

Piecewise convex transformations with no finite invariant measure

by TOMASZ KOMOROWSKI (Lublin)

Abstract. The paper concerns the problem of the existence of a finite invariant absolutely continuous measure for piecewise C^2 -regular and convex transformations $T: [0, 1] \rightarrow [0, 1]$. We show that in the case when $T'(0) = 1$ and $T''(0)$ exists T does not admit such a measure. This result is complementary to the ones contained in [3] and [5].

1. Introduction. Consider a semidynamical system described by the difference equation $x_{n+1} = T(x_n)$, where $T: [0, 1] \rightarrow [0, 1]$ is a transformation. We are interested in whether or not there is an absolutely continuous finite T -invariant measure. The case when T is piecewise smooth and $|T'| > 1$ was thoroughly investigated by several authors; see for instance [4], pp. 119–128, where it is shown that under the above assumptions T has a finite invariant measure. Moreover, the system is exact with respect to such a measure provided that each smooth piece of T is onto. The case when $|T'|$ may assume the value 1 on a certain finite set of points where $T(x) = x$ was studied among others by F. Schweiger and M. Thaler (see [6]–[9]). They proved some interesting properties of such transformations, e.g. the existence of absolutely continuous σ -finite invariant measures and exactness with respect to them. However, the essential assumption they made in all papers was that there is a partition of $[0, 1]$ into a finite number of intervals I_1, \dots, I_n for which $\overline{T(I_k)} = [0, 1]$. Here we do not require that T be onto. Instead, we assume that T is piecewise convex. This case is not covered by the results of F. Schweiger and M. Thaler. The transformations considered here need not even satisfy the Markov property. A map $T: [0, 1] \rightarrow [0, 1]$ is *Markov* if one can find a finite or countable collection $\{I_k\}_{k \geq 1}$ of disjoint open intervals such that

- (a) T is defined in $\bigcup_{k \geq 1} I_k$ and $I \setminus \bigcup_{k \geq 1} I_k$ has measure zero,
- (b) $T|_{I_k}$ is strictly monotonic and extends to a C^2 function on I_k for each k ,
- (c) if $T(I_k) \cap I_j \neq \emptyset$ then $T(I_k) \supseteq I_j$, and
- (d) there is an R such that $\bigcup_{k=1}^R T^n(I_k) \supseteq I_j$ for every k and j .

Transformations having the Markov property were considered by R. Bowen in [1].

The piecewise convex transformations were already treated by A. Lasota and J. Yorke in [5]. They showed that if $T'(0) > 1$ then T has a unique invariant measure. Moreover, the system is exact with respect to this measure (see [4]). The case when $T'(0) = 1$ was studied by P. Kacprowski in [3]. He showed that if $T''(0)$ does not exist then T possesses such a measure provided that T' satisfies a certain special integral condition (for more details see [3]). Our main result is a natural complement of those two. We prove that if $T''(0)$ exists and $T'(0) = 1$ then T has no finite invariant measure. Our method is essentially different from the method of the jump transformation used by F. Schweiger and M. Thaler. The idea of the proof is the following: for any density f we choose a sequence of nonnegative functions $\{g_n\}_{n \geq 1}$ such that

$$(1.1) \quad P^n f \geq g_n, \quad n \geq 1,$$

where P is the Frobenius–Perron operator for T . The functions $\{g_n\}_{n \geq 1}$ may be of a very special form when T admits a finite invariant measure. Assuming that T has such a measure and $T'(0) = 1$ and $T''(0)$ exists we find a density f and $\{g_n\}_{n \geq 1}$ satisfying (1.1) for which $\lim_{N \rightarrow +\infty} \int_0^1 g_n(x) dx = +\infty$, which is impossible since P maps the set of densities into itself. As a corollary we deduce that for any density f and $\varepsilon > 0$

$$\lim_{N \rightarrow +\infty} N^{-1} \sum_{n=0}^N \int_{\varepsilon}^1 P^n f dx = 0.$$

