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S.I.S.S.A.
VIA BEIRUT 4
34014 TRIESTE, ITALY

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Representing and absolutely representing systems

by

V. M. KADETS (Kharkov) and Yu. F. KOROBEĬNIK (Rostov-na-Donu)

Abstract. We introduce various classes of representing systems in linear topological spaces and investigate their connections in spaces with different topological properties. Let us cite a typical result of the paper. If H is a weakly separated sequentially separable linear topological space then there is a representing system in H which is not absolutely representing.

A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space (everywhere below the word "space" means "linear topological space") H over a field Φ is called a *basis* in H (see e.g. [6]) if for each x in H there exists a uniquely determined sequence $\{\eta_k\}_{k=1}^{\infty}$ of scalars from Φ such that the series $\sum_{k=1}^{\infty} \eta_k x_k$ converges to x (everywhere below $\Phi = \mathbb{C}$ or $\mathbb{R}$). A basis X in a locally convex space H is said to be *absolute* if for each x in H the corresponding series $\sum_{k=1}^{\infty} \eta_k x_k$ converges absolutely in H (to x). As is well known, there exist bases in Banach spaces which are not absolute. On the other hand, according to the Dynin–Mityagin theorem [6], each basis in a nuclear Fréchet space is absolute. A. A. Talalyan [7] introduced representing systems in a complete metrizable space as a natural generalization of bases. A sequence $X = (x_k)_{k=1}^{\infty}$ of elements of a space H is called a *representing system* (r.s.) if each x in H can be represented in the form of a series

$$x = \sum_{k=1}^{\infty} \alpha_k x_k \qquad (1)$$

converging in H. The class of spaces having at least one r.s. is much wider than the class of spaces with basis. According to [1], every nuclear Fréchet space not isomorphic to ω has a quotient space without a basis. As for r.s., we can give a criterion for a space to have an r.s. We say that a space H is *sequentially separable* if there exists a "universal" sequence $V = \{v_k\}_{k=1}^{\infty}$ in H such that for each x in H one can find a subsequence $(v_{n_k})_{k=1}^{\infty}$ tending to x in H. For example, every separable space with a countable defining

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set of zero neighbourhoods is sequentially separable. In particular, every separable Fréchet space is sequentially separable. The following result is rather simple.

THEOREM 1. *A space H has at least one r.s. iff H is sequentially separable.*

Proof. Let H have a "universal" sequence $V=(v_k)_{k=1}^\infty$. We put $V_1=(v_k-v_l : k>l\ge 1)$, $V_2=V_1\cup V$. Arrange V_2 in the sequence $\{w_m\}_{m=1}^\infty=\{v_1,v_2-v_1,v_2,v_3-v_2,v_3-v_1,v_3,\ldots\}$. If $x\in H$, then there exists a subsequence $\{n_k\}_{k=1}^\infty$ of integers such that $n_k\uparrow\infty$,

$$x=\lim_{k\to\infty}v_{n_k}=v_{n_1}+(v_{n_2}-v_{n_1})+(v_{n_3}-v_{n_2})+\ldots=\sum_{m=1}^\infty\alpha_m w_m\,,$$

where the α_m are 0 or 1. Consequently, $(w_m)_{m=1}^\infty$ is an r.s. in H. Conversely, suppose that there exists an r.s. $(x_k)_{k=1}^\infty$ in H. If $\Phi=\mathbb{C}$, we put $D=\{\sum_{k=1}^n(\gamma_k+i\delta_k)x_k : n=1,2,\ldots\}$, where γ_k and δ_k run through all rational numbers (if $\Phi=\mathbb{R}$, we put $\delta_k=0$, $\forall k\ge1$). If we arrange the elements of D in a sequence, we obtain the required "universal" sequence in H.

Everywhere below we assume that the space H is sequentially separable. According to [3, 4] the sequence $(x_k)_{k=1}^\infty$ in a locally convex space (l.c.s.) H is said to be an *absolutely representing system* (a.r.s.) in H if each x in H can be represented by a series (1) absolutely converging in H. It is easy to see that if an l.c.s. H has at least one a.r.s., then H is separable. On the other hand, one can easily prove that if the topology of a separable l.c.s. H is separated and defined by a countable set of seminorms, then H has at least one a.r.s.

