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On the generalized Avez method

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Abstract. A generalization of the Avez method of construction of an invariant measure is presented.

0. Introduction. In [9] Lasota and Pianigiani formalized the method of construction of an invariant measure used by Avez in [1]. In the present paper some generalization of this method is presented. This method can also be used for construction of an invariant measure in the situation considered by Schweiger [10].

1. Construction. Let $T : X \to X$ be a measurable transformation of a compact topological space X. Let $(\Omega; \Sigma; P)$ be a probability space. Let $\{S_\omega\}_{\omega\in\Omega}$ be a family of continuous right inverses of T such that $\omega \mapsto S_\omega$ viewed as a map to the space $C(X;X)$ with the topology of uniform convergence is measurable. To construct a T-invariant measure we first define a sequence of functionals on $C(X)$ by

$$(1) \qquad A_n f = \int_{\Omega^n} f(S_{\omega_1} \ldots S_{\omega_n})\, P(d\omega_1) \ldots P(d\omega_n)$$

where x is an arbitrary point in X. We define Af to be the Banach limit of $\{A_n f\}$ [2], i.e.

$$(2) \qquad Af = \operatorname*{Lim}_{n\to\infty} A_n f\,.$$

From the linearity and contractivity of the Banach limit it follows that A is a bounded linear functional on $C(X)$ and in consequence there exists a measure μ on X such that

$$(3) \qquad Af = \int_X f(y)\, \mu(dy)\,.$$

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PROPOSITON. *The measure constructed above is T-invariant, i.e.* $\mu(T^{-1}(E)) = \mu(E)$ *for every* $E \in \mathcal{B}(X)$.

Proof. It is sufficient to prove that for every continuous function f on X

$$\int_X f(y)\,\mu(dy) = \int_X f(Ty)\,\mu(dy).$$

This is equivalent to

$$Af = A(f \circ T).$$

To prove the latter formula it is sufficient to notice that

$$\begin{aligned} A_n(f \circ T) &= \int_{\Omega^n} f(TS_{\omega_1}S_{\omega_2}\ldots S_{\omega_n}x)P(d\omega_1)P(d\omega_2)\ldots P(d\omega_n) \\ &= \int_{\Omega^n} f(S_{\omega_2}\ldots S_{\omega_n}x)P(d\omega_1)P(d\omega_2)\ldots P(d\omega_n) = A_{n-1}f \end{aligned}$$

and that $\operatorname{Lim} a_n = \operatorname{Lim} a_{n-1}$. ∎

2. Properties of the generalized Avez measure

THEOREM 1. *If for every nonempty open set* $\mathcal{U} \subset C(X;X)$ *(in the sense of uniform topology) there exists n such that* $P(S_{\omega_1}\ldots S_{\omega_n} \in \mathcal{U}) > 0$ *then the measure μ is positive on nonempty open sets in X.*

Proof. It is sufficient to prove that $Af > 0$ for each nonnegative function f positive on some nonempty set. Let f be such a function and let $\varepsilon > 0$ be such that $U = \{x : f(x) > \varepsilon\} \neq \emptyset$. Let $\Gamma = \{(\omega_1, \ldots, \omega_n) : \forall y,\ S_{\omega_1}\ldots S_{\omega_n}y \in U\}$ and let $k > n$. Then

$$\begin{aligned} A_k f &= \int_{\Omega^k} f(S_{\omega_1}\ldots S_{\omega_k}x)P(d\omega_1)\ldots P(d\omega_k) \\ &= \int_{\Omega^n}\Big(\int_{\Omega^{k-n}} f(S_{\omega_1}\ldots S_{\omega_k}x)P(d\omega_{n+1})\ldots P(d\omega_k)\Big)P(d\omega_1)\ldots P(d\omega_n) \\ &= \Big(\int_{\Gamma} + \int_{\Omega^n\setminus\Gamma}\Big)\Big(\int_{\Omega^{k-n}} f(S_{\omega_1}\ldots S_{\omega_k}x)P(d\omega_{n+1})\ldots \\ &\qquad\qquad \ldots P(d\omega_k)\Big)P(d\omega_1)\ldots P(d\omega_n) \\ &\geq \varepsilon P(\forall y,\ S_{\omega_1}\ldots S_{\omega_n}y \in U). \end{aligned}$$

Since $\{S : \forall y,\ Sy \in U\}$ is open and nonempty in $C(X;X)$, $A_k f$ is greater than some positive constant independent of k, which completes the proof. ∎

3. Examples

EXAMPLE 1. If $\Omega = \{1, \ldots, n\}$ and P is the classical probability we obtain the construction given in [9].

