nikodym and Corson.*

THEOREM B. *A Banach space E is weakly compactly generated if (and only if) its dual unit ball is Corson compact and if it is GSG, i.e. there is a continuous linear map $T: X \to E$ with dense range, defined on an Asplund space X.*

In Section 3 we reduce Theorem B to the case where the Banach space X is not only GSG but is already itself an Asplund space. This will imply that in the setting of Theorem A a compact space which is Radon–Nikodym and Corson can be weak* embedded into the dual of a Banach space F which is Asplund and has weak* Corson compact dual unit ball. The technique of this section is to apply the factorization method of [DFJP].

In Section 4 we show that a Banach space as above (i.e. one which is Asplund and has weak* Corson compact dual unit ball) is weakly compactly generated and therefore its dual unit ball is a weak* Eberlein compact. This will be proved with the help of "long sequences of projections" as developed in [V].

Finally, in Section 5 we pull the strings together and give the proofs for Theorems A and B.

Similar results have been obtained at about the same time by C. Stegall.

We now collect the notions used in this paper:

If $(X, \mathcal{T})$ is a topological space and d is a metric on X, which need not be related to $\mathcal{T}$ in any way, we say that the space $(X, \mathcal{T})$ is *fragmented by d* if, for each nonempty subset A of X and for each positive ε, there exists a $\mathcal{T}$-open subset U of X such that $U \cap A$ is nonvoid and d-diam$(U \cap A) \leqslant \varepsilon$.

A compact Hausdorff space $(X, \mathcal{T})$ is called *fragmented* (resp. *Radon–Nikodym*) if there is a metric (resp. a $\mathcal{T}$-l.s.c. metric) d on X fragmenting $(X, \mathcal{T})$.

A topological space $(X, \mathcal{T})$ is *countably determined* (resp. *K-analytic*) if there is a metrizable and separable (resp. complete, metrizable and separable) space P and a set-valued map T from P onto X with compact values and upper semicontinuous.

A compact topological space is called a *Gul'ko* (resp. *Talagrand*) *compact* if it is homeomorphic to a weak* compact subset of the dual of a weakly countably determined (resp. weakly K-analytic) Banach space.

If $(X, \mathcal{T})$ is a topological space, we denote by $C_p(X)$ the space of all real continuous functions defined on X endowed with the pointwise topology. A compact topological space K is Gul'ko (resp. Talagrand) if, and only if, $C_p(K)$ is countably determined (resp. K-analytic).

We denote by ω_0 the first infinite ordinal. For a topological space $(X, \mathcal{T})$ the *density character*, dens(X), is the first cardinal λ such that there is a dense subset A of $(X, \mathcal{T})$ with cardinality λ. For a set A, we denote by $|A|$ its cardinal number.
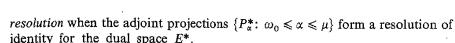
A *projectional resolution of identity* on a Banach space E is a "long sequence of projections" $\{P_\alpha: \omega_0 \leqslant \alpha \leqslant \mu\}$, where μ is the first ordinal whose cardinality equals the density character of E, which satisfies:

    (i) $\|P_\alpha\| = 1$ for every α.

    (ii) $P_\alpha P_\beta = P_\alpha = P_\beta P_\alpha$ if $\omega_0 \leqslant \alpha \leqslant \beta \leqslant \mu$.

    (iii) dens$(P(E)) \leqslant |\alpha|$ for every α.

    (iv) $\overline{\bigcup\{P_{\beta+1}(E): \beta < \alpha\}}^{\|\ \|} = P_\alpha(E)$ for every α.

    (v) P_μ is the identity operator on E.

A projectional resolution of identity $\{P_\alpha: \omega_0 \leqslant \alpha \leqslant \mu\}$ is called a *shrinking resolution* when the adjoint projections $\{P_\alpha^*: \omega_0 \leqslant \alpha \leqslant \mu\}$ form a resolution of identity for the dual space E^*.

2. Talagrand's example is not a Radon–Nikodym compact. Let K be a subset of the cube I^Γ. We say that K is *solid* if it satisfies the following condition: If $x \in K$, $y \in I^\Gamma$ and

$$y(\gamma) = x(\gamma) \quad \text{or} \quad y(\gamma) = 0 \quad \text{for every } \gamma \text{ in } \Gamma,$$

then $y \in K$.

PROPOSITION 2.1. *Let K be a solid compact subset of I^Γ. The metric*

$$d_\Gamma(x, y) = \sup\{|x(\gamma) - y(\gamma)|: \gamma \in \Gamma\}, \quad x \text{ and } y \text{ in } I^\Gamma,$$

fragments K if, and only if, K is contained in $c_0(\Gamma)$.

Proof. If K is fragmented by d_Γ, then every x in K belongs to $c_0(\Gamma)$. Otherwise, we could find a countable infinite subset A of Γ, an $\varepsilon > 0$ and a point x in K such that $|x(\gamma)| > \varepsilon$ for every γ in A. For every subset P of A we define a point x_P of K by $x_P(\gamma) = x(\gamma)$ if $\gamma \in P$ and $x_P(\gamma) = 0$ otherwise. The subset of K defined by $\{x_P: P \subset A\}$ is uncountable and such that $d_A(x_P, x_Q) > \varepsilon$ if $P \neq Q$. Therefore, d_A is not separable on K and the metric d_Γ does not fragment K because of Namioka's theorem [N] mentioned in the introduction.

On the other hand, every weakly compact subset of a Banach space is fragmented by the norm; if K is contained in $c_0(\Gamma)$ then K is a weakly compact subset of it and the norm of $c_0(\Gamma)$ induces the metric d_Γ on K. ∎

There is a class of solid compact spaces which has been used for the construction of suitable examples. This is the class of spaces that result from an *adequate family of sets*.