2. Notations. We denote by \mathcal{B} the σ -algebra of Borel sets contained in $[0, 1]$ and by m the Lebesgue measure defined on \mathcal{B} . $L^1[0, 1]$ is the space of all \mathcal{B} -measurable functions integrable with respect to m on $[0, 1]$ and $D \subseteq L^1[0, 1]$ is the set of all functions satisfying

$$f \geq 0, \quad \int_0^1 f(t) dt = 1.$$

Functions from D are called *densities*. A transformation $T: [0, 1] \rightarrow [0, 1]$ is said to be *measurable* and *nonsingular* if for every $A \in \mathcal{B}$, $T^{-1}(A) \in \mathcal{B}$ and

$$m[A] = 0 \quad \text{implies} \quad m[T^{-1}(A)] = 0.$$

For any measurable and nonsingular transformation T we define the *Frobenius–Perron operator* $P_T: L^1[0, 1] \rightarrow L^1[0, 1]$ by

$$\int_A P_T f dm = \int_{T^{-1}(A)} f dm, \quad A \in \mathcal{B}, f \in L^1[0, 1].$$

Hence

$$\int_0^1 P_T f dm = \int_0^1 f dm, \quad f \in L^1[0, 1],$$

$$P_T f \geq 0 \quad \text{when} \quad f \geq 0.$$

It is well known (see [4]) that

$$(P_T f)^+ \leq P_T f^+, \quad f \in L^1[0, 1].$$

A measure μ defined on \mathcal{B} is said to be *T-invariant* if:

$$\begin{aligned} &\mu \text{ is absolutely continuous with respect to } m, \\ &0 < \mu[[0, 1]] < +\infty, \quad \mu[A] = \mu[T^{-1}(A)], \quad \text{for all } A \in \mathcal{B}. \end{aligned}$$

It is easy to observe (see [4]) that T admits an invariant measure if and only if there exists $f_* \in D$ such that $P_T f_* = f_*$.

Finally, to avoid misunderstandings we recall the definition of a convex function. Let $I \subseteq \mathbb{R}$ be an interval. We say that a function $S: I \rightarrow \mathbb{R}$ is *convex* if

$$S(\alpha x + (1 - \alpha)y) \leq \alpha S(x) + (1 - \alpha)S(y),$$

for all $x, y \in I$ and $\alpha \in [0, 1]$.

3. Auxiliary results. Let $T: [0, 1] \rightarrow [0, 1]$ be a transformation for which

- (i) there is a partition $0 = a_0 < a_1 < \dots < a_N = 1$ such that $T|_{[a_{k-1}, a_k]}$ is C^1 and convex for $k = 1, \dots, N$,
- (ii) $T(a_{k-1}) = 0$, $T'(a_{k-1}) > 0$, $k = 1, \dots, N$,
- (iii) $T'(0) = 1$ and $T'(x) > 1$ for $x \in (0, a_1)$,
- (iv) $T''(0)$ exists.

Denote by \mathcal{T} the class of transformations satisfying (i)–(iv). It is easy to observe that $T^n \in \mathcal{T}$ for all $n \in \mathbb{N}$ provided that $T \in \mathcal{T}$. For any $T \in \mathcal{T}$ we denote by $0 = b_0^n < b_1^n < \dots < b_{s_n}^n = 1$ the partition corresponding to T^n . We have

$$\{b_0^n, b_1^n, \dots, b_{s_n}^n\} = T^{-n}(\{a_0, a_1, \dots, a_{N-1}\}).$$

From (ii) we see that

$$\{a_0, a_1, \dots, a_{N-1}\} \subseteq T^{-1}(\{a_0, a_1, \dots, a_{N-1}\})$$

and in consequence

$$(3.1) \quad \begin{aligned} \{b_0^n, b_1^n, \dots, b_{s_n}^n\} &\subseteq \{b_0^{n+1}, b_1^{n+1}, \dots, b_{s_{n+1}}^{n+1}\} \\ &= T^{-1}(\{b_0^n, b_1^n, \dots, b_{s_n}^n\}). \end{aligned}$$

