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Banach spaces and bilipschitz maps

by

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Abstract. We show that a normed space E is a Banach space if and only if there is no bilipschitz map of E onto $E \setminus \{0\}$.

1. Introduction. A map $f : X \to Y$ between metric spaces X and Y is *bilipschitz* if there is a number $M \geq 1$ such that

$$|x-y|/M \leq |f(x) - f(y)| \leq M|x-y|$$

for all $x, y \in X$. We also say that f is *M-bilipschitz*. The inverse $f^{-1} : fX \to X$ of an M-bilipschitz map is also M-bilipschitz. A bilipschitz map preserves Cauchy sequences and maps complete sets onto complete sets. In particular, if E and E' are Banach spaces and if $f : E \to E'$ is bilipschitz, then fE is closed in E'. Hence f cannot map a Banach space E onto an open proper subset of E. The purpose of this note is to show that this property characterizes the Banach spaces in the class of all normed vector spaces. We formulate the result below; for some variations see Remark 6.

2. THEOREM. *A normed space E (real or complex) is a Banach space if and only if there is no bilipschitz map of E onto $E \setminus \{0\}$.*

3. *Notation.* The norm of a vector $x \in E$ is written as $|x|$. We let $B(x, r)$ and $\overline{B}(x, r)$ denote the open and the closed ball in E, respectively, with center x and radius r. The boundary sphere $\partial B(x, r)$ is written as $S(x, r)$.

The proof of Theorem 2 will be based on the following elementary construction:

4. LEMMA. *Let $a, b \in E$, and let $r \geq 2|a-b|$. Then there is a homeomorphism $h : E \to E$ such that*

(1) $h(a) = b$,
(2) $h(x) = x$ *if* $|x - a| \geq r$,
(3) h *is M-bilipschitz with* $M = 1 + 2|a-b|/r$.

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Proof. Geometrically, $h|\overline{B}(a,r)$ is the cone of the identity map of $S(a,r)$ with a and b as the coning points. This means that for $y \in S(a,r)$, h maps the line segment $[a,y]$ affinely onto $[b,y]$. The map $h|\overline{B}(a,r)$ has the explicit expression

$$h(x) = x + (1 - |x-a|/r)(b-a).$$

For $|x-a| \geq r$ we set $h(x) = x$. Elementary estimates give

$$(1 - |a-b|/r)|x-y| \leq |h(x) - h(y)| \leq (1 + |a-b|/r)|x-y|$$

for $x, y \in \overline{B}(a,r)$. Since $1/(1-t) \leq 1 + 2t$ for $0 \leq t \leq 1/2$, it follows that $h|\overline{B}(a,r)$ is M-bilipschitz with M as in (3). This and the convexity of E imply that h is M-bilipschitz in E.

5. Proof of Theorem 2. As explained in the introduction, a bilipschitz image of a Banach space E is always complete, and hence it cannot be $E \setminus \{0\}$. Conversely, assume that E is a noncomplete normed space. We shall prove the stronger result that given $M > 1$, there is a surjective M-bilipschitz map $f : E \setminus \{0\} \to E$ such that $f(x) = x$ for $|x| \geq 1$. This map will be constructed as the limit of a sequence $h_j : E \to E$.

Choose a sequence of numbers $t_1, t_2, \ldots$ such that $0 < t_j \leq 1$ and such that the infinite product of all numbers $M_j = 1 + t_j$ is at most M. Let $a_1, a_2, \ldots$ be a nonconvergent Cauchy sequence in E. Passing to a subsequence, we may assume that $|a_{j+1} - a_j| \leq 2^{-j} t_j$ for all j. Replacing a_j by $a_j - a_1$ we can further assume that $a_1 = 0$. Writing

$$B_j = B(a_j, 2^{1-j})$$

we obtain a decreasing sequence of balls $B(0,1) = B_1 \supset B_2 \supset \ldots$ Applying Lemma 4 we find homeomorphisms $h_j : E \to E$ such that

(1) $h_j(a_j) = a_{j+1}$,
(2) $h_j(x) = x$ if $x \notin B_j$,
(3) h_j is M_j-bilipschitz.

Let $g_j : E \to E$ be the composite homeomorphism $h_j \circ \ldots \circ h_1$. Then $g_j(0) = a_{j+1}$, $g_j(x) = x$ outside B_1, and g_j is M-bilipschitz.

