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PSEUDOCOMPACTNESS — FROM COMPACTIFICATIONS TO MULTIPLICATION OF BOREL SETS

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0. Introduction. All the spaces considered below are assumed to be completely regular and Hausdorff. For a space X, denote by $K(X)$ the family of all compactifications of X; βX stands for the Čech–Stone compactification. If $\alpha X \in K(X)$, let $C_\alpha(X)$ stand for the set of those functions $f \in C^*(X)$ which are continuously extendable over αX. For $f \in C_\alpha(X)$, let f^α be the continuous extension of f over αX and, for $F \subset C_\alpha(X)$, let $F^\alpha = \{f^\alpha : f \in F\}$.

Suppose that $F \subset C^*(X)$. Define $Z_F(X)$ as the family of all sets of the form $\bigcap_{i=1}^{\infty} \bigcup_{j=1}^{n_i} f_{i,j}^{-1}([a_{i,j}; b_{i,j}])$ where $f_{i,j} \in F$ and $a_{i,j} \le b_{i,j}$ ($a_{i,j}, b_{i,j} \in \mathbb{R}$) for $i \in \mathbb{N}$ and $j = 1, \ldots, n_i$ ($n_i \in \mathbb{N}$). Denote by $B_F(X)$ the smallest σ-algebra of subsets of X, containing $Z_F(X)$. Let $S_F(X)$ stand for the collection of all sets that are obtained from $Z_F(X)$ by the Souslin operation (cf. [11]). For $\alpha X \in K(X)$, put $Z_\alpha(X) = Z_F(X)$, $B_\alpha(X) = B_F(X)$ and $S_\alpha(X) = S_F(X)$ with $F = C_\alpha(X)$.

Let $\mathcal{E}(X)$ be the family of all $F \subset C^*(X)$ such that the diagonal mapping $e_F = \varDelta_{f \in F} f$ is a homeomorphic embedding. If $F \in \mathcal{E}(X)$, then the closure of $e_F(X)$ in $\mathbb{R}^{|F|}$ is a compactification of X called *generated* by F and denoted by $e_F X$. By a slight modification of the proof of Theorem 6 of [13] we get

0.1. THEOREM. *$F \subset C^*(X)$ is in $\mathcal{E}(X)$ if and only if $Z_F(X)$ is a closed base for X.*

In the light of 0.1, if $\alpha X \in K(X)$ and $F \subset C^*(X)$ are such that $Z_F(X) = Z_\alpha(X)$, then $F \in \mathcal{E}(X)$. Unfortunately, from $Z_F(X) = Z_\alpha(X)$ we cannot deduce that αX is generated by F. For instance, if X is Lindelöf, we have $Z_\alpha(X) = Z_\beta(X)$ for any $\alpha X \in K(X)$ (cf. [12, 3.10]). However, it was shown in [12, 3.4] that any compactification αX of a *pseudocompact* space X is the Wallman-type compactification which arises from the normal base $Z_\alpha(X)$. This yields

0.2. THEOREM. *For any compactifications αX and γX of a pseudocompact space X, we have: $\alpha X \le \gamma X$ if and only if $Z_\alpha(X) \subset Z_\gamma(X)$.*

The major portion of our work deals with describing, in terms of $Z_F(X)$ and $B_F(X)$, as well as of $S_F(X)$, all the sets $F \subset C^*(X)$ which generate a fixed compactification of X. Our methods lead us to the problem of multiplying Borel sets. Namely, let $B(X)$ denote the smallest σ-algebra containing all open subsets of X. For σ-algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$ of subsets of spaces X and Y, respectively, let $\mathcal{A}_X \times \mathcal{A}_Y$ be the smallest σ-algebra of subsets of $X \times Y$ which contains all rectangles $C \times D$ with $C \in \mathcal{A}_X$ and $D \in \mathcal{A}_Y$. If $B(X \times Y) = B(X) \times B(Y)$, then we say that the Borel sets of X and Y multiply. We shall finish the paper with answering the question when the Borel sets of perfectly normal pseudocompact spaces multiply.