EXAMPLE 2. Let $X = W_\lambda$ from [8], i.e. X is the set of all continuous functions defined on $[0;1]$ vanishing at 0 and satisfying $|v''(x)| \leq |x|^{\lambda-2}$ for all x. Let $T_t v(x) = e^{\lambda t} v(xe^{-t})$ and let S_ω be the right inverse to $T_{\ln 2}$ given by

$$(4)\quad S_\omega v(x) = \begin{cases} 2^{-\lambda} v(2x), & x \in [0; \frac{1}{2}], \\ 2^{-\lambda}[v(1) + v'(1)(x - \frac{1}{2})] + \omega[\lambda(\lambda-1)]^{-1}(x - \frac{1}{2})^\lambda, & x > \frac{1}{2}, \end{cases}$$

where $\omega \in \Omega = [0;1]$. Considering Ω with the Lebesgue measure we obtain an invariant measure for $T_{\ln 2}$.

EXAMPLE 3. Let V_α be the set of all absolutely continuous functions defined on [0;1] vanishing at 0 and satisfying $|v'(x)| \leq \alpha |x|^{\lambda-1}$ where $\lambda > 1$ is given. Let $T_t v(x) = e^{\lambda t} v(xe^{-t})$. In [4] an invariant measure for $T_{\ln 2}$ is constructed by the following method. Let $\{p_i\}_{i=1,2,\ldots}$ be a sequence of positive numbers such that $\sum_{i=1}^{\infty} p_i = 1$. We construct a sequence of maps $k_n : [0;1] \to \mathbb{N}$ with $m(\{x : k_1(x) = i_1, \ldots, k_n(x) = i_n\}) = p_{i_1} \ldots p_{i_n}$ for every $i_1, \ldots, i_n$. We also consider some sequence of polynomials $\{\sigma_n\}_{n=1,2,\ldots}$ such that the formula

$$S_n v(x) = \begin{cases} 2^{-\lambda} v(2x) & \text{for } x \leq \frac{1}{2}, \\ 2^{-\lambda} v(1) + \sigma_n(2x - 1) & \text{for } x \geq \frac{1}{2}, \end{cases}$$

defines a right inverse to T. It is proved in [4] that for every $x \in [0;1]$ the set $\bigcap_{n=1}^{\infty} S_{k_1(x)} \ldots S_{k_n(x)}(V_\alpha)$ has exactly one element. The unique element of this set is denoted by $\Phi(x)$. The T-invariant measure is defined as the transport of the Lebesgue measure by Φ.

THEOREM 2. *The measure obtained in* [4] *satisfies the conditions of Theorem* 1.

Proof. Let $\Omega = \mathbb{N}$ and let $\mu(E) = \sum_{i \in E} p_i$. Let $X = V_a$. From the Arzelà lemma V_α is compact. Let $\mathcal{U}$ be an open set in $C(V_\alpha; V_\alpha)$. Clearly there exist some $v_0 \in V_\alpha$ and $\varepsilon > 0$ such that $\{S : \forall v \in V_\alpha, Sv \in U(v_0; \varepsilon)\} \subseteq \mathcal{U}$ where $U(v_0; \varepsilon) = \{v \in V_\alpha : \forall x, |v(x) - v_0(x)| < \varepsilon\} \subseteq \mathcal{U}$.

