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THE DIVERGENCE PHENOMENA OF INTERPOLATION TYPE OPERATORS IN L^p SPACE

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Let $L^p_{[-1,1]}$, $1 \le p < \infty$, be the class of real p-integrable functions on $[-1,1]$, $L^\infty_{[-1,1]} = C_{[-1,1]}$ the class of all real continuous functions on $[-1,1]$. Denote by $C^r_{[-1,1]}$ the space of real functions on $[-1,1]$ which have r continuous derivatives, and by $C^\infty_{[-1,1]}$ the space of real functions on $[-1,1]$ which are infinitely differentiable.

For $f \in L^p_{[-1,1]}$, let $E_n(f)_p$ be the best approximation to f by polynomials of degree n in L^p space.

Our works [1], [5] concern the divergence phenomena of trigonometric Lagrange interpolation approximations in comparison with best approximations in L^p space; the paper [1] contains the following theorem:

Let $1 \le p < \infty$. Suppose that $\{X_n\}$, $X_n = \{x_{n,j}\}_{j=0}^{2n}$, is a given sequence of real distinct (by $a \ne b$ we mean that $a \not\equiv b \pmod{2\pi}$) nodes and $\{\lambda_n\}$ is any given positive decreasing sequence. Then there exists an infinitely differentiable function f with period 2π such that

$$\limsup_{n\to\infty} \frac{\|f - L_n^X(f)\|_{L^p_{[0,2\pi]}}}{\lambda_n^{-1} E_n^*(f)_p} > 0\,,$$

where $L_n^X(f,x)$ is the n-th trigonometric Lagrange interpolating polynomial of $f(x)$ with nodes X_n and $E_n^*(f)_p$ is the best approximation to f by trigonometric polynomials of degree n.

Here and throughout, we write

$$\|f\|_{L^p_{[a,b]}} = \Big(\int_a^b |f(x)|^p dx\Big)^{1/p}, \qquad 1 \le p < \infty\,,$$
$$\|f\|_{[a,b]} = \|f\|_{L^\infty_{[a,b]}} = \max_{a \le x \le b} |f(x)|\,,$$

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$$\|f\|_{L^p} = \|f\|_{L^p_{[-1,1]}}, \qquad 1 \le p < \infty .$$

In spite of this counterexample, there do exist several positive results in this direction. For example, in [2], V. P. Motornyĭ discussed the rate of convergence of the $L_n(f,x)$ to $f(x)$ in L^1, expressed in terms of the sequence of best approximations of the function in L^1; he proved that if f is absolutely continuous with period 2π, $f' \in L^1_{[0,2\pi]}$, and $E_n^0(f')_1$ is the best approximation to f' by trigonometric polynomials of degree n with mean value zero in L^1, then

$$\|f - L_n(f)\|_{L^1_{[0,2\pi]}} = O(n^{-1} \log n E_n^0(f')_1) ,$$

where $L_n(f,x)$ is the nth trigonometric Lagrange interpolating polynomial to f with nodes $x_{n,j} = 2j\pi/(2n+1)$ for $j = 0, 1, \ldots, 2n$.

In L^p space for $1 < p < \infty$, K. I. Oskolkov [3] showed the following better estimate. Let f be absolutely continuous with period 2π, and $f' \in L^p_{[0,2\pi]}$ for $1 < p < \infty$; then

$$\|f - L_n(f)\|_{L^p_{[0,2\pi]}} = O(n^{-1} E_n^*(f')_p) .$$

One might ask what happens to other interpolation operators? More generally, to "interpolation type" operators? In this paper, by "interpolation type" operators we mean operators $I_n^r(f, X, x)$ of the form

$$I_n^r(f, X, x) = \sum_{k=0}^{r} \sum_{j=1}^{n_k} f^{(k)}(x_{n,j}^k) l_{n,j}^k(x)$$

for $f \in C^r_{[-1,1]}$, where $X_n = \bigcup_{k=0}^r \{x_{n,j}^k\}_{j=1}^{n_k}$ is a sequence of real nodes within $[-1,1]$, $\{x_{n,j}^r\} \not\subseteq \{-1,1\}$,