DEFINITION ([T], [P]). A family $\mathfrak{C}$ of subsets of a nonempty set T is said to be *adequate* if:

    (i) $A \in \mathfrak{C}$ and $B \subseteq A$ implies $B \in \mathfrak{C}$;

    (ii) if A is a subset of T such that every finite subset of A belongs to $\mathfrak{C}$ then A belongs to $\mathfrak{C}$;

    (iii) for every point t in T the set $\{t\}$ belongs to $\mathfrak{C}$.

Given an adequate family $\mathfrak{C}$, the set of characteristic functions

$$K(\mathfrak{C}) = \{\chi_A: A \in \mathfrak{C}\}$$

is a compact subset of the product space $\{0, 1\}^T$. We often identify A with χ_A, and thus consider K as a space of subsets of T equipped with the topology τ inherited from $\{0, 1\}^T$.

TALAGRAND'S EXAMPLE. We shall work with the set $T = \mathbf{N}^\mathbf{N}$ of sequences of positive integers together with the adequate families of subsets given by

$$\mathfrak{A}_0 = \{\{\sigma\}: \sigma \in \mathbf{N}^\mathbf{N}\} \cup \varnothing,$$

and for every positive integer n,

$$\mathfrak{A}_n = \{A \subset \mathbf{N}^\mathbf{N}: \sigma, \varrho \in A \text{ and } \sigma \neq \varrho \Rightarrow \sigma|_n = \varrho|_n \text{ and } \sigma|_{n+1} \neq \varrho|_{n+1}\}$$

where for each sequence σ in $\mathbf{N}^\mathbf{N}$ and each positive integer n,

$$\sigma|_n = (\sigma(1), \ldots, \sigma(n)).$$

We denote by K_0 the compact set $K(\mathfrak{A}_0)$ and by K_n the compact set $K(\mathfrak{A}_n)$ for every positive integer n. We also define $\mathfrak{A} = \bigcup \{\mathfrak{A}_n: n = 1, 2, \ldots\}$ which is also an adequate family and gives us Talagrand's example K:

$$K = K(\mathfrak{A}) = \bigcup \{K_n: n = 1, 2, \ldots\}.$$

Observe that $\mathfrak{A}_n \cap \mathfrak{A}_m = \mathfrak{A}_0$ if $n \neq m$, and so $K_n \cap K_m = K_0$ if $n \neq m$.

THEOREM 2.2. *The compact space K is a Talagrand compact that admits a fragmenting metric and is not a Radon–Nikodym compact space.*

We begin with some intuitive arguments which suggest that it is not possible to find any lower semicontinuous metric fragmenting K.

Identifying (K, τ) with a subset of $(l^\infty(\mathbf{N}^\mathbf{N}), \sigma^*)$ consider the metric ϱ_0 on K induced by the norm $\| \ \|_\infty$ of $l^\infty(\mathbf{N}^\mathbf{N})$ which is nothing but the discrete metric on K.

Note that (K_0, τ) is homeomorphic to the one-point compactification of the discrete set $\mathbf{N}^\mathbf{N}$ (the point at infinity is the point 0) and therefore the discrete metric ϱ_0 on K_0 fragments K_0. (This is easily seen directly; it may also be deduced more abstractly from the fact that $K_0 \subset c_0(\mathbf{N}^\mathbf{N})$ and from the fragmentability of weakly compact sets.)

But ϱ_0 does not fragment K and in fact it does not fragment any K_n, for $n \geqslant 1$, as follows from the above proposition.

In our search for a τ-l.s.c. metric on K which fragments all of K (this search will turn out to be unsuccessful) a first step would be to find, for $n \in \mathbf{N}$, a τ-l.s.c. metric ϱ_n on K which fragments $K_0 \cup \ldots \cup K_n$. This can indeed be done: Define

$$T_n: l^\infty(\mathbf{N}^\mathbf{N}) \to l^\infty(\mathbf{N}^\mathbf{N}),$$

$$T_n f(\sigma) = \big((\sigma(1)+1) \ldots (\sigma(n+1)+1)\big)^{-1} f(\sigma), \quad f \in l^\infty(\mathbf{N}^\mathbf{N}), \ \sigma \in \mathbf{N}^\mathbf{N}.$$

Then T_n is a weak* continuous injective map from $l^\infty(\mathbf{N}^\mathbf{N})$ into itself such that $T_n(K_0 \cup \ldots \cup K_n) \subseteq c_0(\mathbf{N}^\mathbf{N})$. Hence the metric

$$\varrho_n(A, B) = \|T_n(\chi_A) - T_n(\chi_B)\|_\infty, \quad A, B \in K,$$

is a l.s.c. metric on (K, τ) that fragments $K_0 \cup \ldots \cup K_n$.

But the crux is that ϱ_n does not fragment K_{n+l} for $l \in \mathbf{N}$ as one deduces from the above proposition. On the other hand, the metric ϱ_{n+1} does fragment K_{n+1} too, but the reader should note that ϱ_{n+1} induces a strictly coarser metric than ϱ_n on K_n.

The obvious thing to do now is to paste the ϱ_n's together in such a way as to obtain a metric ϱ which fragments all of K. For example, one may define

$$\varrho(A, B) = \begin{cases} \varrho_n(A, B) & \text{if } A \text{ and } B \text{ are in } K_n \backslash K_0, \\ 1 & \text{otherwise.} \end{cases}$$

This metric indeed fragments all of K: Let $F \subseteq K$ be nonvoid. Either $F \subseteq K_0$, in which case F contains a τ-open singleton; or there is a positive integer n such that $F \cap (K_n \backslash K_0) \neq \emptyset$. In the latter case we can find, for $\varepsilon > 0$, a τ-open subset V of K contained in $K_n \backslash K_0$ such that $V \cap F \neq \emptyset$ and ϱ-diam$(V \cap F) = \varrho_n$-diam$(V \cap F) < \varepsilon$.