Let

$$\psi_k^n(x) = \begin{cases} (T_k^n)^{-1}(x), & x \in T^n([b_{k-1}^n, b_k^n]), \\ b_k^n, & x \in [0, 1] \setminus T^n([b_{k-1}^n, b_k^n]) \end{cases}$$

where $T_k^n = T^n|_{[b_{k-1}^n, b_k^n]}$, $k = 1, \dots, s_n$. From the definition it is easy to observe that ψ_k^n , $k = 1, \dots, s_n$, are increasing and $(\psi_k^n)'$, $k = 1, \dots, s_n$, understood as right derivatives defined in $[0, 1]$ are decreasing. Moreover (i)–(iii) imply

$$(3.2) \quad \psi_1^n(0) = 0,$$

$$(3.3) \quad (\psi_1^n)'(0) = 1 \quad \text{and} \quad (\psi_1^n)'(x) < 1, \quad x \in (0, 1].$$

Since $T_1^n = T^n|_{[0, b_1^n]}$, for $x \in [0, b_1^n]$ we have

$$\psi_1^n(x) = (\psi_1^1)^n(x), \quad \text{for } x \in T^n([0, b_1^1]).$$

From (iii) we see that $b_1^n \in (0, a_1)$, and $T(b_1^n) > b_1^n$ for $n \geq 1$. Hence (3.3) implies that

$$(3.4) \quad b_1^{n+1} = \psi_1(b_1^n) < b_1^n, \quad n \in \mathbb{N}.$$

Since, by (3.1), $b_1^n \in \{b_0^{n+1}, b_1^{n+1}, \dots, b_{s_n}^{n+1}\}$, (3.4) gives us

$$b_2^{n+1} \leq b_1^n, \quad n \in \mathbb{N}.$$

From (3.2) and (3.3) we obtain

$$\lim_{k \rightarrow +\infty} (\psi_1^n)^k(t) = 0 \quad \text{for any } t \in (0, 1] \text{ and } n \in \mathbb{N},$$

so

$$\lim_{n \rightarrow +\infty} b_1^n = \lim_{n \rightarrow +\infty} b_2^n = 0.$$

Denote by P the Frobenius–Perron operator for T . Then the Frobenius–Perron operator for T^n is

$$P^n f(x) = \sum_{p=1}^{s_n} f(\psi_p^n(x)) (\psi_p^n)'(x)$$

and maps the class of decreasing densities defined on $[0, 1]$ into itself (for more details see [4]).

Let $Q = P^k$, and let g be any decreasing density. From (3.2) we have $\varrho > 0$ such that ψ_1^k and ψ_2^k are C^1 -regular on $[0, \varrho]$. Hence for any $x \in [0, \varrho]$

$$\begin{aligned} Qg(x) &\geq g(\psi_1^k(x))(\psi_1^k)'(x) + g(\psi_2^k(x))(\psi_2^k)'(x) \\ &\geq g(\psi_1^k(x))(\psi_1^k)'(x) + g(\psi_2^k(x))(\psi_2^k)'(\varrho). \end{aligned}$$

Setting $L = (\psi_2^k)'(\varrho)$ we obtain

$$(3.5) \quad Qg(x) \geq g(\psi_1^k(x))(\psi_1^k)'(x) + Lg(\psi_2^k(x)), \quad x \in [0, \varrho].$$

Now suppose that f is a decreasing density. Then all the functions $f_n = Q^n f$ are decreasing and nonnegative. Applying (3.5) to f_{n+1} and f_n we get for any $x \in [0, \varrho]$

$$(3.6) \quad f_{n+1}(x) \geq f_n(\psi_1^k(x))(\psi_1^k)'(x) + Lf_n(\psi_2^k(x)),$$

$$(3.6a) \quad f_n(x) \geq f_{n-1}(\psi_1^k(x))(\psi_1^k)'(x) + Lf_{n-1}(\psi_2^k(x)).$$

From (3.2) and (3.3) it follows that $\psi_1^k([0, \varrho]) \subseteq [0, \varrho]$. Since for any $x \in [0, \varrho]$ we have $\psi_1^k(x) \in [0, \varrho]$ from (3.6a) we see that

$$f_n(\psi_1^k(x)) \geq f_{n-1}((\psi_1^k)^2(x))(\psi_1^k)'(\psi_1^k(x)) + Lf_{n-1}(\psi_2^k(\psi_1^k(x))).$$