$$\sum_{k=0}^{r} n_j = n+1 ,$$

and $l_{n,j}^k(x)$, $j = 1, \ldots, n_k$, $k = 0, 1, \ldots, r$, are polynomials of degree not greater than n. Furthermore, if f is a polynomial of degree $\le n$, then $I_n^r(f, X, x) = f(x)$. In particular, if $r = 0$,

$$l_{n,j}^0(x) = \frac{\Omega_n(x)}{\Omega_n'(x_{n,j})(x - x_{n,j})}, \qquad \Omega_n(x) = \prod_{k=1}^{n+1} (x - x_{n,k}) ,$$

then $I_n^r(f, X, x)$ becomes the nth Lagrange interpolating polynomial with nodes $\{x_{n,j}\}_{j=1}^{n+1}$; if $r = 1$,

$$l_{m,j}^0(x) = \left(1 - \frac{\Omega_n''(x_{n,j})}{\Omega_n'(x_{n,j})}(x - x_{n,j})\right) \left(\frac{\Omega_n(x)}{\Omega_n'(x_{n,j})(x - x_{n,j})}\right)^2 ,$$

$$l^1_{m,j}(x) = (x - x_{n,j})\left(\frac{\Omega_n(x)}{\Omega'_n(x_{n,j})(x - x_{n,j})}\right)^2,$$

then $I^r_n(f, X, x)$ becomes the Hermite–Fejér interpolating polynomial of degree $m = 2n + 1$ with nodes $\{x_{n,j}\}_{j=1}^{n+1}$; and so on.

In the present paper we refine the idea used in [1] and prove the following

THEOREM. *Let $1 \le p < \infty$. Suppose that $\{X_n\}$ is a given sequence of real distinct nodes within $[-1, 1]$, and $\{\lambda_n\}$ is any given positive decreasing sequence. Then there exists a function $f \in C^\infty_{[-1,1]}$ such that*

$$\limsup_{n\to\infty} \frac{\|f - I^r_n(f, X)\|_{L^p}}{\lambda_n^{-1} E_n(f^{(r)})_p} > 0\,.$$

Proof. Without loss of generality assume that $-1 < x^r_{n,1} < 1$. Fix n. Considering the nonnegative function

$$g_n(x) = (1 + x)(1 - x)^{(1-x^r_{n,1})/(1+x^r_{n,1})},$$

we note that $g_n(x)$ strictly increases on $[-1, x^r_{n,1}]$ and strictly decreases on $[x^r_{n,1}, 1]$; accordingly we can choose a sufficiently large natural number T_n such that for all x in $[-1, 1] \setminus (x^r_{n,1} - \delta_n, x^r_{n,1} + \delta_n)$ and all $m \ge T_n$,

$$g^m_n(x) \le \frac{1}{2n} \max_{1\le j\le n_r} \|l^r_{n,j}\|^{-1}_{L^p} \eta_n g^m_n(x^r_{n,1}),$$

where

$$\delta_n := \min_{2\le j\le n_r} |x^r_{n,j} - x^r_{n,1}|, \qquad \eta_n := \|l^r_{n,1}\|_{L^p}.$$

In particular, for all $2 \le j \le n_r$,

$$g^m_n(x^r_{n,j}) \le \frac{1}{2n} \max_{1\le j\le n_r} \|l^r_{n,j}\|^{-1}_{L^p} \eta_n g^m_n(x^r_{n,1}). \tag{1}$$

Let N_n be a natural number not less than T_n, and

$$N^*_n = \frac{1 - x^r_{n,1}}{1 + x^r_{n,1}} N_n\,.$$

Write

$$h_n(x) = g_n^{-N_n}(x^r_{n,1}) \int_{-1}^{x} dt_1 \int_{-1}^{t_1} dt_2 \ldots \int_{-1}^{t_{r-1}} g_n^{N_n}(t_r)\, dt_r\,.$$

