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Radial segments and conformal mapping of an annulus onto domains bounded by a circle and a k-circle

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Abstract. Let $f(z)$ be a conformal mapping of an annulus $A(R) = \{1 < |z| < R\}$ and let $f(A(R))$ be a ring domain bounded by a circle and a k-circle. If $R(\varphi)=\{w : \arg w = \varphi\}$, and $\ell(\varphi) - 1$ is the linear measure of $f(A(R)) \cap R(\varphi)$, then we determine the sharp lower bound of $\ell(\varphi_1) + \ell(\varphi_2)$ for fixed φ_1 and φ_2 $(0 \le \varphi_1 \le \varphi_2 \le 2\pi)$.

1. Introduction. We denote the *chordal distance* between the points w_1 and w_2 in the extended complex w-plane $\overline{\mathbb{C}}$ by $q(w_1, w_2)$, that is,

$$q(w_1, w_2) = |w_1 - w_2|/\sqrt{(1 + |w_1|^2)(1 + |w_2|^2)}$$

if w_1 and w_2 are both finite, and

$$q(w_1, \infty) = 1/\sqrt{1 + |w_1|^2}.$$

We define the *chordal cross ratio* of quadruples w_1, w_2, w_3, w_4 in $\overline{\mathbb{C}}$ by

$$X(w_1, w_2, w_3, w_4) = \frac{q(w_1, w_2)q(w_3, w_4)}{q(w_1, w_3)q(w_2, w_4)}. \tag{1.1}$$

A Jordan curve Γ in $\overline{\mathbb{C}}$ is called a *k-circle*, where $0 < k \le 1$, if for all ordered quadruples of points on Γ,

$$X(w_1, w_2, w_3, w_4) + X(w_2, w_3, w_4, w_1) \le 1/k\,. \tag{1.2}$$

This definition of a k-circle was introduced by Blevins [2]. It is well known that a k-circle is a quasicircle (see [1]). One of the simplest k-circles is $\{w : |\arg w| = \arcsin k\}$. Throughout the note the value of arcsin and arccos is restricted between 0 and $\pi/2$.

In this note we consider the class $C(k)$ of conformal mappings $w = f(z)$ of an annulus $A(R) = \{1 < |w| < R\}$ whose images $D_f = f(A(R))$

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are ring domains with inner boundary $f(|z|=1)=\{|w|=1\}$ and outer boundary Γ a k-circle. Let $R(\theta)=\{w : \arg\ w=\theta\}$ and let $\ell(\theta)-1$ be the linear measure of $R(\theta)\cap f(A(R))$. Let $D(k,d_0)$ be the ring domain with $\operatorname{Mod} D(k,d_0)=\log R$ and with inner boundary $\{|w|=1\}$ and outer boundary $\{w : |\arg(w+d_0)|=\pi-\arcsin k\}$. Let $f_0(z)$ be a function mapping $A(R)$ onto $e^{i\beta}D(k,d_0)$ and set

$$T(w)=\frac{w_1}{\overline{w}_1}\cdot\frac{1+\overline{w}_1 w}{w+w_1},$$

where

$$\beta=\arcsin(\sin\theta/(d_0(d_1+\sqrt{d_1^2-1}))),$$
$$w_1=(d_1+\sqrt{d_1^2-1})e^{i\theta}, \qquad d_1=\sqrt{d_0^2\cos^2\theta+\sin^2\theta}.$$

We show the following theorem dealing with radial segments.

THEOREM. *Under the above assumptions, we have the inequalities*

$$\ell(\theta)+\ell(\pi-\theta)\ge 2(d_1+\sqrt{d_1^2-1}) \tag{1.3}$$

for $0\le\theta\le\arccos(\sqrt{d_0^2-1}/(2d_0))$, *while*

$$\ell(\theta)+\ell(\pi-\theta)\ge 2d_0 \tag{1.4}$$

for $\arccos(\sqrt{d_0^2-1}/(2d_0))<\theta\le\pi/2$.