Replacing $f_n(\psi_1^k(x))$ in (3.6) by the foregoing expression we get

$$\begin{aligned} f_{n+1}(x) &\geq [f_{n-1}((\psi_1^k)^2(x))(\psi_1^k)'(\psi_1^k(x)) + Lf_{n-1}(\psi_2^k(\psi_1^k(x)))](\psi_1^k)'(x) + Lf_n(\psi_2^k(x)) \\ &= f_{n-1}((\psi_1^k)^2(x))((\psi_1^k)^2)'(x) + Lf_{n-1}(\psi_2^k(\psi_1^k(x)))(\psi_1^k)'(x) + Lf_n(\psi_2^k(x)). \end{aligned}$$

We may apply again (3.5) to $f_{n-1}(x)$ for $x \in [0, \varrho]$ with $(\psi_1^k)^2(x)$ instead of x . Repeating the above procedure n times we obtain finally

$$\begin{aligned} f_{n+1}(x) &\geq f((\psi_1^k)^{n+1}(x))((\psi_1^k)^{n+1})'(x) \\ &\quad + \sum_{p=0}^n Lf_p(\psi_2^k((\psi_1^k)^{n-p}(x))((\psi_1^k)^{n-p})'(x) \quad \text{for } x \in [0, \varrho]. \end{aligned}$$

Since we have assumed that f is a decreasing density all the functions f_p , $1 \leq p \leq n$, are decreasing. As $\psi_2^k(x) \leq b_2^k$ for $x \in [0, 1]$, we see that

$$(3.7) \quad f_{n+1}(x) \geq \sum_{p=0}^n Lf_p(b_2^k)((\psi_1^k)^{n-p})'(x), \quad x \in [0, \varrho].$$

In the sequel we will need the following lemma:

LEMMA 3.1. Assume that $T \in \mathcal{F}$ and P is the Frobenius–Perron operator for T . Suppose that there exists a density f_* invariant under P . Then for any continuous and decreasing $f \in D$ satisfying $f(1) > 0$ there are $\sigma, \kappa > 0$ for which

$$P^n f(\sigma) \geq \kappa, \quad n \in \mathbb{N}.$$

Proof. It suffices to prove that there is $\sigma > 0$ for which

$$\liminf_{n \rightarrow +\infty} P^n f(\sigma) > 0.$$

Suppose not. Then for an arbitrary $\sigma > 0$ we have a sequence $n_1 < n_2 < \dots$ of natural numbers for which $\lim_{k \rightarrow +\infty} P^{n_k} f(\sigma) = 0$. Thus

$$P^{n_k} f \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

uniformly on $[\sigma, 1]$. For any $\varepsilon > 0$ there exists $M > 0$ satisfying

$$\int_0^1 (f_* - Mf)^+ dx < \varepsilon.$$

Now,

$$\begin{aligned} \int_{\sigma}^1 f_* dx &\leq \int_{\sigma}^1 (P^{n_k} f_* - M P^{n_k} f)^+ dx + M \int_{\sigma}^1 P^{n_k} f dx \\ &\leq \int_{\sigma}^1 P^{n_k} (f_* - Mf)^+ dx + M \int_{\sigma}^1 P^{n_k} f dx \\ &\leq \varepsilon + M \int_{\sigma}^1 P^{n_k} f dx. \end{aligned}$$

As $k \rightarrow +\infty$ we obtain

$$\int_{\sigma}^1 f_{\star} dx \leq \varrho \quad \text{for any pair } \varrho, \sigma > 0.$$

This proves that $\int_{\sigma}^1 f_{\star} dx = 0$, $\sigma > 0$, which is impossible since $f_{\star} \in D$. This contradiction ends the proof.

In the case when P has an invariant density we can transform (3.7). From Lemma 3.1 we find a decreasing density f and $\sigma > 0$ for which $P^n f(\sigma) \geq \kappa$, $n \in \mathbb{N}$. Let k be so large that $b_2^k < \sigma$. Setting $Q = P^k$, $f_n = Q^n f$ and applying (3.7) we obtain

$$(3.8) \quad f_{n+1}(x) \geq A \sum_{p=0}^n ((\psi_1^k)^p)'(x) \quad \text{for } x \in [0, \varrho]$$

where $A = L\kappa > 0$.