Then $h_n \in C^r_{[-1,1]}$ and we clearly have

$$\|h^{(r)}_n\| = h^{(r)}_n(x^r_{n,1}) = 1\,, \tag{2}$$

and for $2 \le j \le n_r$, by (1),

$$0 \le h^{(r)}_n(x^r_{n,j}) \le \frac{1}{2n} \eta_n \max_{1\le j\le n_r} \|l^r_{n,j}\|^{-1}_{L^p}\,. \tag{3}$$

On the other hand, a calculation gives

$$\|g_n^{N_n}\|_{L^p} = 2^{N_n+N_n^*+1/p}\left(\frac{\Gamma(N_np+1)\Gamma(N_n^*p+1)}{\Gamma(N_np+N_n^*p+2)}\right)^{1/p} \le Cg_n^{N_n}(x_{n,1}^r)N_n^{-1/(2p)}\,,$$

where here and throughout the paper, C always indicates a positive constant independent of n which may have different values in different places. So

$$\|h_n^{(r)}\|_{L^p} \le CN_n^{-1/(2p)}\,, \tag{4}$$

and for $0 \le s \le r-1$,

$$\|h_n^{(s)}\| \le 2^{r-1}\|h_n^{(r)}\|_{L^1} \le CN_n^{-1/2}\,. \tag{5}$$

We now establish that

$$\|h_n - I_n^r(h_n, X)\|_{L^p} \ge \tfrac{1}{2}\eta_n - Cn\varrho_nN_n^{-1/(2p)}\,, \tag{6}$$

where

$$\varrho_n := \max_{1\le j\le n_k,\, 0\le k\le r-1}\{1, \|l_{n,j}^k\|_{L^p}\}\,.$$

In fact, from the definition,

$$I_n^r(h_n, X, x) = \sum_{k=0}^{r}\sum_{j=1}^{n_k} h_n^{(k)}(x_{n,j}^k)l_{n,j}^k(x)\,.$$

By (2), (3) and (5),

$$\begin{aligned}\|h_n - I_n^r(h_n, X)\|_{L^p} &\ge \eta_n - \sum_{j=2}^{n_r} h_n^{(r)}(x_{n,j}^r)\|l_{n,j}^r\|_{L^p} \\ &\quad - \sum_{k=0}^{r-1}\sum_{j=1}^{n_k} h_n^{(k)}(x_{n,j}^k)\|l_{n,j}^k\|_{L^p} - \|h_n\|_{L^p} \\ &\ge \tfrac{1}{2}\eta_n - Cn\varrho_nN_n^{-1/(2p)}\,,\end{aligned}$$

thus (6) is proved. Without loss suppose that $\lambda_n \le 1$. Now choose

$$N_n = [\lambda_n^{-2p}n^{4p}(4\varrho_n^{2p}\eta_n^{-2p}+1)+T_n]\,.$$

Then for sufficiently large n, (6) becomes

$$\|h_n - I_n^r(h_n, X)\|_{L^p} \ge \tfrac{1}{4}\eta_n\,, \tag{7}$$

and (4) becomes

$$\|h_n^{(r)}\|_{L^p} \le C\lambda_n\eta_n\,. \tag{8}$$

Because $h_n \in C_{[-1,1]}^r$, select an algebraic polynomial f_n^* with sufficiently large degree $M_n \ge n$ such that (cf., for example, A. F. Timan [4]) for

$0 \le s \le r$,

(9) $$\|h_n^{(s)} - (f_n^*)^{(s)}\| \le n^{-1}\eta_n\lambda_n(1+\|I_n^r\|)^{-1},$$

where for bounded operators B on $C_{[-1,1]}$,

$$\|B\| := \sup_{f\in C_{[-1,1]},\, \|f\|=1} \{\|Bf\|\}.$$

Hence by (8) and (9),

$$\begin{aligned}\|(f_n^*)^{(r)}\|_{L^p} &\le \|(f_n^*)^{(r)} - h_n^{(r)}\| + \|h_n^{(r)}\|_{L^p}\\ &\le n^{-1}\eta_n\lambda_n + C\eta_n\lambda_n \le C\eta_n\lambda_n,\end{aligned}$$