However, the problem with the above metric ϱ is that—unlike the metrics ϱ_n—it fails to be τ-l.s.c. Let us see why: Fix two distinct points μ and σ in K_0, both different from 0, and let $(A_n)_{n=1}^\infty$ and $(B_n)_{n=1}^\infty$ be elements of $K_n \backslash K_0$ such that A_n contains μ, so it starts at $\mu|_n$, and B_n contains σ, so it starts at $\sigma|_n$; i.e. for $\alpha \in A_n$, $\alpha|_n = \mu|_n$ and for $\beta \in B_n$, $\beta|_n = \sigma|_n$. Then

$$\tau\text{-}\lim_n A_n = \{\mu\}, \qquad \tau\text{-}\lim_n B_n = \{\sigma\}$$

and

$$\lim_n \varrho(A_n, B_n) = \lim_n \varrho_n(A_n, B_n) = 0,$$

which readily shows that ϱ fails to be τ-l.s.c. in view of $\varrho(\{\mu\}, \{\sigma\}) = 1$.

Summing up what we have shown so far about Talagrand's space K: There is a fragmenting metric on K, namely ϱ, but this metric fails to be τ-l.s.c. In the rest of this section we shall show that no fragmenting metric d on K can be τ-l.s.c. for reasons which are essentially the same as the argument given for ϱ above. This will prove Theorem 2.2.

So suppose that there is a τ-l.s.c. fragmenting metric d on K and let us work towards a contradiction:

The metric d induces a topology on K finer than τ [JNR]. Thus the identity from $(K_0 \backslash \{0\}, d)$ into $(K_0 \backslash \{0\}, \tau)$ is continuous and the last space has the discrete topology. So this is an homeomorphism, whence for every σ in $\mathbf{N}^\mathbf{N}$ there is a positive number δ_σ such that

$$d(\{\sigma\}, x) > \delta_\sigma > 0$$

for every x in K_0. In what follows we shall work with the numbers δ_σ and we shall see how, in some sense, the discreteness of K_0 spreads over all of K. We need some preparatory lemmas:

LEMMA 1. *For every σ in $\mathbf{N}^\mathbf{N}$ there is a positive integer n_σ such that if A and B are elements of K_m, $m \geqslant n_\sigma$ and $\sigma \in (A \backslash B) \cup (B \backslash A)$, then $d(A, B) \geqslant \delta_\sigma/2$.*

Proof. It is enough to note that any sequence (A_n) in K with A_n in K_n has a cluster point in K_0. Then arguing by contradiction we get the conclusion from the lower semicontinuity of the metric d and the definition of the numbers δ_σ.

Indeed, suppose that the assertion does not hold true. We can find an increasing sequence of positive integers $m_1 < m_2 < \ldots$ and subsets A_n and B_n in K_{m_n} such that $\sigma \in A_n$, $\sigma \notin B_n$, and

$$d(A_n, B_n) < \delta_\sigma / 2.$$

The sequence (A_n) converges to $\{\sigma\}$ in K and (B_n) has a cluster point x in K_0 different from $\{\sigma\}$. By the lower semicontinuity of d, we know that $d(\{\sigma\}, x) \le \delta_\sigma / 2$, which contradicts the previous choice of the number δ_σ. ∎

We shall now work with the product space $\mathbf{N}^\mathbf{N}$ of discrete spaces N. Note that if a subset $\mathfrak{H}$ of $\mathbf{N}^\mathbf{N}$ is not contained in a σ-compact subset and $\mathfrak{H} = \bigcup \{\mathfrak{H}_n: n = 1, 2, \ldots\}$, then there is a positive integer r such that $\mathfrak{H}_r$ is not contained in a σ-compact subset.

LEMMA 2. *There exists a subset* $\Sigma' \subseteq \mathbf{N}^\mathbf{N}$, *which is not contained in a σ-compact subset, and positive integers p and q, such that for any A and B in K_m with $m \ge q$ and $[(A \backslash B) \cup (B \backslash A)] \cap \Sigma'$ nonvoid*

$$d(A, B) > 1/(2p).$$

Proof. Consider the identity

$$\mathbf{N}^\mathbf{N} = \bigcup \{\{\sigma \in \mathbf{N}^\mathbf{N}: \delta_\sigma > 1/n \text{ and } n_\sigma < k\}: n, k = 1, 2, \ldots\},$$

where n_σ is given by Lemma 1 above. As $\mathbf{N}^\mathbf{N}$ is not σ-compact, there are positive integers p and q such that

$$\Sigma' = \{\sigma \in \mathbf{N}^\mathbf{N}: \delta_\sigma > 1/p \text{ and } n_\sigma < q\}$$

is not contained in a σ-compact subset. ∎

LEMMA 3. *Let $\mathfrak{X}$ be a subset of $\mathbf{N}^\mathbf{N}$ which is not contained in a σ-compact subset and q a fixed positive integer. Then there is a finite sequence $(s_1, \ldots, s_k)$ with $k \ge q$ such that*

$$\{n \in \mathbf{N}: \text{ for some } \sigma \text{ in } \mathfrak{X}, \ \sigma|_{k+1} = (s_1, \ldots, s_k, n)\}$$

is infinite.