The following lemma will be useful in the sequel.

LEMMA 3.2. Assume that $\psi: [0, 1] \rightarrow [0, 1]$ satisfies the following conditions:

- (i) ψ is C^1 -regular on $[0, \varrho]$ for some $\varrho \in (0, 1]$.
- (ii) $\psi(0) = 0$ and $\psi(x) = \psi(\varrho)$, $x \geq \varrho$.
- (iii) $0 < \psi'(x) < 1$ for $x \in (0, \varrho]$.
- (iv) $\psi'(0) = 1$ and $\psi''(0)$ exists.

Then for every $t \in (0, 1]$

$$(3.9) \quad \lim_{n \rightarrow +\infty} n\psi^n(t) = \frac{2}{|\psi''(0)|} \quad \text{if } |\psi''(0)| > 0,$$

$$(3.10) \quad \lim_{n \rightarrow +\infty} n\psi^n(t) = +\infty \quad \text{if } \psi''(0) = 0.$$

Proof. First we assume that $\psi''(0) \neq 0$. From (iii) and (iv) we see that $\psi''(0) < 0$. Let $t \in (0, 1]$ be fixed. Set $a_n = n\psi^n(t)$ for $n \in \mathbb{N}$. From the Taylor formula for ψ at 0 we obtain

$$\psi(x) = x - \frac{x^2}{2} |\psi''(0)| (1 + o(x)),$$

where $\lim_{x \rightarrow 0^+} o(x) = 0$. Thus

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \left(1 + \frac{1}{n}\right) \frac{\psi^n(t) - \frac{1}{2} [\psi^n(t)]^2 |\psi''(0)| \varepsilon_n}{\psi^n(t)} \\ &= \left(1 + \frac{1}{n}\right) \left[1 - \frac{a_n \varepsilon_n}{nG}\right] = 1 + b_n, \end{aligned}$$

where

$$\varepsilon_n = 1 + \sigma(\psi^n(t)) \rightarrow 1, \quad n \rightarrow +\infty,$$

$$(3.11) \quad b_n = \frac{1}{n} \left[1 - \frac{a_n \varepsilon_n}{G} \right] - \frac{a_n \varepsilon_n}{G n^2}, \quad n \in \mathbf{N}, \quad G = \frac{2}{|\psi''(0)|}.$$

Since $\lim_{n \rightarrow +\infty} \psi^n(t) = 0$ we see that

$$\lim_{n \rightarrow +\infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow +\infty} \left(1 + \frac{1}{n} \right) \frac{\psi(\psi^n(t))}{\psi^n(t)} = \psi'(0) = 1.$$

Let $\varepsilon > 0$. Take N_0 so large that

$$(3.12) \quad G + \varepsilon/2 > G/\varepsilon_n > G - \varepsilon/2 \quad \text{for } n \geq N_0,$$

$$(3.13) \quad \frac{G + \varepsilon}{G - \varepsilon} \geq \frac{a_{n+1}}{a_n} \geq \frac{G - \varepsilon}{G + \varepsilon} \quad \text{for } n \geq N_0.$$

Let $N_1 = \max\{N_0, 2G/\varepsilon\}$.

CLAIM. There is $N_2 \geq N_1$ such that

$$(3.14) \quad \frac{(G + \varepsilon)^2}{G - \varepsilon} \geq a_n \geq \frac{(G - \varepsilon)^2}{G + \varepsilon} \quad \text{for } n \geq N_2.$$

To prove (3.14) first observe that neither

$$(3.15) \quad a_n < \frac{(G - \varepsilon)^2}{G + \varepsilon} \quad \text{for all } n \geq N_1$$

nor

$$(3.16) \quad \frac{(G + \varepsilon)^2}{G - \varepsilon} < a_n \quad \text{for all } n \geq N_1$$

can be true. Indeed, if (3.15) took place then from (3.11) and (3.12) we would obtain

$$b_n \geq \frac{1}{n} \left[1 - \frac{(G - \varepsilon)^2}{(G - \varepsilon/2)(G + \varepsilon)} \right] - \frac{(G - \varepsilon)^2}{(G - \varepsilon/2)(G + \varepsilon)n^2}.$$

So $\sum_{n=1}^{+\infty} b_n = +\infty$. Hence

$$\frac{a_{N+1}}{a_1} = \prod_{n=1}^N \frac{a_{n+1}}{a_n} = \prod_{n=1}^N (1 + b_n) \rightarrow +\infty, \quad N \rightarrow +\infty,$$

and this would contradict (3.15).