1. Subsets of $C^*(X)$ generating compactifications

1.1. LEMMA. *For any $\alpha X \in K(X)$ and $F \in \mathcal{E}(X)$ with $e_F X = \alpha X$, we have $Z_F(X) = Z_\alpha(X)$.*

Proof. It suffices to show that if $A = f^{-1}(0)$ where $f \in C_\alpha(X)$ then $A \in Z_F(X)$. It follows from [13, Prop. 2 and Thm. 2] that, for any $i \in \mathbb{N}$, there exist $f_{i,j,k} \in F$ and real numbers $a_{i,j,k} < b_{i,j,k} \le c_{i,j,k} < d_{i,j,k}$ ($j = 1, \ldots, m_i$; $k = 1, \ldots, n_i$) such that

$$f^{-1}\left(\left[-\frac{1}{i+1}; \frac{1}{i+1}\right]\right) \subset B_i = \bigcup_{j=1}^{m_i} \bigcap_{k=1}^{n_i} f_{i,j,k}^{-1}([b_{i,j,k}; c_{i,j,k}]),$$

$$f^{-1}\left(\left(-\infty; -\frac{1}{i}\right] \cup \left[\frac{1}{i}; \infty\right)\right) \subset \bigcap_{j=1}^{m_i} \bigcup_{k=1}^{n_i} f_{i,j,k}^{-1}((-\infty; a_{i,j,k}] \cup [d_{i,j,k}; \infty)).$$

Then $A = \bigcap_{i=1}^{\infty} B_i$, hence $A \in Z_F(X)$ because $B_i \in Z_F(X)$ for $i \in \mathbb{N}$.

1.2. LEMMA. *Let $F \subset C^*(X)$ and $A \subset X$. Suppose that either A is pseudocompact, or X is pseudocompact and $A \in Z_\beta(X)$. Then $X \setminus A \in S_F(X)$ if and only if $A \in Z_F(X)$.*

Proof. Assume that $W = X \setminus A$ has the Souslin representation of the form $W = \bigcup_{\sigma \in \mathbb{N}^\omega} \bigcap_{n=1}^{\infty} A(\sigma|n)$ with $A(\sigma|n) \in Z_F(X)$ for all $\sigma \in \mathbb{N}^\omega$ and $n \in \mathbb{N}$ (cf. [11]). Since any z-filter in a pseudocompact space has the countable intersection property (cf. [8, 5H]), for any $\sigma \in \mathbb{N}^\omega$ there exists $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{m} A(\sigma|n) \subset W$. Put $n(\sigma) = \min\{m \in \mathbb{N} : \bigcap_{n=1}^{m} A(\sigma|n) \subset W\}$ and $T_m = \{\sigma \in \mathbb{N}^\omega : n(\sigma) = m\}$ for $\sigma \in \mathbb{N}^\omega$ and $m \in \mathbb{N}$. Let $M = \{m \in \mathbb{N} : T_m \neq \emptyset\}$. Then

$$W = \bigcup_{m \in M} \bigcup_{\sigma \in T_m} \bigcap_{n=1}^{m} A(\sigma|n).$$

This implies that W is a countable union of members of $Z_F(X)$. Let

$$W = \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{n_{i,j}} f_{i,j,k}^{-1}([a_{i,j,k}; b_{i,j,k}])$$

with $f_{i,j,k} \in F$ and $a_{i,j,k} \le b_{i,j,k}$ ($a_{i,j,k}, b_{i,j,k} \in \mathbb{R}$). Using the countable intersection property of z-filters in pseudocompact spaces we deduce that for any $i \in \mathbb{N}$, there exists $m_i \in \mathbb{N}$ with

$$A \subset A_i = \bigcup_{j=1}^{m_i} \bigcap_{k=1}^{n_{i,j}} \bigcup_{m=1}^{m_i} f_{i,j,k}^{-1}\Big(\Big(-\infty; a_{i,j,k} - \frac{1}{m}\Big] \cup \Big[b_{i,j,k} + \frac{1}{m}; \infty\Big)\Big).$$

Then $A = \bigcap_{i=1}^{\infty} A_i$, so $A \in Z_F(X)$.

1.3. THEOREM. *Let X be a pseudocompact space and let $F \in \mathcal{E}(X)$. For any $G \subset C(X)$ the following conditions are equivalent*:

(i) $G \in \mathcal{E}(X)$ *and* $e_F X \le e_G X$;
(ii) $Z_F(X) \subset Z_G(X)$;
(iii) $B_F(X) \subset B_G(X)$;
(iv) $S_F(X) \subset S_G(X)$.