Proof. Fix q and write

$$\mathfrak{X} = \bigcup \{\{\sigma \in \mathfrak{X}: \sigma|_q = (b_1, \ldots, b_q)\}: (b_1, \ldots, b_q) \in \mathbf{N}^q\}.$$

Since $\mathfrak{X}$ is not contained in a σ-compact subset, there is some $(a_1, \ldots, a_q) \in \mathbf{N}^q$ such that

$$\mathfrak{X}_q := \{\sigma \in \mathfrak{X}: \sigma|_q = (a_1, \ldots, a_q)\}$$

is not contained in a σ-compact subset. Now suppose that the assertion of the lemma is false. Then the set

$$\{m \in \mathbf{N}: \text{ for some } \sigma \in \mathfrak{X}_q, \ \sigma(q+1) = m\}$$

$$= \{m \in \mathbf{N}: \text{ for some } \sigma \in \mathfrak{X}, \ \sigma|_{q+1} = (a_1, \ldots, a_q, m)\}$$

is finite. Let a_{q+1} denote the maximum of this set. So we have

$$\mathfrak{X}_q \subset \{\sigma \in \mathfrak{X}: \sigma(i) \le a_i, \ i = 1, \ldots, q+1\}.$$

Proceeding by recurrence, suppose we have found integers $a_{q+1}, \ldots, a_{q+n}$ such that

$$\mathfrak{X}_q \subset \{\sigma \in \mathfrak{X}: \sigma(i) \le a_i, \ i = 1, \ldots, q+n\}.$$

For every finite sequence $(s_1, \ldots, s_{q+n})$ with $s_i \le a_i$, $i = 1, \ldots, q+n$, the set

$$\{m \in \mathbf{N}: \text{ for some } \sigma \in \mathfrak{X}, \ \sigma|_{q+n+1} = (s_1, \ldots, s_{q+n}, m)\}$$

is finite, and we denote it by $N(s_1, \ldots, s_{q+n})$. The set of integers

$$\bigcup \{N(s_1, \ldots, s_{q+n}): s_i \le a_i, \ i = 1, \ldots, q+n\}$$

is finite and we can find its maximum, which we denote by a_{q+n+1}. We thus have

$$\mathfrak{X}_q \subset \{\sigma \in \mathfrak{X}: \sigma(i) \le a_i, \ i = 1, \ldots, q+n+1\}.$$

Finally, we get a sequence (a_n) in $\mathbf{N}^\mathbf{N}$ such that $\mathfrak{X}_q$ is contained in the compact set $\{\sigma \in \mathfrak{X}: \sigma(i) \le a_i, \ i = 1, 2, \ldots\}$, which is a contradiction. ∎

Proof of Theorem 2.2. We apply Lemma 3 to the set Σ' and the positive integer q given by Lemma 2. There exists a finite sequence $(s_1, \ldots, s_k)$ with $k \ge q$ such that

$$\{n \in \mathbf{N}: \text{ for some } \sigma \in \Sigma', \ \sigma|_{k+1} = (s_1, \ldots, s_k, n)\}$$

is an infinite subset of N, which we denote by P.

Denote by F the Banach space of all continuous and d-Lipschitz functions on K_k with the norm $\| \ \|_{\mathrm{Lip}} + \| \ \|_\infty$. Then K_k is a weak* compact subset of the dual F^*, and the dual norm also fragments K_k according to a theorem of Ghoussoub and Maurey [GM, Th. VII.1] (see also [JNR]). Now, if $\sigma \in \mathbf{N}^\mathbf{N}$ and e_σ denotes the canonical projection from $\{0, 1\}^{\mathbf{N}^\mathbf{N}}$ onto the coordinate σ, then the family of functions

$$\mathfrak{F} = \{(1/(2p))e_\sigma: \sigma \in \Sigma'\},$$

where p is given by Lemma 2, is a bounded subset of F. Indeed, the only thing to check is the Lipschitz condition. Take a pair of elements A and B in K_k and σ a fixed element in Σ'. Then we have

$$|e_\sigma(A) - e_\sigma(B)| = 0 \quad \text{if } \sigma \in A \cap B \text{ or } \sigma \notin A \cup B,$$

$$|e_\sigma(A) - e_\sigma(B)| = 1 \quad \text{if } \sigma \in (A \backslash B) \cup (B \backslash A).$$

In each case, from Lemma 2, we have

$$|(1/(2p))e_\sigma(A) - (1/(2p))e_\sigma(B)| \le d(A, B)$$

because $d(A, B) > 1/(2p)$ if $\sigma \in (A \backslash B) \cup (B \backslash A)$. Now we arrive at a contradiction:

Let R be the restriction mapping from $[0, 1]^{\mathbf{N}^N}$ into $[0, 1]^{\mathfrak{F}}$. We obtain a solid compact subset $R(K_k)$ of the cube $[0, 1]^{\mathfrak{F}}$ which is fragmented by $d_{\mathfrak{F}}$, but which is not contained in $c_0(\mathfrak{F})$. Indeed, for every q in P there is some σ_q in Σ' with the first $k+1$ coordinates equal to $(s_1, \ldots, s_k, q)$, and the set $A = \{\sigma_q : q \in P\}$ belongs to K_k and $R(A)$ is not in $c_0(\mathfrak{F})$. ∎

R e m a r k. We shall see that the arguments given for Talagrand's example above are already essentially applicable to the general case, i.e. the impossibility of a Corson compact to be Radon–Nikodym without being already Eberlein.

Let us indicate briefly why: By its very definition Talagrand's space is a subset of $\Sigma(\mathbf{N}^N)$, whence it is obvious that it is a Corson compact. What we have done in the above proof—assuming K is a Radon–Nikodym compact—is the following: We have embedded K (or at least K_k) into the dual of a Banach space F such that K is fragmented by the norm of F^* and such that the coordinate functionals e_σ on K are in F.

In the general case of a Corson compact K we shall proceed similarly: Represent K as a subset of $\Sigma(\Gamma)$ for some set Γ and—assuming K is a Radon–Nikodym compact space—embed it into a dual Banach space F^* with the Radon–Nikodym property. But we have to do this in such a way that the coordinate functionals $(e_\gamma : \gamma \in \Gamma)$ are contained in F. To do so we apply the interpolation method of [DFJP] (Section 3 below), together with an embedding result due to R. Pol and S. Gul'ko [P] (Section 4 below). Next we exploit the fact that F^* has the RNP plus the countability assumption appearing in the very definition of $\Sigma(\Gamma)$ to construct a projectional resolution $(P_\alpha)_{\alpha \in I}$ of the identity in F such that the adjoint maps $(P_\alpha^*)_{\alpha \in I}$ are a projectional resolution of the identity for F^*. This argument is taken from [V] and will be given in Section 4 below.