For $0\le\theta\le\theta_0$, *equality is attained only for the function* $F(z)=T(f_0(z))$ *up to a rotation around the origin, where* θ_0 *is a positive constant depending only on* k, *and determined in the proof of the theorem.*

We remark that this theorem can be reformulated as an estimate for $\ell(\varphi_1)+\ell(\varphi_2)$ $(0\le\varphi_1\le\varphi_2\le 2\pi)$. For example, (1.3) is equivalent to

$$\ell(\varphi_1)+\ell(\varphi_2)\ge 2(d_2+\sqrt{d_2^2-1}) \tag{1.5}$$

with $d_2=\sqrt{(1+d_0^2+(1-d_0^2)\cos(\varphi_2-\varphi_1))/2}$. Let $w=f(z)$ be a conformal mapping of an annulus $A(R)$ (with Γ not necessarily a k-circle). Mityuk [8] obtained the lower bound of $\ell(\theta)+\ell(\pi+\theta)$ $(0\le\theta\le\pi)$. Our theorem yields his result by considering the special case of $\varphi_2-\varphi_1=\pi$ and letting $k\to 0$.

2. Fundamental lemma. In this section we will verify the following fundamental lemma on the Koebe region for the class $C(k)$.

FUNDAMENTAL LEMMA. *Let* $w=f(z)$ *be a function in* $C(k)$. *Then the distance* $d(\Gamma,0)$ *between the origin and* Γ *satisfies the inequality*

$$d(\Gamma,0)\ge d_0. \tag{2.1}$$

Equality holds in (2.1) *if and only if* D_f *is* $D(k,d_0)$ *up to a rotation around the origin.*

This lemma can be restated as follows: The Koebe region for the class $C(k)$ is generated by functions f arising from f_0 by rotations around the origin.

P r o o f o f t h e f u n d a m e n t a l l e m m a. First we verify this lemma under the condition that $\Gamma = f(|z| = R)$ contains the point at infinity.

Let w' be a point on Γ such that $|w'| = d(\Gamma, 0)$ $(=a)$. We consider the circular symmetrization D_f^* of D_f with respect to the positive real axis.

The following statement is due to Blevins [2]: If Γ contains the point at infinity and a point w' with $|w'| = a$, then the circular symmetrization D_f^* of D_f with respect to the positive real axis is contained in the domain $D(k,a) = \{w : |\arg(w+a)| < \pi - \arcsin k\} \cap \{|w| > 1\}$.

Using this and a well known Jenkins result on circular symmetrization [6] together with the monotonicity property of the module, we obtain the inequalities

$$\operatorname{Mod} D_f \le \operatorname{Mod} D_f^* \le \operatorname{Mod} D(k,a) \tag{2.2}$$

where equality $\operatorname{Mod} D_f = \operatorname{Mod} D(k,a)$ holds if and only if D_f is obtained from $D(k,a)$ by a rotation around the origin. From the relation

$$\operatorname{Mod} D_f = \operatorname{Mod} D(k,d_0) (= \log R)\,, \tag{2.3}$$

$$\operatorname{Mod} D_f \le \operatorname{Mod} D(k,a) \tag{2.4}$$

and monotonicity of the module, we have

$$a \ge d_0\,, \tag{2.5}$$

which implies the desired inequality (2.1). It is trivial that equality holds in (2.1) if and only if D_f is $D(k,d_0)$ up to a rotation around the origin (see [6]).

Now we consider the case when Γ does not contain the point at infinity. Without loss of generality we can assume $a = d(\Gamma,0) \in \Gamma$. For a negative point $-d$ on Γ, the Möbius transformation $\zeta(w) = (1+dw)/(w+d)$ maps the points a and $-d$ to $(1+ad)/(a+d)(< a)$ and the point at infinity, respectively. This means that the minimum of $d(\Gamma,0)$ is attained (if and) only if Γ contains the point at infinity. Therefore the inequality (2.1) holds even when Γ does not contain the point at infinity.

3. Proof of the theorem.

Let $w_1 = r_1 e^{i\theta}$ and $w_2 = r_2 e^{i(\pi-\theta)}$ $(= -r_2 e^{-i\theta})$ be the points on Γ such that the segments $(e^{i\theta}, r_1 e^{i\theta})$ and $(-e^{-i\theta}, -r_2 e^{-i\theta})$ are in D_f. Without loss of generality we can assume $r_1 = a$, $r_2 = at$ $(a > 0,\ t \ge 1)$, because the case with $r_1 \ge r_2$ can be proved analogously.

