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AN EQUATIONAL BASIS IN FOUR VARIABLES FOR THE THREE-ELEMENT TOURNAMENT

BY

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1. Introduction. As in E. Fried [1] and H. L. Skala [3], we can associate with a *tournament* $\langle T; <\rangle$ (T with a binary relation $<$ such that for all $a, b \in T$ exactly one of $a = b$, $a < b$, and $b < a$ holds) an algebra $\langle T; \wedge, \vee\rangle$ by the rule: if $x < y$, then $x = x \wedge y = y \wedge x$ and $y = x \vee y = y \vee x$, and $x = x \wedge x = x \vee x$ for all x.

In this algebra $\langle T; \wedge, \vee\rangle$, neither $\wedge$ nor $\vee$ is associative unless $\langle T; <\rangle$ is a chain, that is, $<$ is transitive. However, the two operations are idempotent, commutative; the absorption identities hold, and a weak form of the associative identities holds. In E. Fried and G. Grätzer [2], such algebras were named "weakly associative lattices."

More formally, following E. Fried [1] and H. L. Skala [3], an algebra $\langle A; \wedge, \vee\rangle$ is called a *weakly associative lattice* (WA-lattice) iff it satisfies the following set of identities:

$$x \wedge x = x, \quad x \vee x = x \qquad \text{(idempotency)}; \tag{1}$$

$$x \wedge y = y \wedge x, \quad x \vee y = y \vee x \qquad \text{(commutativity)}; \tag{2}$$

$$x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x \qquad \text{(absorption identities)}; \tag{3}$$

$$((x \wedge z) \vee (y \wedge z)) \vee z = z, \quad ((x \vee z) \wedge (y \vee z)) \wedge z = z \qquad \text{(weak associativity)}. \tag{4}$$

Define "dual" to mean interchanging $\wedge$ and $\vee$. The dual of a WA-lattice is a WA-lattice. The set of identities (1)–(4) is self-dual (i.e., the dual of

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every identity in the set (1)–(4) is in the set). For a polynomial p, we denote by $\widetilde{p}$ its dual.

The smallest example of a nontransitive tournament is the three-element cycle $\langle \{0,1,2\}; <\rangle$ in which $0 < 1$, $1 < 2$, and $2 < 0$. In the corresponding algebra Z, neither $\wedge$ nor $\vee$ is associative.

Z plays the same role for tournaments as the two-element lattice does for distributive lattices. A tournament (algebra) $\langle T; \wedge, \vee\rangle$ is not a chain iff it contains Z as a subalgebra.

Let $\mathbf{Z}$ be the variety generated by Z. Note that $\mathbf{Z}$ is self-dual: if an algebra is in $\mathbf{Z}$, so is its dual.

Let L be a WA-lattice, $a, b \in L$. We denote by $\Theta(a,b)$ the smallest congruence relation in L under which a and b are congruent. The following is a description of $\Theta(a,b)$ in any $L \in \mathbf{Z}$ (E. Fried and G. Grätzer [2, Theorem 1]):

Characterization Theorem of $\Theta(a,b)$ in $\mathbf{Z}$. *Let $L \in \mathbf{Z}$, let $a, b, c, d \in L$, and let $a \leq b$, $c \leq d$. Then $c \equiv d$ $(\Theta(a,b))$ iff the following two equations hold*:

$$a \wedge (c \wedge b) = a \wedge (d \wedge b), \qquad (a \vee c) \vee b = (a \vee d) \vee b\,.$$

One of the main results of E. Fried and G. Grätzer [2] is a characterization of $\mathbf{Z}$ in terms of $\Theta(a,b)$; we will need this in our proof:

Characterization Theorem of $\mathbf{Z}$. *Let $\mathbf{K}$ be a variety of WA-lattices in which for any $A \in \mathbf{K}$, $a, b, c, d \in A$, $a \leq b$, $c \leq d$, and $c \equiv d$ $(\Theta(a,b))$ imply that $a \wedge (c \wedge b) = a \wedge (d \wedge b)$, and $(a \vee c) \vee b = (a \vee d) \vee b$. Then $\mathbf{K} \subseteq \mathbf{Z}$.*

In E. Fried and G. Grätzer [2], a finite set of identities was exhibited that form an equational basis of $\mathbf{Z}$. The identities are in five variables; so from this result we can conclude that if every five-generated subalgebra of an algebra belongs to $\mathbf{Z}$, then so does the algebra. The question was raised whether "five" could be improved to "four." ("Three" is obviously impossible, since every three-variable identity that holds in $\mathbf{Z}$ also holds in any tournament.) In this paper, we answer this question in the affirmative.

2. The identities. We build our identities from the following polynomial:

$$r(x,y,z) = (x \wedge y) \wedge ((x \vee y) \wedge z),$$

and its dual. We consider the following identities:

$$r(x,y,z \wedge t) = (r(x,y,z) \wedge \widetilde{r}(x,y,t)) \wedge (r(x,y,t) \wedge \widetilde{r}(x,y,z))\,, \tag{5}$$

$$\begin{aligned} r(x,y,z \vee t) = {} & [(r(x,y,z) \vee \widetilde{r}(x,y,t)) \wedge (r(x,y,t) \vee \widetilde{r}(x,y,z))] \\ & \wedge (r(x,y,z) \vee r(x,y,t))\,, \end{aligned} \tag{6}$$

and their duals (7) and (8), respectively.

LEMMA. *Identities* (5)–(8) *hold in* Z.