Finally, we shall see that $(P_\alpha^*)_{\alpha \in I}$ leads to a weak*-to-weak continuous injection from F^* into $c_0(\Gamma)$, thus showing that ball(F^*) (and therefore K) is Eberlein and the Banach space F is weakly compactly generated.

3. Interpolation spaces for weak* compact subsets.

Let E be a Banach space and D an absolutely convex and weak* compact subset of E^*. Denote by $|\cdot|$ the *seminorm on E dual to D*, i.e.

$$|x| = \sup\{\langle x, y\rangle : y \in D\},$$

and by $(G, |\cdot|)$ the Banach space obtained by completing the equivalence classes modulo $|\cdot|$ of E; let j be the canonical map $j: E \to G$. For $1 \leqslant p < \infty$ denote by F_p the [DFJP]-interpolation space

$$F_p = \left\{x \in G : \|x\|_{F_p} = \left(\sum_{n=1}^\infty \|x\|_n^p\right)^{1/p} < \infty\right\}$$

and

$$F_{c_0} = \left\{x \in G : \|x\|_{F_{c_0}} = \max\{\|x\|_n\} < \infty \text{ and } \lim_n \|x\|_n = 0\right\}$$

where $\|\cdot\|_n$ is the norm on G whose unit ball is given by

$$\text{ball}(G, \|\cdot\|_n) = \overline{2^n j(\text{ball}(E, \|\cdot\|)) + 2^{-n}\text{ball}(G, |\cdot|)}^{(G, |\cdot|)}.$$

We shall need the following notion:

DEFINITION. A class $\mathscr{E}$ of compact Hausdorff spaces is called a *perfect class* if $\mathscr{E}$ is stable under taking continuous images, countable products, and closed subspaces.

Typical examples of perfect classes are Eberlein [L], Talagrand [T], Gul'ko [T], Corson compact spaces (see the survey papers of [Ne] and [P] for a detailed account of all these examples), fragmented compact spaces ([N], [R]), the class $\mathfrak{C}$ considered by Stegall [S3], etc. The fact that the continuous image of an Eberlein compact is also Eberlein was proved by Benyamini, Rudin and Wage [BRW], and for Corson compact spaces the result is due to Gul'ko [Gu1], Michael and Rudin [MR]. Let us mention here that it is an open problem whether a continuous image of a Radon–Nikodym compact is also Radon–Nikodym.

Our main objective here is to prove the following:

PROPOSITION 3.1. *Let E be a Banach space and D an absolutely convex and weak* compact subset of E^* that belongs to a perfect class $\mathscr{E}$. Then the unit ball of $(F_p)^*$ as well as the unit ball of $(F_{c_0})^*$ equipped with the weak* topology belong to $\mathscr{E}$.*

P r o o f. Define, for $1 \leqslant p < \infty$,

$$\Sigma_p = \Sigma_p(G, \|\cdot\|_n) = \left\{\bar{x} = (x_n) : x_n \in G \text{ and } \|\bar{x}\| = \left(\sum_{n=1}^\infty \|x_n\|_n^p\right)^{1/p} < \infty\right\}$$

and

$$\Sigma_{c_0} = \Sigma_{c_0}(G, \|\cdot\|_n)$$
$$= \left\{\bar{x} = (x_n) : x_n \in G, \|\bar{x}\| = \sup\{\|x_n\|_n\} < \infty \text{ and } \lim_n \|x_n\| = 0\right\}.$$

By definition F_p (resp. F_{c_0}) is isometrically isomorphic to the diagonal of Σ_p (resp. Σ_{c_0}), i.e. those $(x_n) \in \Sigma_p$ such that $x_1 = x_2 = \ldots$ Hence the unit ball of $(F_p)^*$ (resp. $(F_{c_0})^*$) is a continuous image of the unit ball of $(\Sigma_p)^*$ (resp. $(\Sigma_{c_0})^*$) with the weak* topology, and it therefore will suffice to show that the unit balls of $(\Sigma_p)^*$ and $(\Sigma_{c_0})^*$ equipped with their weak* topologies are in the class $\mathscr{E}$.

Note that the identity on G induces a continuous injection from Σ_1 into Σ_p for $1 \leqslant p < \infty$ and into Σ_{c_0} of norm 1 and with dense range. Hence the unit balls of $(\Sigma_p)^*$ and $(\Sigma_{c_0})^*$ are weak* homeomorphic to subsets of the unit ball of $(\Sigma_1)^*$ and it therefore will suffice to show that the latter ball is in $\mathscr{E}$.

Finally, observe that the dual unit ball of $(G, \|\cdot\|_n)^*$ may naturally be identified with a subset of $2^n D$ and therefore belongs to $\mathscr{E}$. As $(\text{ball}(\Sigma_1)^*, \text{weak}^*)$ is homeomorphic to $\prod_{n=1}^\infty (\text{ball}(G, \|\cdot\|_n)^*, \text{weak}^*)$ we conclude that

(ball(Σ_1)*, weak*) is in $\mathscr{E}$, thus finishing the proof. ∎

COROLLARY 3.2. *Let K be a weak* compact subset of a dual Banach space E^* which is norm fragmented and such that its weak* closed absolutely convex hull D belongs to a perfect class $\mathscr{E}$ of compact spaces. Then for every $1 < p < \infty$ the spaces F_p and F_{co} obtained by the above interpolation are Asplund and their dual unit balls belong to $\mathscr{E}$.*

Proof. The fact that D is norm fragmented if K is so follows from a result of I. Namioka [N, Th. 2.5], whence a theorem of C. Stegall [S2] applies to yield that F_p as well as F_{co} is an Asplund space (compare also [N]). By Proposition 3.1 above the unit balls of $(F_p)^*$ and $(F_{co})^*$ with their weak* topologies are also in $\mathscr{E}$. ∎

4. Banach spaces with weak* Corson compact dual unit balls.

A topological space X belongs to the class Σ if there is a set Γ and a continuous injection of X into $\Sigma(\Gamma)$. Banach spaces with weak* dual in Σ have been introduced and studied by Valdivia [V]. Namely, he proves that any Banach space E with $(E^*, \sigma(E^*, E))$ in Σ admits a projectional resolution of identity, and in the case of E being an Asplund space too, the projectional resolution of identity can be chosen to be shrinking and the Banach space E is weakly compactly generated. In [V], the hypothesis was used that $(E^*, \sigma(E^*, E))$ is in Σ while in the present context the hypothesis that (ball(E^*), σ^*) is in Σ (i.e. is a Corson compact) is natural. Fortunately the two concepts coincide as will be shown in the subsequent proposition, which uses arguments of R. Pol and S. Gul'ko ([P], [Gu2]).