Now, we prove that there exists $n \in \mathbb{N}$ such that $P(S_{\omega_1} \ldots S_{\omega_n} \in \mathcal{U}) > 0$. Let n be such that $(\alpha/\lambda)2^{-n\lambda} < \varepsilon/4$. Evidently $|v(x) - v_0(x)| < \varepsilon/2$ for every $v \in V_\alpha$. Let now $k_1, \ldots, k_n$ be positive integers such that $|\sigma_{k_i}(x) - 2^{\lambda i} v_0(2^{-i}x + 2^{-i})| < (1/n)\varepsilon 2^{\lambda i}$ for $x \in [2^{-i}; 2^{-i+1}]$. Clearly for every $v \in V_\alpha$, $S_{k_1} \ldots S_{k_n} v(x) = 2^{-\lambda i} \sigma_{k_i}(2^i x - 1) + s_i$ for $x \in [2^{-i}; 2^{-i+1}]$. It follows that $S_{k_1} \ldots S_{k_n} v \in U(v_0; \varepsilon)$ for every $v \in V_\alpha$, and in consequence $P(S_{\omega_1} \ldots S_{\omega_n} \in \mathcal{U}) \geq p_{k_1} \ldots p_{k_n}$.

Now, we must prove that the measure from [4] is the same as the measure from the present paper. Let v be an arbitrary function from V_α and let A_n be the functional defined by (1). Since $S_{k_1}\dots S_{k_n}(V_\alpha)\cap S_{k'_1}\dots S_{k'_n}(V_\alpha)=\emptyset$ for every $(k_1,\dots,k_n)\neq(k'_1,\dots,k'_n)$, we have $\mu_n(S_{k_1}\dots S_{k_m}(V_\alpha))=p_{k_1}\dots p_{k_m}$ for any $n>m$. Moreover, $\int_X f(v)\,\mu(dv)=\int_0^1 f(\Phi x)\,dx$. Let $v_0\in V_\alpha$ and let $\Phi_n x=S_{k_1(x)}\dots S_{k_n(x)}v_0$. It is easy to notice that

$$\begin{aligned}\int_0^1 f(\Phi_n x)\,dx &= \int_0^1 f(S_{k_1(x)}\dots S_{k_n(x)}v_0)\,dx\\ &= \sum_{r_1\dots r_n=1}^{\infty} p_{r_1}\dots p_{r_n} f(S_{r_1}\dots S_{r_n}v_0)=\int_X f(v)\,\mu_n(dv)\,.\end{aligned}$$

Since $S_{k_1(x)}\dots S_{k_n(x)}(V_\alpha)\subseteq U(S_{k_1(x)}\dots S_{k_n(x)};\varepsilon)$ for sufficiently large n, we have $\Phi_n(x)\to\Phi(x)$ uniformly for every $x\in\Phi$. This completes the proof. ∎

4. An invariant measure for the tangens function. In the previous section we have constructed an invariant measure on a compact space. This method can also be used if the space X is only locally compact. In [11] F. Schweiger has proved the ergodicity of the function tan with respect to Lebesgue measure. In this paper the existence of an invariant measure is proved, and the method of proof gives a very nice example of application of the method presented in this paper. The author does not know if the measure constructed is absolutely continuous. The classical theorems cannot be used, because the derivative of this function is not strongly greater than one.

THEOREM 3. *Let $\{p_n\}_{n\in\mathbb{Z}}$ be a family of positive numbers the sum of which is equal to* 1. *Then there exists a probability measure μ on $\mathbb{R}$ invariant with respect to the function $x\mapsto\tan x$, ergodic and such that $\mu(-\pi/2+n\pi;\pi/2+n\pi)=p_n$ for every $n\in\mathbb{Z}$.*

Proof. Let $V(k):\mathbb{R}\to\mathbb{R}$ be defined by $V(k)x=\arctan x+k\pi$. Clearly $\operatorname{im}V(k)=((k-\frac12)\pi;(k+\frac12)\pi)$. Let the linear functional A_n on the space $C_0(\mathbb{R})$ of continuous functions with compact support be given by

$$A_n u=\sum_{k_1,\dots,k_n\in\mathbb{Z}} u(V(k_1)\dots V(k_n)x)p_{k_1}\dots p_{k_n} \tag{5}$$

where x is some given real number. Clearly $|A_n u|\le\|u\|$. Let

$$Au=\operatorname*{Lim}_{n\to\infty} A_n u\,. \tag{6}$$

From the properties of Banach limits it follows that A is a positive linear functional on $C_0(\mathbb{R})$. Hence there exists a positive Borel measure μ such that $Au=\int_{\mathbb{R}} u(y)\,\mu(dy)$. Since $\mathbb{R}$ is σ-compact and $|Au|\le\sup_{x\in\mathbb{R}}|u(x)|$, μ

is finite. It is obvious, by the same argument as above, that μ is invariant with respect to the function tan.