Suppose that $x \in E$ with $|x| \geq 2^{-k} M$. Then $|g_k(x) - a_{k+1}| \geq 2^{-k}$, which means that $g_k(x) \notin B_{k+1}$. It follows that $g_j(x) = g_k(x)$ for all $j \geq k$. Hence the sequence $g_j(x)$ converges to $g_k(x)$. Consequently, the limit

$$f(x) = \lim_{j\to\infty} g_j(x)$$

exists for all $x \in E \setminus \{0\}$. Since each g_j is M-bilipschitz, so is $f : E \setminus \{0\} \to E$. Moreover, $f(x) = x$ for $|x| \geq 1$.

It remains to show that f is a map onto E. Let $y \in E$. Since the sequence (a_j) has no convergent subsequence, there is a number $s > 0$ and a positive integer n such that $|a_j - y| \geq s$ for $j \geq n$. Choose $k \geq n$ with $2^{1-k} \leq s$, and write $x = g_k^{-1}(y)$. Then $y \notin B_k$, and hence $g_j(x) = y$ for all $j \geq k$. This implies that $f(x) = y$ and completes the proof of Theorem 2.

6. Remark. There are plenty of variations of Theorem 2. As noted in the proof, the map $E \to E \setminus \{0\}$ can be chosen to be M-bilipschitz with an arbitrary $M > 1$ and to be the identity outside the unit ball. In the other direction, we may replace the class of bilipschitz maps by any larger class of maps which preserve the completeness of a set. For example, we can take the continuous maps satisfying only the inequality

$$|f(x) - f(y)| \geq |x-y|/M. \tag{6.1}$$

Replacing f by Mf we see that this inequality can also be replaced by

$$|f(x) - f(y)| \geq |x-y|. \tag{6.2}$$

Another choice is the class of quasisymmetric maps (see [1, 2.24]). These observations are summarized as follows:

Let i and j be integers with $1 \leq i \leq 5$ and $1 \leq j \leq 3$. Then the following statement is true:

A normed space E is not a Banach space if and only if E can be mapped by some continuous map satisfying the condition A_i onto some set $F \subset E$ satisfying the condition B_j. Here A_i and B_j are given by the following lists:

A_1. f is M-bilipschitz with a given $M > 1$, and $f(x) = x$ for $|x| \geq 1$,
A_2. f is bilipschitz,
A_3. (6.1),
A_4. (6.2),
A_5. f is quasisymmetric;
B_1. $F = E \setminus \{0\}$,
B_2. F is open and $F \neq E$,
B_3. F is not closed.

7. Remark. Restricting the map $f : E \setminus \{0\} \to E$ of the proof of Theorem 2, we obtain a bilipschitz homeomorphism $f_1 : B_1 \setminus \{0\} \to B_1$, $B_1 = B(0,1)$. In the terminology of [3], f_1 is not quasihyperbolic or even solid. Hence Theorem 4.8 of [3] is not true in noncomplete normed spaces. It seems to the author that the theory of free quasiconformality in Banach spaces, initiated in [3], has no useful extension to arbitrary normed spaces.

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On linear operators having supercyclic vectors

by

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Abstract. We show that for a real separable Banach space X there are operators in $B(X)$ having supercyclic vectors if and only if $\dim X \le 2$ or $\dim X = \infty$.

1. Introduction. Let $(X, \|\cdot\|)$ be a real (or complex) Banach space and $B(X)$ the set of linear continuous mappings from X into itself. Let $T \in B(X)$. A vector $x \in X$ is called (a) *cyclic*, (b) *supercyclic*, (c) *hypercyclic* if the orbit

$$\mathrm{Orb}(T, x) := \{T^n x : n \in \mathbb{N}_0\}$$

satisfies

(a) $\overline{\mathrm{span}(\mathrm{Orb}(T, x))} = X$,

(b) $\overline{\{\lambda y : y \in \mathrm{Orb}(T, x),\ \lambda \in \mathbb{R}(\mathbb{C})\}} = X$,

(c) $\overline{\mathrm{Orb}(T, x)} = X$

(see [5]).

As far as we know it is still an open problem whether there is an operator with hypercyclic vectors in every separable infinite-dimensional Banach space, and it is well known that there are none in finite dimensions (see [8]). In this paper we will characterize those separable Banach spaces which have operators with supercyclic vectors. Of course, a Banach space having such operators is separable. The main result is:

THEOREM 1. *Let $(X, \|\cdot\|)$ be a real separable Banach space. Then there exist operators in $B(X)$ having supercyclic vectors if and only if*

$$\dim X \in \{0, 1, 2\} \quad \textit{or} \quad \dim X = \infty.$$

To prove Theorem 1 we will use methods of the theory of universal functions developed by K.-G. Große-Erdmann [4].

For further properties of the operator classes defined above compare, e.g., [1], [2], [5], [6] and [8].

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