PROPOSITION 4.1. *The weak* dual $(E^*, \sigma(E^*, E))$ of a Banach space E is in the class Σ if, and only if, the dual ball is a weak* Corson compact space. Moreover, in that case there is a set $\{e_\gamma : \gamma \in \Gamma\} \subset E$ such that the mapping T from E^* into $\mathbf{R}^\Gamma$ defined by $T(f) = (\langle e_\gamma, f \rangle)_{\gamma \in \Gamma}$ is injective, weak*-to-pointwise continuous, and such that $T(E^*) \subset \Sigma(\Gamma)$.*

Proof. Let V denote the unit ball of E^* endowed with the weak* topology. Suppose that V is Corson compact. A theorem of R. Pol says that $C_p(V)$ is the continuous image of some closed subset of the product space $L(\tau)^{\mathbf{N}}$, where $L(\tau)$ is a union of a discrete space T of cardinality τ and a "point at infinity" p whose neighbourhoods are of the form $\{p\} \cup (T \backslash C)$, where C is countable [P]. The Banach space E with the weak topology is a closed subspace of $C_p(V)$ by Grothendieck's completeness theorem. Therefore there is a closed subspace F of $L(\tau)^{\mathbf{N}}$ and a continuous map ψ from F onto $(E, \sigma(E, E^*))$. If we apply the results of Gul'ko and Pol at this point [P, Prop. 1.4, p. 23], we find a set Γ and a continuous linear injection S from $C_p(F)$ into $\Sigma(\Gamma)$. If we denote by R the map from $C_p(E, \sigma(E, E^*))$ into $C_p(F)$ defined by $R(f) = f \circ \psi$, the composition $S \circ R$ is injective and continuous from $C_p(E, \sigma(E, E^*))$ into $\Sigma(\Gamma)$. The restriction to the dual E^* gives us a linear, continuous and one-to-one map from

$(E^*, \sigma(E^*, E))$ into $\Sigma(\Gamma)$. The linearity of this map means that it is defined through a set of coordinate functionals $\{e_\gamma : \gamma \in \Gamma\} \subset E$. ∎

For the convenience of the reader we shall include here a proof of the fact that an Asplund space with weak* Corson compact unit dual ball is weakly compactly generated, which is a corollary of the last proposition and the results of [V]. Our proof follows the arguments in [V] (compare also [OV]). Let us also remark here that our Theorem B is more general than this particular result. Indeed, if we apply Theorem 4.2 to Banach spaces of continuous functions on a compact set K we only find that every scattered and Corson compact space is an Eberlein compact ([A], [V]).

THEOREM 4.2. *If an Asplund space E has a Corson compact weak* unit dual ball, then E is weakly compactly generated.*

Proof. Let T be the mapping determined by $\{e_\gamma : \gamma \in \Gamma\}$, defined in the last proposition. We shall consider the multivalued map

$$\varphi : E^* \to 2^E, \qquad \varphi(f) = \{e_\gamma : \langle e_\gamma, f \rangle \neq 0\},$$

which has the following properties:

(φ1) *For every f in E^* the set $\varphi(f)$ is countable.*

In the duality (E, E^*), for a subset C of E (resp. of E^*) we shall denote by $C^\perp$ the orthogonal of C in E^* (resp. in E). With this notation:

(φ2) *For every subset $B \subset E^*$, $\varphi(B)^\perp \cap \overline{B}^{\text{weak}^*} = \{0\}$.*

Indeed, if f is in the weak* closure of B and $f \neq 0$, there is some γ in Γ such that $\langle e_\gamma, f \rangle \neq 0$, hence $\langle e_\gamma, b \rangle \neq 0$ for some b in B and $e_\gamma \in \varphi(B)$, so f does not belong to $\varphi(B)^\perp$.

We shall say that a pair of **Q**-linear subspaces A in E and B in E^* is a *norming pair* when for every x in A and every f in B we have

$$\|x\| = \sup \{|\langle x, g \rangle| : g \in \text{ball}(E^*) \cap B\},$$

$$\|f\| = \sup \{|\langle y, f \rangle| : y \in \text{ball}(E) \cap A\}.$$

Whenever we have a norming pair (A, B) in a Banach space E, the norm closures $\overline{A}$ and $\overline{B}$ are a norming pair too. Observe that if (A, B) is a norming pair, then for every $x \in A$ and $y \in B^\perp$ we have the inequality $\|x\| \leqslant \|x + y\|$, analogously for every $f \in A^\perp$ and $g \in B$, $\|g\| \leqslant \|g + f\|$.

(φ3) *For every norming pair (A, B) in E with $\varphi(B) \subset A$ there is a norm one projection P in E with range $\overline{A}$ and kernel $B^\perp$.*

Indeed, the hypothesis we have together with (φ2) implies that the weak* closure of B meets $A^\perp$ only in zero. So $\overline{A} + B^\perp$ is dense in the Banach space E. The above inequalities give us the projection P.