To prove that $\mu((k\pi-\pi/2;k\pi+\pi/2))=p_k$ we have to consider functions φ and ψ satisfying

$$\varphi \le \chi_{B(k)} \le \psi \tag{7}$$

where $\chi_{B(k)}$ denotes the indicator of the interval $B(k)=(k\pi-\pi/2;k\pi+\pi/2)$ [10]. It is sufficient to show that $A\varphi \le p_k \le A\psi$ for any such φ,ψ. Clearly

$$A_n\varphi \le \sum_{S_{k_1}\ldots S_{k_n}x\in B(k)} p_{k_1}\cdots p_{k_n} \le A_n\psi\,.$$

From the construction of the sequence $\{p_n\}$ it follows that $V(k_1)\ldots V(k_n)x \in B(k)$ if and only if $k_1=k$. Hence

$$\begin{aligned}\sum_{S_{k_1}\ldots S_{k_n}x\in B(k)} p_{k_1}\cdots p_{k_n} &= p_k \sum_{k_2,\ldots,k_n\in\mathbb{Z}} p_{k_2}\cdots p_{k_n}\\ &= p_k\Big(\sum_{j\in\mathbb{Z}} p_j\Big)^n = p_k\,.\end{aligned} \tag{8}$$

Thus $A_n\varphi \le p_k \le A_n\psi$ and in consequence $A\varphi \le p_k \le A\psi$. Thus $\mu((k\pi-\pi/2;k\pi+\pi/2))=p_k$ and so μ is a probability measure. ∎

5. Some properties of the measure μ. We now prove

LEMMA 1. *If μ is the measure constructed in the theorem, then $\mu(B(k_1,\ldots,k_n))=p_{k_1}\ldots p_{k_n}$ where $B(k_1,\ldots,k_n)$ is defined as in* [10].

Proof. Consider functions φ and ψ belonging to $CB(\mathbb{R})$ and satisfying

$$\varphi \le \chi_{B(k_1,\ldots,k_n)} \le \psi\,. \tag{9}$$

Since $B(k_1,\ldots,k_n)=B(k_1)\cap(\tan)^{-1}B(k_2,\ldots,k_n)$ we have: $x\in B(k_1,\ldots,k_n)$ if and only if $x=S_{k_1}y$ for some $y\in B(k_2,\ldots,k_n)$. By simple induction it follows that $B(k_1,\ldots,k_n)=\operatorname{im}V(k_1)\ldots V(k_n)$. Hence for every $m\ge n$ we have

$$A_m\varphi \le p_{k_1}\cdots p_{k_n} \le A_m\psi\,, \tag{10}$$

which finishes the proof. ∎

LEMMA 2. *The measure μ does not depend on the choice of the starting point x.*

Proof. Since the sets of the form $B(k_1,\ldots,k_n)$ are a basis of the natural topology in $\mathbb{R}$ and every open set in $\mathbb{R}$ is a countable union of such sets the lemma is obvious. ∎

LEMMA 3. *The measure μ is ergodic.*

Proof. We present another construction of μ. Let μ_0 be the measure on $\mathbb{Z}$ given by

$$\mu_0(A) = \sum_{k\in A} p_k \tag{11}$$

and let μ_1 be the corresponding product measure on $\mathbb{Z}^{\mathbb{N}}$. Let $\Phi : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$ be defined by

$$\Phi(k_1, k_2, \ldots) \in \bigcap_{n=1}^{\infty} B(k_1, \ldots, k_n). \tag{12}$$