Proof. That (iv)$\Rightarrow$(ii) follows from 1.2. To show that (i)$\Leftrightarrow$(ii), it suffices apply 0.1, 0.2 and 1.1.

1.4. DEFINITION. We shall say that sets $C, D \subset X$ are *separated* by a family $\mathcal{A}$ of subsets of X if there exists $A \in \mathcal{A}$ such that either $C \subset A \subset X \setminus D$ or $D \subset A \subset X \setminus C$.

1.5. THEOREM. *Let X be a pseudocompact space and let $F \in \mathcal{E}(X)$. A function $f \in C(X)$ is continuously extendable over $e_F X$ if and only if, for any real numbers $c < d$, the sets $C = f^{-1}((-\infty; c])$ and $D = f^{-1}([d; \infty))$ are separated by $S_F(X)$.*

Proof. Suppose that $A \in S_F(X)$ and $C \subset A \subset X \setminus D$. Arguing similarly to the proof of 1.2, we can show that there exist functions $f_{i,j,k} \in F$ and real numbers $a_{i,j,k} \le b_{i,j,k}$ such that

$$C \subset \bigcup_{i=1}^{\infty} \bigcap_{j=1}^{\infty} \bigcup_{k=1}^{m_{i,j}} f_{i,j,k}^{-1}([a_{i,j,k}; b_{i,j,k}]) \subset X \setminus D.$$

Since any z-filter in X has the countable intersection property, there exist positive integers n_i and p such that

$$C \subset \bigcup_{i=1}^{p} \bigcap_{j=1}^{n_i} \bigcup_{k=1}^{m_{i,j}} \bigcap_{n=1}^{n_i} f_{i,j,k}^{-1}\Big(\Big[a_{i,j,k} - \frac{1}{n+1}; b_{i,j,k} + \frac{1}{n+1}\Big]\Big),$$

$$D \subset \bigcap_{i=1}^{p} \bigcup_{j=1}^{n_i} \bigcap_{k=1}^{m_{i,j}} \bigcup_{n=1}^{n_i} f_{i,j,k}^{-1}\bigg(\bigg(-\infty; a_{i,j,k} - \frac{1}{n}\bigg] \cup \bigg[b_{i,j,k} + \frac{1}{n}; \infty\bigg)\bigg).$$

Theorem 4 of [4] completes the proof.

1.6. COROLLARY. *Suppose that X is pseudocompact, $F \in \mathcal{E}(X)$ and $G \subset C(X)$. Then the following conditions are equivalent*:

(i) *$G \in \mathcal{E}(X)$ and $e_F X \leq e_G X$*;

(ii) *any two disjoint members of $Z_F(X)$ are separated by $S_G(X)$*;

(iii) *for any function $f \in F$ and real numbers $c < d$, the sets $f^{-1}((-\infty; c])$ and $f^{-1}([d; \infty))$ are separated by $S_G(X)$.*

Proof. It suffices to apply 1.5 and [13, Thm. 2]

1.7. THEOREM. *Let X be pseudocompact. Then a set $F \subset C(X)$ belongs to $\mathcal{E}(X)$ if and only if, for any closed set $A \subset X$ and any $x \in X \setminus A$, the sets $\{x\}$ and A are separated by $S_F(X)$.*

Proof. Consider any zero-set $A \subset X$ and any $x \in X \setminus A$. If A and $\{x\}$ are separated by $S_F(X)$ then arguing similarly to the proof of 1.5, we can show that there exists $Z \in Z_F(X)$ with $A \subset Z \subset X \setminus \{x\}$. Now use 0.1.

1.8. THEOREM. *A Tikhonov space X is pseudocompact if and only if $Z_\alpha(X) \neq Z_\beta(X)$ for any $\alpha X \in K(X)$ with $\alpha X \neq \beta X$.*

Proof. Suppose that X is not pseudocompact. In view of [7, 3.10E] there exists a nonempty zero-set Z in βX with $Z \subset \beta X \setminus X$. If αX is obtained from βX by identifying the set Z with a point, then $Z_\alpha(X) = Z_\beta(X)$. Theorem 0.2 concludes the proof.