In [4] the second author posed the problem of connections between r.s. and a.r.s., and conjectured that in a nuclear Fréchet space each r.s. must be an a.r.s. In this article we show that the connection between these two classes is quite different from the connection between the corresponding classes of bases. In particular, the above mentioned conjecture is false. We first give some definitions. An l.t.s. H is said to be *weakly separated* if for any $x\ne0$ in H there exists $\varphi\in H'$ such that $\varphi(x)\ne0$ (i.e. the weak topology $\sigma(H,H')$ is separated).

A sequence $X=(x_k)_{k=1}^\infty$ is said to be a *finitely representing system* (f.r.s.) in a space H if $\operatorname{span}X=H$. Also, X is a *weak absolutely representing system* (w.a.r.s.) in H if for each x in H there exists a series $\sum_{k=1}^\infty a_kx_k$ such that for any φ in H' the series $\sum_{k=1}^\infty a_k\varphi(x_k)$ converges absolutely in Φ to $\varphi(x)$.

THEOREM 2. *If H is a sequentially separable weakly separated space, then either each r.s. in H is an f.r.s., or there exists an r.s. in H which is not a w.a.r.s.*

Proof. Let $Y=(y_k)_{k=1}^\infty$ be an r.s. in H such that $\operatorname{span}Y\ne H$. W.l.o.g. one can assume that $y_1\ne0$. There exists a functional $\mathcal{F}$ in H' such that $\mathcal{F}(y_1)=1$. We put

$$Lx=x-y_1\mathcal{F}(x)\,,\quad x\in H\,,$$
$$t_1=y_1\,,\quad t_k=Ly_k\,,\quad k=2,3,\ldots$$

For each x in H we have $x=\sum_{k=1}^\infty\beta_ky_k$, whence $Lx=\sum_{k=1}^\infty\beta_ky_k-y_1\sum_{k=1}^\infty\beta_k\mathcal{F}(y_k)=\sum_{k=2}^\infty\beta_kt_k$. This implies that $(t_k)_{k=2}^\infty$ is an r.s. in $LH:=\{w=Lx : x\in H\}$. Let y be any element in H. Then $y=[y-y_1\mathcal{F}(y)]+y_1\mathcal{F}(y)=Ly+y_1\mathcal{F}(y)$, and so $(t_k)_{k=1}^\infty$ is an r.s. in H.

Fix $x_0\in H\setminus\operatorname{span}Y$. For each $n\ge1$ there exists $f_n\in H'$ such that $f_n(x_0)=1$, $f_n(t_k)=0$, $1\le k\le n$. We put

$$u_1=\frac{t_1}{2}\,,\quad u_k=\frac{2^{-k}t_k}{1+\max\limits_{1\le j\le k}|f_j(t_k)|}\,,\quad k=2,3,\ldots;$$
$$x_1=u_1\,,\quad x_k=u_1+u_k\,,\quad k=2,3\ldots$$

Note that $(u_k)_{k=1}^\infty$ is an r.s. in H.

Now let $\Gamma=(\gamma_m)_{m=1}^\infty$ be a sequence in H consisting only of elements $(x_m)_{m=1}^\infty$ and such that for any $k\ge1$ there exists an infinite subsequence $(\gamma_{k_m})_{m=1}^\infty$ with $\gamma_{k_m}=x_k$, $m=1,2,\ldots$ (in other words, each x_k appears among the γ_m infinitely many times). We shall show that Γ is an r.s. in H. If $x\in H$, we have $x=\sum_{k=1}^\infty b_ku_k$ where the series converges in H. We choose integers n_k such that