If (3.16) occurred then from (3.11), (3.12) we would get

$$b_n \leq \frac{1}{n} \left[1 - \frac{(G + \varepsilon)^2}{(G + \varepsilon/2)(G - \varepsilon)} \right]$$

and $\sum_{n=1}^{+\infty} b_n = -\infty$. Hence

$$\frac{a_{N+1}}{a_1} = \prod_{n=1}^N \frac{a_{n+1}}{a_n} = \prod_{n=1}^N (1 + b_n) \rightarrow 0, \quad N \rightarrow +\infty.$$

This would contradict (3.16). Thus we have proved that there are $n \geq N_1$ such that (3.14) holds. It now suffices to show that if (3.14) holds for some $n \geq N_1$ then it does for $n+1$. We fix such an n and consider several cases.

If $a_n < G + \varepsilon$ then

$$a_{n+1} \leq a_n \frac{G + \varepsilon}{G - \varepsilon} < \frac{(G + \varepsilon)^2}{G - \varepsilon}.$$

Let now

$$\frac{(G + \varepsilon)^2}{G - \varepsilon} \geq a_n \geq G + \varepsilon.$$

We have

$$\frac{a_{n+1}}{a_n} \leq 1 + \frac{1}{n} \left[1 - \frac{\varepsilon_n a_n}{G} \right] \leq 1 + \frac{1}{n} \left[1 - \frac{G + \varepsilon}{G + \varepsilon/2} \right] < 1$$

and

$$a_{n+1} < a_n \leq \frac{(G + \varepsilon)^2}{G - \varepsilon}.$$

Thus we have proved that if $a_n \leq (G + \varepsilon)^2 / (G - \varepsilon)$ then $a_{n+1} \leq (G + \varepsilon)^2 / (G - \varepsilon)$.

Assume now that

$$\frac{(G - \varepsilon)^2}{G + \varepsilon} < a_n < G - \varepsilon.$$

Then

$$\frac{a_{n+1}}{a_n} \geq 1 + \frac{1}{n} \left[1 - \frac{\varepsilon_n a_n}{G} - \frac{\varepsilon_n a_n}{Gn} \right] \geq 1 + \frac{1}{n} \left[1 - \frac{G - \varepsilon}{G - \varepsilon/2} - \frac{G - \varepsilon}{n(G - \varepsilon/2)} \right] \geq 1$$

and so

$$a_{n+1} \geq a_n \geq \frac{(G - \varepsilon)^2}{G + \varepsilon}.$$

If $a_n \geq G - \varepsilon$ then from (3.13)

$$a_{n+1} \geq a_n \frac{G - \varepsilon}{G + \varepsilon} \geq \frac{(G - \varepsilon)^2}{G + \varepsilon}.$$

Thus if $a_n \geq (G - \varepsilon)^2 / (G + \varepsilon)$ then

$$a_{n+1} \geq \frac{(G - \varepsilon)^2}{G + \varepsilon}.$$

Hence

$$\frac{(G + \varepsilon)^2}{G - \varepsilon} \geq \limsup_{n \rightarrow +\infty} a_n \geq \liminf_{n \rightarrow +\infty} a_n \geq \frac{(G - \varepsilon)^2}{G + \varepsilon}.$$

Since $\varepsilon > 0$ was arbitrarily chosen we see that

$$\lim_{n \rightarrow +\infty} a_n = G.$$

Observe that (3.10) follows immediately from (3.9). Indeed, when ψ satisfies (i)–(iv) and $\psi''(0) = 0$ then for any $\varepsilon > 0$ we can find ψ_ε satisfying (i)–(iv) such that

$$(3.17) \quad \psi(t) \geq \psi_\varepsilon(t), \quad t \in [0, 1], \quad \psi''(0) = -\varepsilon.$$

Hence from (3.9), (3.17) and (iii)

$$\liminf_{n \rightarrow +\infty} n\psi^n(t) \geq \liminf_{n \rightarrow +\infty} n\psi_\varepsilon^n(t) = 2/\varepsilon.$$

Since $\varepsilon > 0$ was arbitrarily chosen we get finally (3.10).