It was noticed in [12, 3.10] that $Z_\alpha(X) = Z_\beta(X)$ for any $\alpha X \in K(X)$ if and only if either $|\beta X \setminus X| \leq 1$ or X is Lindelöf. Let us give an example of a locally compact space X that is neither Lindelöf nor almost compact (cf. [8, 6J]) but $B_\alpha(X) = B_\beta(X)$ for any $\alpha X \in K(X)$.

1.9. EXAMPLE. Consider the interval $(-2; -1]$ with the usual topology and the space of ordinals $[0; \omega_1)$ with the order topology. Let X be their free union. Then $B_\omega(X) = B_\beta(X)$ with ωX standing for the one-point compactification.

For $\alpha X \in K(X)$, we denote by $w(S_\alpha(X))$ the smallest infinite cardinal κ for which there exists a family $\mathcal{A} \subset S_\alpha(X)$ such that $|\mathcal{A}| \leq \kappa$ and any member of $S_\alpha(X)$ is obtained from $\mathcal{A}$ by the Souslin operation. Let $w(B_\alpha(X))$ stand for the smallest infinite cardinal κ for which there exists $\mathcal{A} \subset B_\alpha(X)$ such that $|\mathcal{A}| \leq \kappa$ and $B_\alpha(X)$ is the σ-algebra generated by $\mathcal{A}$. Finally, let $w(Z_\alpha(X))$ be the smallest infinite cardinal κ for which there exists $\mathcal{A} \subset Z_\alpha(X)$ such that $|\mathcal{A}| \le \kappa$ and $Z_\alpha(X)$ is the smallest family containing $\mathcal{A}$ and closed under finite unions and countable intersections.

1.10. Theorem. *For any compactification αX of a pseudocompact space X, we have $w(\alpha X) = w(S_\alpha(X)) = w(B_\alpha(X)) = w(Z_\alpha(X))$.*

Proof. By [2, 4.2], there exists $F \in \mathcal{E}(X)$ with $|F| \le w(\alpha X)$ and $e_F X = \alpha X$. According to 1.1, $w(Z_\alpha(X)) \le |F| + \omega = w(\alpha X)$. For $\kappa \ge \omega$, let $\mathcal{A} \subset S_\alpha(X)$ with $|\mathcal{A}| \le \kappa$ be such that each member of $S_\alpha(X)$ is obtained from $\mathcal{A}$ by the Souslin operation. For $A \in \mathcal{A}$, choose a collection $\mathcal{H}_A = \{H_A(\sigma|n) : \sigma \in \mathbb{N}^\omega \text{ and } n \in \mathbb{N}\} \subset Z_\alpha(X)$ with $A = \bigcup_{\sigma \in \mathbb{N}^\omega} \bigcap_{n=1}^\infty H_A(\sigma|n)$. To each $H \in \mathcal{H}_A$ assign some $g_{A,H} \in C_\alpha(X)$ such that $H = g_{A,H}^{-1}(0)$. The collection $G = \{g_{A,H} : A \in \mathcal{A} \text{ and } H \in \mathcal{H}_A\}$ satisfies $|G| \le \kappa$ and $S_G(X) = S_\alpha(X)$. In view of 1.3, $G \in \mathcal{E}(X)$ and $e_G X = \alpha X$. Hence $w(\alpha X) \le w(S_\alpha(X))$. The obvious inequalities $w(S_\alpha(X)) \le w(B_\alpha(X)) \le w(Z_\alpha(X))$ complete the proof.

2. Multiplication of Borel sets. Let X and Y be Tikhonov spaces. For $\alpha X \in K(X)$ and $\gamma Y \in K(Y)$, denote by $\alpha \times \gamma(X \times Y)$ the compactification $\alpha X \times \gamma Y$ of $X \times Y$. If $f \in C(X)$ and $g \in C(Y)$, we put $f_X(x, y) = f(x)$ and $g_Y(x, y) = g(y)$ for any $(x, y) \in X \times Y$.