We consider the Möbius transformation

$$h(w) = \frac{w_1}{\overline{w}_1} \cdot \frac{\overline{w}_1 w - 1}{w_1 - w}\,, \tag{3.1}$$

which maps $f(A(R))$ onto $D(\Gamma')$ with inner boundary $\{|h|=1\}$ and outer boundary Γ'. Since the chordal cross ratio is invariant under Möbius transformations, Γ' is also a k-circle. Substituting $w=w_1$ and $w=w_2$ into (3.1) we have the inequalities

$$h(w_1)=\infty,\quad h(w_2)=-e^{2i\theta}\frac{a^2te^{-2i\theta}+1}{ae^{i\theta}+ate^{-i\theta}}\,. \tag{3.2}$$

Now the fundamental lemma and $|h(w_2)|\ge d_0$ imply

$$\frac{1+2a^2t\cos 2\theta+a^4t^2}{a^2(1+2t\cos 2\theta+t^2)}\ge d_0^2\,, \tag{3.3}$$

$$a^4t^2-a^2(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta)+1\ge 0\,. \tag{3.4}$$

From (3.4) we easily obtain either

$$\begin{aligned} a^2\ge{}&\frac{d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta}{2t^2}\\ &+\frac{\sqrt{(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta)^2-4t^2}}{2t^2}\end{aligned} \tag{3.5}$$

or

$$\begin{aligned} a^2\le{}&\frac{d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta}{2t^2}\\ &-\frac{\sqrt{(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta)^2-4t^2}}{2t^2}\,.\end{aligned} \tag{3.6}$$

Using the fundamental lemma we now show that (3.6) never holds: Let A and B be positive constants such that $A\pm\sqrt{A^2-1}=(B\pm\sqrt{B^2-1})^2$. Then $B=\sqrt{(A+1)/2}$. If $A=(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta)/2t^2$, we have

$$\begin{aligned} B^2&=\frac{A+1}{2}=\frac{d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta}{4t^2}+\frac{1}{2}\\ &=d_0^2\frac{1+t^2}{4t^2}+\frac{(d_0^2-1)\cos 2\theta}{2t}+\frac{1}{2}\le\frac{d_0^2}{2}+\frac{d_0^2-1}{2}+\frac{1}{2}=d_0^2\,.\end{aligned} \tag{3.7}$$

On the other hand, the inequality (3.6) implies

$$a^2\le A-\sqrt{A^2-1}=(B-\sqrt{B^2-1})^2\le B^2\le d_0^2\,, \tag{3.8}$$

contradicting $a\ge d_0>1$, because $a=d_0$ would imply $d_0=B=1$.

Now we utilize (3.5) to obtain

$$\begin{aligned} (r_1+r_2)^2&=a^2(1+t)^2\\ &\ge\frac{(1+t)^2}{2t^2}(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta\\ &\quad+\sqrt{(d_0^2(1+2t\cos 2\theta+t^2)-2t\cos 2\theta)^2-4t^2})\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&= \frac{(1+t)^2}{t}\left(d_0^2\left(\frac{1+t^2}{2t}+\cos 2\theta\right)-\cos 2\theta\right.\\
&\qquad \left.+\sqrt{\left(d_0^2\left(\frac{1+t^2}{2t}+\cos 2\theta\right)-\cos 2\theta\right)^2-1}\right)\\
&\geq 4(d_0^2(1+\cos 2\theta)-\cos 2\theta+\sqrt{(d_0^2(1+\cos 2\theta)-\cos 2\theta)^2-1})\\
&= 4(d_1+\sqrt{d_1^2-1})^2 \qquad \left(d_1=\sqrt{d_0^2\cos^2\theta+\sin^2\theta}\right),
\end{aligned}
$$

which implies $r_1+r_2 \geq 2(d_1+\sqrt{d_1^2-1})$. Since $\ell(\theta)\geq r_1$ and $\ell(\pi-\theta)\geq r_2$, we obtain the desired inequality (1.3). Using the fundamental lemma and (3.9), we conclude that equality in (1.3) is attained only if $t=1$, $r_1=r_2=\ell(\theta)=\ell(\pi-\theta)=d_1+\sqrt{d_1^2-1}$, and only if $f(A(R))$ is a rotation of $D(k,d_0)$ around the origin.