4. The main theorem

THEOREM 4.1. *Assume that $T \in \mathcal{F}$ and $T'(0) = 1$. Then there is no T -invariant measure.*

Proof. Suppose that T admits an invariant measure. Then there are a density f and a positive number A for which (3.8) holds. Thus we obtain

$$1 = \int_0^1 f_{n+1}(x) dx \geq A \sum_{p=0}^n (\psi_1^k)^p(\varrho), \quad n \in \mathbb{N},$$

so

$$(4.1) \quad 1/A \geq \sum_{p=0}^{+\infty} (\psi_1^k)^p(\varrho).$$

However, ψ_1^k satisfies the assumptions of Lemma 3.2 and from this we see that (4.1) does not hold. This contradiction ends the proof.

Remark 4.2. From the nonexistence of a finite invariant measure it follows that there exists an increasing sequence of sets $A_1 \subseteq A_2 \subseteq \dots$ for which

$$(4.2) \quad [0, 1] = \bigcup_{k \geq 1} A_k,$$

$$(4.3) \quad \lim_{N \rightarrow +\infty} N^{-1} \sum_{n=1}^N \int_{A_k} P^n f dx = 0 \quad \text{for any } k \in \mathbb{N} \text{ and } f \in D$$

(see [2]). Since $g_N = N^{-1} \sum_{n=1}^N P^n \mathbf{1}$ is a decreasing density, it is easy to calculate that for every $\varepsilon > 0$

$$g_N(x) \leq 1/\varepsilon \quad \text{if } x \in [\varepsilon, 1] \text{ (see [3]).}$$

From (4.2), (4.3) we have

$$(4.4) \quad \lim_{N \rightarrow +\infty} N^{-1} \sum_{n=1}^N \int_{\varepsilon}^1 P^n 1 dx = 0.$$

By standard computations we deduce from (4.4) that for every $f \in D$

$$\lim_{N \rightarrow +\infty} N^{-1} \sum_{n=1}^N \int_{\varepsilon}^1 P^n f dx = 0.$$

References

- [1] R. Bowen, *Invariant measures for Markov maps of the interval*, Comm. Math. Phys. 69 (1979), 1–17.
- [2] S. R. Foguel, *The Ergodic Theory of Markov Processes*, van Nostrand Reinhold, New York 1969.
- [3] P. Kacprowski, *On the existence of invariant measures for piecewise smooth transformations*, Ann. Polon. Math. 40 (1983), 179–184.
- [4] A. Lasota and M. C. Mackey, *Probabilistic Properties of Deterministic Systems*, Cambridge University Press, 1985.
- [5] A. Lasota and J. A. Yorke, *Exact dynamical systems and the Frobenius-Perron operators*, Trans. Amer. Math. Soc. 273 (1982), 375–384.
- [6] F. Schweiger, *Numbertheoretical endomorphisms with σ -finite invariant measure*, Israel J. Math. 21 (1975), 308–318.
- [7] —, *Some remarks on ergodicity and invariant measures*, Michigan Math. J. 22 (1975), 308–318.
- [8] M. Thaler, *Estimates of the invariant densities of endomorphisms with indifferent fixed points*, Israel J. Math. 37 (1980), 303–314.
- [9] —, *Transformations on $[0, 1]$ with infinite invariant measure*, *ibid.* 46 (1983), 67–96.

INSTITUTE OF MATHEMATICS, M. CURIE-SKŁODOWSKA UNIVERSITY
Pl. M. Curie-Skłodowskiej 1, 20-031 Lublin, Poland

Reçu par la Rédaction le 04.12.1989
Révisé le 25.03.1990 et le 28.06.1990