The set on the right hand side is always a one-element set and Φ is well-defined. From the definition of Φ it follows that for every $k_1, \ldots, k_n$, $\Phi^{-1}(B(k_1, \ldots, k_n)) = \{\overline{k} \in \mathbb{Z}^{\mathbb{N}} : \overline{k}_i = k_i,\ i = 1, \ldots, n\}$ and in consequence μ is the transport of μ_1 by the transformation Φ. Let S be the shift transformation on $\mathbb{Z}^{\mathbb{N}}$, i.e. $(Sk)_n = k_{n+1}$ for $n \in \mathbb{N}$. To complete the proof it is sufficient to notice that μ_1 is S-invariant and ergodic and the diagram

$$\begin{array}{ccc} \mathbb{Z}^{\mathbb{N}} & \xrightarrow{S} & \mathbb{Z}^{\mathbb{N}} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbb{R} & \xrightarrow{\tan} & \mathbb{R} \end{array}$$

commutes. ∎

Remarks. The measure μ is clearly continuous and positive on non-empty open sets. Clearly, μ is supported on the full Lebesgue-measure set Y of all real numbers for which every iterate of the tan function exists.

6. Remarks. In the last example X was not compact. The structure of the transformation made it possible to construct a measure by our method. Now, we present another method of construction, which can be used in the general case. First, consider the functional A_n on a compact space X. Clearly, there exists a measure μ_n such that

$$A_n f = \int_X f(y)\, \mu_n(dy). \tag{13}$$

This measure is the transport of the nth tensor power of P by the map $\Phi_n : \Omega^n \ni (\omega_1, \ldots, \omega_n) \mapsto S_{\omega_1} \ldots S_{\omega_n} x$. If the sequence $\{\mu_n\}$ is weakly convergent to a probability measure, then this measure is equal to μ, but otherwise we must use the method of Banach limits. Clearly, this method can only be used if the space X is compact.

We can obtain an invariant measure in yet another way. The map Φ_n can be considered as a selector of the set-valued map $\widehat{\Phi}_n : \Omega^n \ni (\omega_1, \ldots, \omega_n) \mapsto S_{\omega_1} \ldots S_{\omega_n}(X)$. Since Ω^n is the image of $\Omega^{\mathbb{N}}$ by a projection, we can consider

$\widehat{\Phi}_n$ to be defined on $\Omega^{\mathbb{N}}$; since P is a probability measure, μ_n is the transport of $P^{\mathbb{N}}$ by a selector of $\widehat{\Phi}_n$. Moreover, for every $(\omega_1, \dots, \omega_n) \in \Omega^{\mathbb{N}}$ the sequence $\{\widehat{\Phi}_n(\omega_1, \dots, \omega_n)\}$ is decreasing (in the sense of inclusion). If we can prove that, with probability one, $\widehat{\Phi}(\omega_1, \dots, \omega_n) = \bigcap_{n=1}^{\infty} \widehat{\Phi}_n(\omega_1, \dots, \omega_n)$ is a one-element set we can also construct the invariant measure as the transport of P by a measurable selector of $\widehat{\Phi}$ (unique modulo probability 0). This method has been used for example in [3], where the local compactness of the space is not assumed. The method used there can be generalized. To illustrate this situation we consider an example.

Let X be a σ-compact topological space. Let $\{\pi_t\}_{t\in\mathbb{R}}$ be a dynamical system on X such that there exists some compact set $K \subseteq X$ satisfying the following conditions:

$$\text{(i)} \qquad \forall t > 0 \quad \pi_t(K) \subseteq \operatorname{int} K\,,$$

$$\text{(ii)} \qquad \exists x_0 \in X \quad \bigcap_{t>0} \pi_t(K) = \{x_0\}\,,$$

$$\text{(iii)} \qquad \bigcup_{t<0} \pi_t(K) = X\,.$$

Let $C(X; x_0)$ be the space of continuous functions on X vanishing at x_0 with the topology of almost uniform convergence. Let T be the dynamical system on $C(X; x_0)$ given by

$$(T_t u)(x) = e^{\lambda t} u(\pi_t x)\,. \tag{14}$$

Let $r > 0$ and let μ be some measure on the space $C_0(\overline{K \setminus \pi_r(K)})$ of all continuous functions on $\overline{K \setminus \pi_r(K)}$ vanishing on $\partial \pi_r(K)$.