Proof. These identities were checked with a computer program. For the reader's convenience, we show a quick way to check them by hand. Let a, b, and c be three distinct elements of Z. It is easily verified that

$$a \wedge (b \wedge c) = a \vee (b \vee c) = a\,.$$

Therefore, if $a \neq b$ in Z, then $a \wedge b < a \vee b$, $r(a, b, z) = a \wedge b$, and $\widetilde{r}(a, b, z) = a \vee b$, for all values of z in Z. This reduces the identities to relations involving only the two elements $x = a$ and $y = b$, which are easily verified.

Otherwise, $x = y$. If $\{x, z, t\}$ is contained in a two-element subset of Z, then the result follows easily since we work in a distributive lattice. Let $\{x, z, t\} = \{0, 1, 2\}$. Since the 3-cycle $0 \to 1 \to 2 \to 0$ is an automorphism of Z, we can assume that $x = 0$. Each identity is symmetric in z and t; therefore, it suffices to take $z = 1$ and $t = 2$. A check of this single case completes the proof.

Let $\mathbf{K}$ be the class of weakly associative lattices satisfying the identities (5)–(8). Then the Lemma can be restated as follows: $\mathbf{Z} \subseteq \mathbf{K}$.

3. The Theorem. Our main result is the following:

THEOREM. *Identities* (1)–(8) *define* $\mathbf{Z}$.

Proof. Let $A \in \mathbf{K}$ and $a, b \in A$. Consider a binary relation $\sim$ on A defined as follows: for $c, d \in A$, let $c \sim d$ iff

$$(*) \qquad r(a, b, c) = r(a, b, d) \quad \text{and} \quad \widetilde{r}(a, b, c) = \widetilde{r}(a, b, d).$$

This is clearly an equivalence relation, and $a \sim b$ holds. We show that $\sim$ has the Substitution Property. Indeed, if $c \sim d$, then for all $e \in A$,

$$\begin{aligned} r(a, b, c \wedge e) &= r(a, b, d \wedge e)\,, & \widetilde{r}(a, b, c \wedge e) &= \widetilde{r}(a, b, d \wedge e)\,, \\ r(a, b, c \vee e) &= r(a, b, d \vee e)\,, & \widetilde{r}(a, b, c \vee e) &= \widetilde{r}(a, b, d \vee e)\,. \end{aligned}$$

To prove the first equation, compute:

$$\begin{aligned} & r(a, b, c \wedge e) = && \text{(by (5))} \\ & (r(a, b, c) \wedge \widetilde{r}(a, b, e)) \wedge (r(a, b, e) \wedge \widetilde{r}(a, b, c)) = && \text{(by } (*)) \\ & (r(a, b, d) \wedge \widetilde{r}(a, b, e)) \wedge (r(a, b, e) \wedge \widetilde{r}(a, b, d)) = && \text{(by (5))} \\ & r(a, b, d \wedge e)\,. \end{aligned}$$

The other three proofs are similar. Thus, $\sim$ is a congruence relation on A.

Now let $c \equiv d$ $(\Theta(a, b))$. Since $a \sim b$ and $\sim$ is a congruence relation, it follows that $c \sim d$; therefore, $(*)$ holds. If, in addition, $a \leq b$ and $c \leq d$, then $(*)$ simplifies to $a \wedge (b \wedge c) = a \wedge (b \wedge d)$ and $b \vee (a \vee c) = b \vee (a \vee d)$.

Thus, in view of the Characterization Theorem of $\mathbf{Z}$, quoted in §2, this proves that $\mathbf{K} \subseteq \mathbf{Z}$; since by the Lemma, $\mathbf{K} \supseteq \mathbf{Z}$, this completes the proof of the Theorem.

COROLLARY 1. *Let A be an algebra. If every four-generated subalgebra of A belongs to $\mathbf{Z}$, then so does A.*

The identities (1)–(8) correspond closely to the identities defining distributive lattices. The identities (1)–(4) define WA-lattices, and (5)–(8) are the distributive identities. One difference shows up in (1)–(4): for lattices, we have three identites for $\vee$, the three dual ones for $\wedge$, and these two sets of identities are connected by the two absorption identities. For WA-lattices, weak associativity involves both operations. We can remedy this situation for $\mathbf{Z}$.

Consider the identities:

$$\begin{aligned} (x \wedge z) \wedge (x \wedge (y \wedge z)) &= (x \wedge z) \wedge ((x \wedge y) \wedge z)\,, \\ (x \vee z) \vee (x \vee (y \vee z)) &= (x \vee z) \vee ((x \vee y) \vee z)\,. \end{aligned} \tag{4'}$$

It is easy to see that (4′) holds in Z, and therefore in $\mathbf{Z}$. The role of (4) is to ensure that $a \vee b$ is the least upper bound of a and b (in the sense that $a \le a \vee b$, $b \le a \vee b$, and if $a \le d$ and $b \le d$, then $a \vee b \le d$); and dually. This readily follows also from (4′).

COROLLARY 2. *The identities* (1)–(3), (4′), (5)–(8) *define* $\mathbf{Z}$.

Finally, we would like to point out a curiosity. The set of identities in [2] characterizing $\mathbf{Z}$ is equivalent to the identities (1)–(8) in this paper. The proof of the equivalence uses the Characterization Theorem of $\mathbf{Z}$ from [2], a result that cannot be proved without some form of the Axiom of Choice. It would be interesting to find a direct equational theoretic proof of the equivalence. The existence of such a proof is known.

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