Following Fabian and Godefroy [FG] we shall now consider the multi-valued map J from E into ball(E^*) defined by

$$J(x) = \{f \in \text{ball}(E^*): \langle x, f \rangle = \|x\|\}.$$

Since E^* has RNP, the selection theorem of Jayne and Rogers [JR, Th. 8] gives a sequence (ψ_n) of norm-to-norm continuous functions from E into E^* which pointwise converges in the norm topology to a mapping ψ_0 that satisfies $\langle x, \psi_0(x) \rangle = \|x\|$ for every $x \in E$. We now define the multivalued map

$$\psi: E \to 2^{E^*}, \qquad \psi(x) = \{\psi_0(x), \psi_1(x), \ldots\},$$

which has the following properties:

(ψ1)   For every x in E the set $\psi(x)$ is countable.

(ψ2)   For every closed subspace V of E the norm closed linear span of the set $\{f|_V: f \in \psi(V)\}$ coincides with V^*.

This property has been used in [FG] in order to obtain a projectional resolution of the identity in the dual of any Asplund space. The proof relies on the weak* version of James' compactness theorem [FG, main theorem].

(ψ3)   For every norming pair (A, B) in E with $\psi(A) \subset B$ there is a norm one projection Q in E^* with range $\bar{B}^{\| \|}$ and kernel $A^{\perp}$.

This follows from (ψ2) since the restriction map from E^* onto $(\bar{A}^{\| \|})^*$ maps B onto $(\bar{A}^{\| \|})^*$.

We have now assembled all the ingredients to construct a shrinking resolution of the identity in the Banach space E:

Step 1. Let λ be a cardinal number and let $A_0 \subset E$ and $B_0 \subset E^*$ be two infinite subsets with $|A_0| \leqslant \lambda$ and $|B_0| \leqslant \lambda$. There is a norming pair (A, B) in E with

$$A_0 \subset A \subset E, \quad B_0 \subset B \subset E^*, \quad |A| \leqslant \lambda, \quad |B| \leqslant \lambda,$$

$$\varphi(B) \subset A, \quad \psi(A) \subset B.$$

We proceed by induction: Let $p \in \mathbf{N}$ and suppose that for every integer m with $0 \leqslant m \leqslant p$, we have subsets

$$A_m \subset E, \quad B_m \subset E^*, \quad |A_m| \leqslant \lambda, \quad |B_m| \leqslant \lambda.$$

We write C_m and D_m for the Q-linear spans of A_m and B_m, respectively. Define

$$B_{p+1} := D_p \cup \{\psi(x): x \in C_p\}, \quad A_{p+1} := C_p \cup \{\varphi(f): f \in D_p\} \cup N_p,$$

where N_p is a subset of cardinality less than or equal to λ which norms every element of D_p. If we take $A = \bigcup\{A_n: n = 0, 1, \ldots\}$ and $B = \bigcup\{B_n: n = 0, 1, \ldots\}$, then (A, B) is a norming pair with the required properties.

Step 2. For a norming pair (A, B) as above there is a norm one projection

P in E with range equal to the norm closure of A and with kernel $B^{\perp}$. Moreover, the adjoint projection P^* has range equal to the norm closure of B in E^*.

That follows from properties (φ3) and (ψ3) above.

We shall say that *the projection P is based on the norming pair (A, B)* to summarize the above situation.

Step 3. If $\{x_\alpha: \alpha < \mu\}$ is a dense subset of E, where μ is the first ordinal number with $|\mu| = \text{dens}(E)$, then there is a projectional resolution of identity $\{P_\alpha: \omega_0 \leqslant \alpha \leqslant \mu\}$ in E where every P_α is based on a norming pair (A_α, B_α) such that

$$|A_\alpha| \leqslant |\alpha|, \quad |B_\alpha| \leqslant |\alpha|, \quad \{x_\beta: \beta < \alpha\} \subset A_\alpha \quad \text{for every } \alpha \in [\omega_0, \mu],$$

$$A_\beta \subset A_\alpha \quad \text{and} \quad B_\beta \subset B_\alpha \quad \text{if } \omega_0 \leqslant \beta \leqslant \alpha \leqslant \mu,$$

$$A_\alpha = \bigcup\{A_\beta: \omega_0 \leqslant \beta < \alpha\} \quad \text{and} \quad B_\alpha = \bigcup\{B_\beta: \omega_0 \leqslant \beta < \alpha\}$$

$$\text{for every limit ordinal } \alpha > \omega_0$$

and the adjoint projections $\{P_\alpha^*: \omega_0 \leqslant \alpha \leqslant \mu\}$ are a projectional resolution of the identity in the dual space E^*.

Let A_0 be equal to $\{x_\alpha: \alpha < \omega_0\}$ and B_0 be equal to $\psi(A_0)$. Step 1 gives us a norming pair $(A_{\omega_0}, B_{\omega_0})$ of countable Q-linear spaces and step 2 a norm one projection P_{ω_0} from E onto the norm closure of A_{ω_0} with adjoint $P_{\omega_0}^*$ from E^* onto the norm closure of B_{ω_0}. We shall proceed by transfinite induction to construct the other projections. Take $\omega_0 < \alpha < \mu$ and assume that for $\omega_0 \leqslant \beta < \alpha$ we have defined the projection P_β based on the norming pair (A_β, B_β), with the corresponding properties. If $\alpha = \gamma + 1$, we take A_0 equal to $A_\gamma \cup \{x_\gamma\}$ and B_0 equal to $\psi(A_\gamma) \cup B_\gamma$. Another application of step 1 produces a norming pair (A_α, B_α) of Q-linear subspaces of cardinality less than or equal to $|\alpha|$, and step 2 gives us a norm one projection P_α from E onto the norm closure of A_α with adjoint P_α^* from E^* onto the norm closure of B_α. If α is a limit ordinal, we take

$$A_\alpha = \bigcup\{A_\beta: \omega_0 \leqslant \beta < \alpha\}, \quad B_\alpha = \bigcup\{B_\beta: \omega_0 \leqslant \beta < \alpha\}.$$

(A_α, B_α) is a norming pair of Q-linear subspaces of cardinality at most $|\alpha|$ with $\varphi(B_\alpha) \subset A_\alpha$ and $\psi(A_\alpha) \subset B_\alpha$. Therefore, step 2 produces a norm one projection P_α from E onto the norm closure of A_α with adjoint P_α^* from E^* onto the norm closure of B_α. Finally, P_μ is the identity operator and we have finished the construction.