and similarly, from (7) and (9),

$$\begin{aligned}\|f_n^* - I_n^r(f_n^*,X)\|_{L^p} &\ge \|h_n - I_n^r(h_n,X)\|_{L^p} - \|f_n^* - h_n\|\\ &\quad - \|I_n^r(h_n,X) - I_n^r(f_n^*,X)\|\\ &\ge C\eta_n - n^{-1}\eta_n\lambda_n(\|I_n^r\|+1)^{-1}(1+\|I_n^r\|) \ge C\eta_n\end{aligned}$$

for large enough n. Set $f_n(x) = \eta_n^{-1} f_n^*(x)$; we thus have

(10) $$\|f_n^{(r)}\|_{L^p} = O(\lambda_n),$$

(11) $$\|f_n - I_n^r(f_n,X)\|_{L^p} \ge C.$$

Select a sequence $\{m_j\}$ by induction. Let $m_1 = 4r$. After m_j, choose

(12) $$m_{j+1} = [(M_{m_j}^*)^2\lambda_{m_j}^{-1/m_j}(\|I_{m_j}^r\|+1) + m_j + 1],$$

where $M_n^* = M_n(\eta_n^{2/n}+1)$. Define

$$f(x) = \sum_{j=1}^{\infty}(M_{m_j}^*)^{-m_j} f_{m_j}(x).$$

Clearly $f \in C_{[-1,1]}^{\infty}$ (since f_{m_j} is a polynomial of degree M_{m_j}) in view of (2) and (9). Together with (12), (11) implies that

$$\begin{aligned}\|f - I_{m_j}^r(f,X)\|_{L^p} &\ge (M_{m_j}^*)^{-m_j}\|f_{m_j} - I_{m_j}^r(f_{m_j},X)\|_{L^p}\\ &\quad - C(\|I_{m_j}^r\|+1)\sum_{k=j+1}^{\infty}(M_{m_k}^*)^{-m_k}\|f_{m_k}\|\\ &\ge C(M_{m_j}^*)^{-m_j} - C(M_{m_{j+1}}^*)^{-m_{j+1}/2} \ge C(M_{m_j}^*)^{-m_j}.\end{aligned}$$

At the same time, by (10) and (12),

$$\begin{aligned}E_{m_j}(f^{(r)})_p &= O\Big((M_{m_j}^*)^{-m_j}\|f_{m_j}^{(r)}\|_{L^p} + \sum_{k=j+1}^{\infty}(M_{m_k}^*)^{-m_k}\|f_{m_k}^{(r)}\|\Big)\\ &= O((M_{m_j}^*)^{-m_j}\lambda_{m_j} + (M_{m_{j+1}}^*)^{-m_{j+1}/2}) = O((M_{m_j}^*)^{-m_j}\lambda_{m_j}).\end{aligned}$$

Altogether,

$$\frac{\|f - I^r_{m_j}(f, X)\|_{L^p}}{\lambda^{-1}_{m_j} E_{m_j}(f^{(r)})_p} \geq C > 0\,,$$

which is the required result. ■

Remark. Considering the Theorem together with Motornyĭ's and Oskolkov's results, we might have reasons to guess that there might be some connections between the interpolation approximation rate of a given function with some kinds of nodes in L^p space and the best approximation rate of a higher derivative of that function in L^p.

REFERENCES

[1] P. B. Borwein, T. F. Xie and S. P. Zhou, *On approximation by trigonometric Lagrange interpolating polynomials II*, Bull. Austral. Math. Soc. 45 (2) (1992), in print.

[2] V. P. Motornyĭ, *Approximation of periodic functions by interpolation polynomials in L_1*, Ukrain. Math. J. 42 (1990), 690–693.

[3] K. I. Oskolkov, *Inequalities of the "large sieve" type and applications to problems of trigonometric approximation*, Analysis Math. 12 (1986), 143–166.

[4] A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Macmillan, New York 1963.

[5] T. F. Xie and S. P. Zhou, *On approximation by trigonometric Lagrange interpolating polynomials*, Bull. Austral. Math. Soc. 40 (1989), 425–428.

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