Definition. The measure μ satisfies the *condition* $(*)$ if and only if there exist a sequence $\{\varrho_n\}$ of positive numbers and $r > 0$ such that

$$\sum_{n=1}^{\infty} \varrho_n e^{-\lambda n r} < \infty\,, \tag{$*1$}$$

$$\sum_{n=1}^{\infty} \mu(\{u \in C_0(\overline{K \setminus \pi_r(K)}) : \sup|u| \geq \varrho_n\}) < \infty\,. \tag{$*2$}$$

Let $\varphi_n : K \to K$ be defined by

$$\varphi_n(x) = \pi_s(x), \quad \text{where} \quad s = \inf\{t > 0 : \pi_t(x) \in \pi_{nr}(K)\}\,.$$

Clearly φ_n is continuous. Let $\Phi : C(X) \to C_0(\overline{K \setminus \pi_r(K)})$ be defined by

$$(\Phi u)(x) = u(x) - (u \circ \varphi)(x)$$

where $\varphi = \varphi_1$.

THEOREM 4. *Let μ be a measure on $C_0(\overline{K \setminus \pi_r(K)})$ satisfying* $(*)$. *Then there exists a T-invariant measure ν on the space $C(X; x_0)$. The measure ν has the following properties*:

(a) *ν is ergodic with respect to T_t.*

(b) *ν is positive on nonempty open sets* (*in the topology of almost uniform convergence*).

(c) *$\nu(E_0) = 0$ where E_0 is the set of all T_t-periodic points.*

Moreover, if μ_0 is a T_t-invariant measure and the measure μ is defined by

$$\mu(E) = \mu_0(\Phi^{-1}(E))$$

then $\mu = \nu$.

Proof. We shall construct a measure ϱ invariant with respect to T_r on the space $C(K; x_0)$. To use the generalized Avez method we shall construct a family of right inverses of T_r. Let $v \in C_0(\overline{K \setminus \pi_r(K)})$. We define S_v by

$$(S_v u)(x) = \begin{cases} e^{-\lambda r} u(\pi_{-r}(x)) & \text{for } x \in \pi_r(K), \\ u(\varphi(x)) + v(x) & \text{for } x \notin \pi_r(K). \end{cases}$$

Clearly $T_r S_v = \mathrm{id}_{C(K)}$. Let $\{v_n\}_{n=1,2,\ldots}$ be a sequence in $C_0(\overline{K \setminus \pi_r(K)})$. Define $\widehat{\Phi}(v_1, v_2, \ldots) = \bigcap_{n=1}^{\infty} S_{v_1} \ldots S_{v_n}(C(K; x_0))$.

We claim that this set has at most one element. Indeed, let $u_1, u_2 \in S_{v_1} \ldots S_{v_n}(C(K; x_0))$. We shall consider the difference $u = u_1 - u_2$ on the set $K \setminus \pi_{nr}(K)$. First, notice that $u_i(x) = h_i(\varphi(x)) + v_1(x)$ for $i = 1, 2$ and for some functions h_i. Since φ is constant along the part of every trajectory of the system $\{\pi_t\}$ contained in $\overline{K \setminus \pi_r(K)}$, also the difference u is constant along the same part of that trajectory. By simple induction one proves that u is constant on the part of every trajectory of $\{\pi_t\}$ contained in $\overline{K \setminus \pi_{nr}(K)}$. Since u is continuous on K and $u(x_0) = 0$ it follows that if $u_1, u_2 \in \widehat{\Phi}(v_1, v_2, \ldots)$ then $u_1 = u_2$.