$$\lim_{k\to\infty}b_k/n_k=0\,. \tag{2}$$

Consider the series

$$\begin{aligned}b_1x_1+\frac{b_2}{n_2}x_2-\frac{b_2}{n_2}x_1+\ldots+\frac{b_2}{n_2}x_2-\frac{b_2}{n_2}x_1\\+\frac{b_3}{n_3}x_3-\frac{b_3}{n_3}x_1+\ldots+\frac{b_3}{n_3}x_3-\frac{b_3}{n_3}x_1+\ldots,\end{aligned} \tag{3}$$

where each difference $(b_k/n_k)x_k-(b_k/n_k)x_1$ is repeated n_k times. If S_m is the mth partial sum of this series, then $S_{1+n_2+\ldots+n_k}=\sum_{l=1}^kb_lu_l$. By (2) the series (3) converges to x. This implies that Γ is an r.s. in H.

Suppose now that there exists a sequence $(c_k)_{k=1}^\infty$ such that

$$\forall f\in H'\quad f(x_0)=\sum_{k=1}^\infty c_kf(\gamma_k)\,,\quad \sum_{k=1}^\infty|c_k|\,|f(\gamma_k)|<\infty\,.$$

Since $\mathcal{F}(\gamma_m)=\frac12$, $\forall m\ge1$, we have $\sum_{k=1}^\infty|\mathcal{F}(\gamma_k)|\,|c_k|=\frac12\sum_{k=1}^\infty|c_k|=A<\infty$. For every $m\ge1$ we put $N_m=\{k\ge1 : \gamma_k=x_m\}$. If $j>1$, we have

$$f_j(x_0)=\sum_{k=1}^{\infty}c_kf_j(\gamma_k)=\sum_{m=1}^{\infty}f_j(x_m)\sum_{k\in N_m}c_k\,,$$

whence

$$1=|f_j(x_0)|\le 2A\sum_{m=j+1}^{\infty}|f_j(u_m)|\le 4A\cdot 2^{-j}<1$$

for j sufficiently large. This contradiction implies that Γ is not a w.a.r.s. in H.

COROLLARY. *Each sequentially separable weakly separated space with an uncountable Hamel basis has an r.s. which is not a w.a.r.s.*

We call a sequence $X=(x_k)_{k=1}^{\infty}$ a *weak representing system* (w.r.s.) in H if for each x in H there exists a series $\sum_{k=1}^{\infty}c_kx_k$ which converges weakly to x : $\forall f\in H'$, $f(x)=\sum_{k=1}^{\infty}c_kf(x_k)$. Let $W=(w_k)_{k=1}^{\infty}$ be any part of the system Γ constructed above such that W contains each x_k at most once (or does not contain it at all). Suppose that W is a w.r.s. in H. Then for every $k\ge 1$ there is a $d_k\in\mathbb{C}$ such that for every $f\in H'$, $f(x_0)=\sum_{k=1}^{\infty}d_kf(w_k)$ (the series converges in $\mathbb{C}$). Since $\mathcal{F}(x_0)=\frac12\sum_{k=1}^{\infty}d_k$, the series $\sum_{k=1}^{\infty}d_k$ converges (in $\mathbb{C}$). For each $k>1$ we have $f_k(x_0)=\sum_{j=1}^{\infty}d_jf_k(x_{n_j})=\sum_{n_j>k}d_jf_k(x_{n_j})$, whence

$$|f_k(x_0)|=1\le\sum_{n_j>k}|d_j|2^{-n_j}\le\max\{|d_j|\,:\,n_j>k\}=\alpha_k\,,$$

a contradiction since $\lim_{k\to\infty}\alpha_k=0$. So W cannot be a w.r.s. in H. We have thus obtained the following complement to Theorem 2:

THEOREM 3. *If H is a sequentially separable weakly separated space, then either each r.s. in H is an f.r.s., or there exists an r.s. in H which is not a w.a.r.s. in H and none of whose linearly independent parts is a w.r.s. in H.*

COROLLARY. *Each sequentially separable weakly separated space with an uncountable Hamel basis has an r.s. none of whose subsystems is a weak basis (all the more, a basis).*

Now consider the case of a space H with a countable Hamel basis. The Hamel basis is then an f.r.s. in H. However, this does not imply that any other r.s. in H is also an f.r.s. It is easily seen that any space with a countable Hamel basis is sequentially separable. The following auxiliary result will be useful later.