2.1. Lemma. *If $F \in \mathcal{E}(X)$ generates αX and $G \in \mathcal{E}(Y)$ generates γY, then $H = \{f_X : f \in F\} \cup \{g_Y : g \in G\}$ generates $\alpha X \times \gamma Y$.*

Proof. By [3, 2.3], it suffices to observe that $H \subset C_{\alpha\times\gamma}(X \times Y)$, and $H^{\alpha\times\gamma}$ separates points of $\alpha X \times \gamma Y$.

2.2. Theorem. *For any $\alpha X \in K(X)$ and $\gamma Y \in K(Y)$, we have $B_\alpha(X) \times B_\gamma(Y) = B_{\alpha\times\gamma}(X \times Y)$.*

Proof. Note that, in the light of 1.1 and 2.1, the σ-algebra $B_{\alpha\times\gamma}(X \times Y)$ is generated by all the sets $f_X^{-1}(0) \cap g_Y^{-1}(0) = f^{-1}(0) \times g^{-1}(0)$ with $f \in C_\alpha(X)$ and $g \in C_\gamma(Y)$.

It was shown in [1] that if $X \times Y$ is either Lindelöf or pseudocompact, then $B_\beta(X) \times B_\beta(Y) = B_\beta(X \times Y)$. Observe that this fact follows immediately from Glicksberg's theorem (cf. [7, 3.12.20(c)]), Theorem 3.10 of [12] and our Theorem 2.2.

2.3. Theorem. *Suppose that X is a countably compact space such that $B(X) \subset S_\beta(X)$. Then X is perfectly normal.*

Proof. In view of 1.2, each closed subset of X is a zero-set, which implies the perfect normality of X.

2.4. Theorem. *Let X and Y be perfectly normal pseudocompact spaces. Then $B(X) \times B(Y) = B(X \times Y)$ if and only if $X \times Y$ is perfectly normal.*

Proof. Since X is first-countable, the space $X\times Y$ is countably compact (cf. [7, 3.10.15]). It follows from 2.2 and Glicksberg's theorem that $B(X)\times B(Y)=B_\beta(X\times Y)$. Therefore our proposition is a consequence of 2.3.

It is well known that every countably compact Hausdorff space with diagonal of type G_δ is metrizable (cf. [5]); however, a pseudocompact perfect space with a G_δ diagonal need not be metrizable (cf. [8, 5I]). In the case of pseudocompactness we get the following metrization theorem:

2.5. Theorem. *A pseudocompact space X is metrizable if and only if $X\times X\setminus \Delta\in S_\beta(X\times X)$, where $\Delta=\{(x,y)\in X\times X : x=y\}$.*

Proof. Let $X\times X\setminus\Delta\in S_\beta(X\times X)$. It follows from 1.2 that Δ is a zero-set in $X\times X$; thus X is first-countable. Hence $X\times X$ is pseudocompact (cf. [7, 3.10.28]). Consequently, $\Delta\in Z_{\beta\times\beta}(X\times X)$. By 1.1 and 2.1, $\Delta=\bigcap_{i=1}^\infty\bigcup_{j=1}^{n_i} f_{i,j}^{-1}(0)\times g_{i,j}^{-1}(0)$ for some $f_{i,j},g_{i,j}\in C(X)$. Then the family $H=\{f_{i,j},g_{i,j} : i\in\mathbb{N},\ j\in\{1,\ldots,n_i\}\}$ separates points of X, which implies the metrizability of X.

2.6. Corollary. *Let X be a perfectly normal pseudocompact space. Then $B(X\times X)=B(X)\times B(X)$ if and only if X is metrizable.*

Denote by $P(Y)$ the collection of all subsets of Y. There exists a pseudocompact space Z such that $|Z|=2^\omega$, $B(Z)=P(Z)$ and $B(Z\times Z)=P(Z\times Z)$, any subset of Z is of type G_δ but Z fails to be countably compact (cf. [8, 5I]). If we assume CH then $B(Z\times Z)=B(Z)\times B(Z)$ (cf. [9, Thm. 12.5(ii), p. 73] or [10, Thm. 2]). Under the assumption of the negation of CH, it depends on one's set theory whether $B(Z\times Z)=B(Z)\times B(Z)$ (cf. [9, Thm. 12.8, p. 76] and [6]). The above remarks show that, in Corollary 2.6, the assumption of perfect normality cannot be weakened to perfectness.

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