To finish the proof it is sufficient to show that $\widehat{\Phi}(v_1, v_2, \ldots) \neq \emptyset$ with probability 1 (i.e. the measure of the set of all sequences $(v_1, v_2, \ldots)$ for which $\widehat{\Phi}(v_1, v_2, \ldots) \neq \emptyset$ is one). Let $(v_1, v_2, \ldots)$ be a sequence such that $|v_n(x)| \leq \varrho_n$ for $n \geq n_0$. We claim that then $\widehat{\Phi}(v_1, v_2, \ldots) \neq \emptyset$. Define

$$\begin{aligned} h(x) &= v_1(x) \quad \text{for } x \in \overline{K \setminus \pi_r(K)}, \\ h(x) &= e^{-\lambda(n-1)r}[v_n(\pi_{-r(n-1)}(x)) - v_n(\pi_{-r(n-1)}(\varphi_{n-1}(x)))] \\ &\quad + e^{-\lambda(n-2)} v_{n-1}(\pi_{-r(n-2)}(\varphi_{n-2}(x))) \\ &\qquad\qquad \text{for } x \in K_n = \overline{\pi_{r(n-1)}(K) \setminus \pi_{rn}(K)}. \end{aligned}$$

Let $x \in K$. The maximal oscillation of h on the segment of the trajectory of $\{\pi_t\}$ contained in K_n is less than $2\varrho_n e^{-\lambda rn}$ for n sufficiently large. Hence the series

$$\sum_{n=1}^{\infty} \sup\{|v(\pi_t(x)) - v(\pi_s(x))| : \pi_s(x), \pi_t(x) \in K_n\}$$

is uniformly convergent and so $L(x) = \lim_{t\to\infty} h(\pi_t(x))$ exists, is continuous with respect to x and constant along the trajectories of $\{\pi_t\}$. Let $l(x) = h(x) - L(x)$. Obviously $l \in \widehat{\Phi}(v_1, v_2, \ldots)$ and hence this set is nonempty for each sequence $\{v_n\}$ for which $|v_n| \le \varrho_n$ for n sufficiently large. But from the Borel–Cantelli lemma it follows that this condition is satisfied with probability one.

To prove that this measure is positive on open nonempty sets we shall use the method analogous to that of [5]. Let $G(n;\varepsilon) = \{v \in C_0(K) : |v(x)| \le \varepsilon, \forall x \in \pi_{rn}(K)\}$. By the same argument as in [5] (Lemma 5) we can prove that $\varrho(G(n;\varepsilon)) > 0$. Let A denote the set of all functions $v \in G(n;\varepsilon)$ for which there exist a function c defined on $\overline{K \setminus \pi_{rn}(K)}$ and constant along the trajectories of $\{\pi_t\}$ and a function $v_0 \in C_0(K)$ such that for each $x \in \overline{K \setminus \pi_{rn}(K)}, |v(x) - v_0(x) - c(x)| < \varepsilon$. By the same argument as in [5], $\varrho(A) > 0$ and every open set in $C_0(K)$ has a subset of this form. To construct a T_t-invariant measure we use the natural method, i.e. we define the measure κ by the formula

$$\kappa(E) = \int_0^1 \varrho(T_t^{-1}(E))\, dt\,.$$

To construct the required measure on $C(X)$ it is sufficient to use the method of [6].

Remark. The measure constructed in [3] is a special case of the measure from the last theorem. Indeed, let $X = \mathbb{R}_+$, $x_0 = 0$, $\pi_t(x) = xe^{-t}$. If X is compact the set $\widehat{\Phi}(\omega_1, \ldots, \omega_n)$ is always nonempty, but the choice of a selector of $\widehat{\Phi}$ does not necessarily correspond to the map T.

EXAMPLE 4. Let $Tv(x) = 2^{\lambda} v(x/2)$ on the space of all Lipschitz functions satisfying $|v'(x)| \le x^{\lambda-1}$ (without the assumption that $v(0) = 0$). Using the notation of [3] we can obtain, for every sequence $\{k_n\}$, the set of functions $\widehat{\Phi}(k_1, k_2, \ldots)$. Choosing a function in this set satisfying $v(0) = 1$ we obtain a measure which is not T-invariant.

Clearly, if X is compact we need not look for a selector, because we can construct the desired measure by the method presented in Section 1. The selector method is only necessary whenever X is not locally compact because in this situation the Riesz theorem cannot be used. The measure in [3], [4] is constructed by the selector method, but in the specific situation where $\widehat{\Phi}(k_1, k_2, \ldots)$ is a one-element set for almost all $(k_1, k_2, \ldots)$. This situation is also considered in [10].

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