THEOREM 4. *Let H be a weakly separated space with a countable Hamel basis. Suppose that there exists a linearly independent sequence $X=(x_k)_{k=1}^{\infty}$ such that $\sum_{k=2}^{\infty}x_k=0$. Then H has an r.s. which is not a w.a.r.s.*

Proof. Let Q be a countable Hamel basis containing the sequence X (such a basis always exists). Let $Y=(y_k)_{k=1}^{\infty}$ be the sequence of all elements of Q which do not belong to X. We put $v_{2n-1}=y_n$, $v_{2n}=x_{n+1}+2^{-n}x_1$, $V=(v_n)_{n=1}^{\infty}$. We shall show that V is an r.s. in H. If y is any element in H, then $y=\sum_{k=1}^{N_1}t_ky_k+\sum_{m=1}^{N_2}r_mx_m$. We put $r_k=0$ for all $k>N_2$. Then

$$\begin{aligned}\sum_{k=1}^{N_1}t_ky_k&+\sum_{k=1}^{\infty}(x_{k+1}+2^{-k}x_1)\Big(r_1+r_{k+1}-\sum_{j=2}^{N_2}2^{1-j}r_j\Big)\\&=\sum_{k=1}^{N_1}t_ky_k+\sum_{k=1}^{\infty}x_{k+1}\Big(r_1-\sum_{j=2}^{N_2}2^{1-j}r_j\Big)+x_1r_1+\sum_{k=1}^{\infty}x_{k+1}r_{k+1}\\&+x_1\Big(\sum_{k=1}^{\infty}2^{-k}r_{k+1}-\sum_{k=1}^{N_2-1}2^{-k}r_{k+1}\Big)=\sum_{k=1}^{N_1}t_ky_k+\sum_{k=1}^{N_2}r_kx_k=y\,.\end{aligned}$$

Thus V is an r.s. in H. Note that $x_1\notin\operatorname{span}W$, where $W=(x_k:k\ge 2)\cup(y_k\,:\,k\ge 1)$. Assume that $x_1\in\operatorname{span}V$. Then

$$x_1=\sum_{k=1}^{n_1}a_ky_k+\sum_{k=2}^{n_2}b_k(x_k+2^{-k+1}x_1)\,,$$

whence

$$x_1\Big(1-\sum_{k=2}^{n_2}b_k2^{-k+1}\Big)=\sum_{k=1}^{n_1}a_ky_k+\sum_{k=2}^{n_2}b_kx_k\,.$$

Consequently, $\sum_{k=2}^{n_2}b_k2^{-k+1}=1$ and $a_k=0$ for $k\ge 1$, $b_j=0$ for $j\ge 2$, which is a contradiction. This implies that $\operatorname{span}V\ne H$, and according to Theorem 2 there exists an r.s. in H which is not a w.a.r.s. in H.

The topology of a space H is called *regular* if each sequence in H tending to zero lies in a finite-dimensional space.

THEOREM 5. *Let H be a weakly separated space with a countable Hamel basis and with regular topology. Then each r.s. in H is an f.r.s.*

Proof. Let $(h_k)_{k=1}^{\infty}$ be a countable Hamel basis in H and let $(g_k)_{k=1}^{\infty}$ be an r.s. in H. If x_0 is any element in H, then there exists a series $\sum_{k=1}^{\infty}c_kg_k$ converging to x_0 in H. In particular, $\lim_{k\to\infty}c_kg_k=0$, whence $\exists N<\infty$: $\forall n\ge 1\,,\ c_ng_n\in\Phi_N:=\operatorname{span}(h_k)_{k=1}^{N}$.

We put $M=\{k\ge 1\,:\,g_k\in\Phi_N\}$. Then $x_0=\sum_{k\in M}c_kg_k$, and so $x_0\in\operatorname{span}(g_k)_{k=1}^{\infty}$ if M is a finite set.