It follows trivially from the fundamental lemma that

$$
\ell(\theta)\geq d_0, \qquad \ell(\pi-\theta)\geq d_0\,. \tag{3.10}
$$

For $\arccos(\sqrt{d_0^2-1}/(2d_0))<\theta\leq\pi/2$, by a simple calculation, we conclude that

$$
d_1+\sqrt{d_1^2-1}<d_0\,, \tag{3.11}
$$

which implies that the inequality (1.4) is better than (1.3) in this case.

Next we discuss the case of equality in (1.3). For the case of $w_1=a_0e^{i\theta}$, $w_2=-a_0e^{-i\theta}$ ($a_0=d_1+\sqrt{d_1^2-1}$), we have

$$
\begin{aligned}
h(w_2) = -e^{2i\theta}\frac{1+a_0^2e^{-2i\theta}}{a_0(e^{i\theta}+e^{-i\theta})} &= -\frac{a_0^2+e^{2i\theta}}{2a_0\cos\theta}=-d_0e^{i\beta} \qquad (\beta \text{ real})\,, \qquad (3.12)\\
a_0^2+e^{2i\theta} &= 2d_0a_0e^{i\beta}\cos\theta\,,\\
\sin 2\theta &= 2d_0a_0\sin\beta\cos\theta\,,\\
\sin\theta &= d_0a_0\sin\beta\,,\\
\beta &= \arcsin(\sin\theta/(d_0a_0)) \qquad (0\leq\beta<\theta)\,.
\end{aligned}
$$

Now we determine the value θ_0 mentioned in the theorem, as follows: For the extremal function $F(z)$, the point $h(\infty)=-w_1=-a_0e^{i\theta}$ must be contained in the complement of $e^{i\beta}D(k,d_0)$, because the extremal function must be conformal. Considering the rotation around the origin through $\pi-\beta$, we see that the point $a_0e^{i(\theta-\beta)}$ must lie in the closed domain $\{w : |\arg(w-d_0)|\leq\arcsin k\}$. We consider two functions of the angle θ,

$$
Y_1(\theta)=a_0=\sqrt{(d_0^2-1)\cos^2\theta+1}+\sqrt{(d_0^2-1)\cos^2\theta}\,, \tag{3.13}
$$

$$
Y_2(\theta)=d_0k/\sin(\theta_2-\theta) \qquad (\theta_2=\arcsin k)\,, \tag{3.14}
$$

where (3.14) represents the rays $\{w : |\arg(w-d_0)|=\arcsin k\}$ in polar coordinates (Y_2,θ). The functions $Y=Y_1(\theta)$ and $Y=Y_2(\theta)$ are, respectively, strictly decreasing and increasing, and their values run from $d_0 + \sqrt{d_0^2 - 1}$ to 1 $(0 \le \theta \le \pi/2)$ and from d_0 to ∞ $(0 \le \theta \le \theta_2)$, respectively. Therefore the curves $Y = Y_1(\theta)$ and $Y = Y_2(\theta)$ intersect at some point $\theta = \theta_3$ $(< \theta_2)$. Since

$$a_0 = \sqrt{(1 - d_0^2)\sin^2\theta + d_0^2} + \sqrt{(1 - d_0^2)\sin^2\theta + d_0^2 - 1}$$

(which implies that $\beta = \beta(\theta)$ is a strictly decreasing function of θ for $0 \le \theta \le \pi/2$) and $\beta(\theta) < \theta$, the function $\theta - \beta(\theta)$ is non-negative and strictly increasing for $0 \le \theta \le \pi/2$ and varies from 0 to $\pi/2 - \arcsin(1/d_0)$ there. Therefore there exists a constant θ_0 such that $0 \le \theta - \beta \le \theta_3$ for $0 \le \theta \le \theta_0$. Then the point $a_0 e^{i(\theta-\beta)}$ is contained in $\{w : |\arg(w - d_0)| \le \arcsin k\}$ for $0 \le \theta \le \theta_0$.

Since $T(w)$ is the inverse function of (3.1) the function $F(z)$ maps $A(R)$ onto the extremal domain which has two points $w_1 = a_0 e^{i\theta}$ and $w_2 = a_0 e^{i(\pi-\theta)}$ on the boundary $F(|z| = R)$ for $0 \le \theta \le \theta_0$, and so the theorem has been verified.

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