This shrinking resolution implies that E is weakly compactly generated as we are going to show in the next steps that follow the well known argument of [AL].

Step 4. There is a set I and a continuous linear injection T from E^* into $c_0(I)$ which is one-to-one and weak*-to-weak continuous.

Again we shall proceed by transfinite induction on the density character of the Banach space E. If E is separable there is nothing to prove. Suppose the result is true for every space with density character strictly less than the density character of E. Let $\{P_\alpha: \omega_0 \leqslant \alpha \leqslant \mu\}$ be the shrinking resolution of identity constructed above. For every α, $P_\alpha(E)$ is an Asplund space with weak* Corson compact unit dual ball and $\mathrm{dens}(P_\alpha(E)) \leqslant |\alpha| < \mathrm{dens}(E) = |\mu|$. By the induction hypothesis there is a weak*-to-weak linear injection T_α from $P_\alpha^*(E^*)$ into $c_0(I_\alpha)$ for some set I_α with $\|T_\alpha\| \leqslant 1$. If I denotes the disjoint union of the family of sets $\{I_{\omega_0}, I_{\alpha+1}: \omega_0 \leqslant \alpha < \mu\}$, we define the operator T from E^* into $l^\infty(I)$ by

$$(Tf)(\gamma) := T_{\omega_0}\big(P_{\omega_0}^*(f)\big)(\gamma) \qquad \text{if } \gamma \in I_{\omega_0},$$

$$(Tf)(\gamma) := \tfrac{1}{2}T_{\alpha+1}\big((P_{\alpha+1}^* - P_\alpha^*)(f)\big)(\gamma) \quad \text{if } \gamma \in I_{\alpha+1}.$$

The fact that $T(E^*) \subset c_0(I)$ follows from the fact that $\{P_\alpha^*: \omega_0 \leqslant \alpha \leqslant \mu\}$ is a resolution of the identity in E^*. Indeed, given $\varepsilon > 0$ and f in E^* the set $\{\alpha: \omega_0 \leqslant \alpha < \mu \text{ and } \|(P_{\alpha+1}^* - P_\alpha^*)(f)\| > \varepsilon\}$ is finite because for every limit ordinal β we have $\|\,\|\text{-}\lim \{P_\alpha^*(f): \alpha < \beta\} = P_\beta^*(f)$.

Step 5. $(\mathrm{ball}(E^*), \text{weak}^*)$ *is affinely homeomorphic to a weakly compact subset of $c_0(I)$ and E is weakly compactly generated.*

Indeed, the above mapping T is weak*-to-weak continuous. The conclusion now follows from a result of Lindenstrauss [L, Th. 3.3]. ∎

5. Radon–Nikodym and Corson compact spaces. Now, we are ready to prove Theorem A stated in the introduction:

Proof of Theorem A. Suppose K is Radon–Nikodym and Corson compact. By definition K is homeomorphic to a weak* compact subset of the dual E^* of a Banach space such that the dual norm fragments K. It is known that every regular Borel probability measure on K has separable support (see for instance [S], or [JNR]), and that for a Corson compact space K with this property the unit ball of $C(K)^*$ is a Corson compact space [AMN, Th. 3.5]. Therefore the weak* closed absolutely convex hull D of K is a Corson compact, since it is the continuous image of the weak* unit ball of $C(K)^*$. We can now apply Corollary 3.2 to the perfect class of Corson compact spaces and find an Asplund space F with weak* Corson compact dual unit ball such that K is homeomorphic to a weak* compact subset of the dual F^*. Theorem 4.2 says that F is weakly compactly generated and thus K is Eberlein compact. ∎

Remark. Let us mention that the above argument can be simplified if we replace the assumption "K is Corson" by the stronger assumption "K is Gul'ko". Indeed, as it is known that for a Gul'ko compact space K, the weak* unit ball of $C(K)^*$ is Gul'ko compact we do not have to refer to the results of [AMN] in this case. The interpolation space F is then weakly countably determined (apply Corollary 3.2 to the perfect class of Gul'ko compact spaces

and note that F is weakly countably determined iff $(\mathrm{ball}(F^*), \sigma^*)$ is Gul'ko [T]). Hence we do not need the full strength of 4.2 above and can instead apply a result of M. Fabian [F] to conclude as above that F is weakly compactly generated.

We now turn to the proof of Theorem B stated in the introduction.

Proof of Theorem B. Suppose that E is a Banach space such that $(\mathrm{ball}(E^*), \sigma^*)$ is Corson compact and such that there is a continuous linear map from an Asplund space X into E with dense range. By Theorem A, $(\mathrm{ball}(E^*), \sigma^*)$ is Eberlein compact. Note, however, that this only would allow us to conclude that E is a subspace of a weakly compactly generated Banach space. But an inspection of the proof of Theorem A shows that $(\mathrm{ball}(E^*), \sigma^*)$ is in fact affinely homeomorphic to a weakly compact subset of $c_0(\Gamma)$. Hence by a theorem of Lindenstrauss [L, Th. 3.3] we may conclude that E is weakly compactly generated, thus completing the proof of Theorem B. ∎

Remark. It seems to have been unknown so far whether weakly K-analytic or —more generally— weakly countably determined Banach spaces necessarily are GSG. Of course, Theorem B above shows that this is not the case: For any Corson compact space K which fails to be Eberlein compact (e.g. Talagrand's space) the space $C(K)$ fails to be GSG.

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