Assume now that M is infinite. In the set $E=\{g_k:k\in M\}$ one can find linearly independent elements $g_1^0,g_2^0,\ldots,g_r^0$ (with $r\le N$) which form a base in $\operatorname{span}E$. This implies that for any m in M we have $g_m=\sum_{j=1}^{r}d_{m,j}g_j^0$. Let $(\varphi_j)_{j=1}^{r}$ be the system of continuous linear functionals on H, biorthogonal to $(g_m^0)_{m=1}^r$. For any $s \le r$, $\varphi_s(x_0) = \sum_{k\in M} c_k d_{k,s}$ and the series converges for all such s. Moreover,

$$x_0 = \sum_{k\in M} c_k \sum_{m=1}^{r} d_{k,m} g_m^0 = \sum_{m=1}^{r} \Big(\sum_{k\in M} d_{k,m} c_k\Big) g_m^0 .$$

This implies that $(g_k)_{k=1}^\infty$ is an f.r.s. in H.

THEOREM 6. *Let H be a weakly separated space with a countable Hamel basis and with irregular topology. Then H has an r.s. which is not a w.a.r.s.*

Proof. Let $(g_k)_{k=1}^\infty$ be a Hamel basis in H. For $n \ge 1$ we put $\Phi_n = \operatorname{span}(g_k)_{k=1}^\infty$. Since the topology of H is irregular, there exists a sequence $(w_k)_{k=1}^\infty$ tending to zero in H such that $w_k \in \Phi_{n_{k+1}} \setminus \Phi_{n_k}$ for any $k \ge 1$, where $n_k \uparrow \infty$. We put $v_1 = w_2$, $v_k = w_{k+1} - w_k$, $k = 2, 3, \ldots$ The system $V = (v_k)_{k=1}^\infty$ is linearly independent and $\sum_{k=1}^\infty v_k = 0$. Suppose that $w_1 \in \operatorname{span} V : w_1 = \sum_{k=1}^p \alpha_k v_k$. If $p_0 = \max\{k : 1 \le k \le p,\, \alpha_k \ne 0\}$, then $\alpha_{p_0} v_{p_0} = w_1 - \sum_{k=1}^{p_0-1} \alpha_k v_k$, whence $w_{p_0+1} \in \operatorname{span}\{w_s\}_{s=1}^{p_0}$. This contradicts the choice of $(w_k)_{k=1}^\infty$. We have thus shown that the system $\{w_1, v_1, v_2, \ldots\}$ is linearly independent.

According to Theorem 4 there exists an r.s. in H which is not a w.a.r.s.

It should be mentioned that under slightly different assumptions Theorem 6 was proved independently by S. N. Melikhov who also obtained a criterion for the regularity of the topology.

COROLLARY. *If H is a weakly separated space with a countable Hamel basis, then the following assertions are equivalent*:

1) *each r.s. in H is an f.r.s.*;
2) *each r.s. in H is a w.a.r.s.*;
3) *the topology of H is regular.*

In conclusion, we formulate the results for the most important class of linear topological spaces—the separated locally convex spaces. Since the latter are always weakly separated, directly from the previous theorems we obtain the following result:

THEOREM 7. 1. *If H is a sequentially separable separated locally convex space with an uncountable Hamel basis, then there exists a representing system in H which is not a weak absolutely representing system in H and no linearly independent part of it is a weak representing system* (*all the more, a basis*) *in H.*

2. *If H is a locally convex space with a countable Hamel basis, then the following assertions are equivalent*:

1) *each representing system in H is a finitely representing system*;

2) *each representing system in H is a weak absolutely representing system*;
3) *the topology of H is regular.*

The main results of this paper were published (partly without proofs) in [2, 5]. Those two papers also contain the history of the investigations on the subject of this article.

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DEPARTAMENT OF MATHEMATICS
KHARKOV STATE UNIVERSITY
310000 KHARKOV, UKRAINE

DEPARTMENT OF MATHEMATICS
ROSTOV STATE UNIVERSITY
344711 ROSTOV-NA-DONU, RUSSIA

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