ACTA ARITHMETICA
LXIII.1 (1993)

Metric properties of generalized Cantor products

by

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0. Introduction. Generalized Cantor products are algorithms that give a representation of real numbers $x \in [0,1[$ as infinite products of rational ones. They have been developed in [Opp] first. Let us present those we shall consider from the metric point of view in this paper.

The letter k shall denote an integer ≥ 1. For any $x \in [0,1[$, let $r_0(x) \in \mathbb{N}$ and $T(x) \in [0,1[$ be defined by

$$(1)\qquad \frac{r_0(x)-1}{r_0(x)+k-1} \leq x < \frac{r_0(x)}{r_0(x)+k}, \qquad T(x) := x\left(\frac{r_0(x)+k}{r_0(x)}\right).$$

One can see that $r_0(x) = [kx/(1-x)]+1$. Define, for any real number $z \geq 1$,

$$(2)\qquad \begin{aligned} a_z &= (z-1)/(z+k-1)\,, \\ b_z &= a_z/a_{z+1} = a_{(z-1)(z+k)+1}\,, \\ J_z &= [a_z, a_{z+1}[\,. \end{aligned}$$

The sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are strictly increasing from 0 to 1. By definitions we have $\bigcup_{n\geq 1} J_n = [0,1[$, $J_n \cap J_m = \emptyset$ if $n \neq m$ and $T(x) = x a_{n+1}^{-1}$ on J_n. Moreover,

$$T(J_n) = [b_n, 1[\,.$$

Thus, according to the terminology of F. Schweiger (see [Sch]), the triple $(T, [0,1[, (J_n)_{n\geq 1})$ is a measurable fibered system on $[0,1[$ with the Borel σ-algebra B.

Given $k \geq 1$ and $x \in [0,1[$, we define the sequence $(r_t(x))_{t\geq 0}$ as follows:

$$(3)\qquad r_t(x) = r_0(T^{(t)}(x))\,,$$

where $T^{(t)}$ denotes the tth iterate of T ($T^{(0)} = \mathrm{Id}_{[0,1[}$).

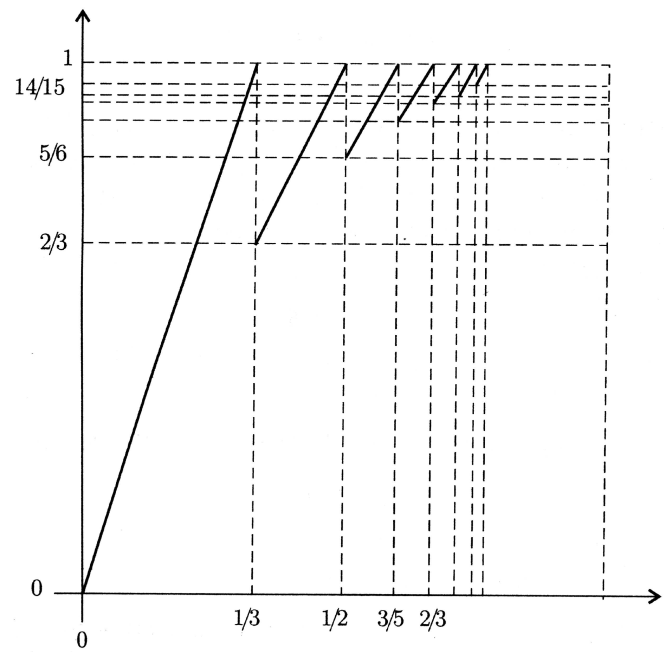
Research partially supported under DRET contract 901636/A000/DRET/DS/SR.

W. Sierpiński ([Sie-1]) and A. Oppenheim ([Opp]) showed that for any integer $k \geq 1$ and any $x \in [0, 1[$, with (3),

$$x = \prod_{i=0}^{\infty} \frac{r_i(x)}{r_i(x) + k}\,. \tag{4}$$

The case $k = 1$ corresponds to Cantor's product (see [Per]). Generalizations of Cantor's product given in [Kn-Kn] do not overlap with those from [Sie-1] or [Opp], and do not arise from fibered systems on $[0, 1[$.

Graph of T for $k = 2$



Euler's formula (see [MF-VP]) and Escott's formula ([Esc], [Sie-2])

$$\sqrt{\frac{x-1}{x+1}} = \prod_{n=0}^{\infty} \frac{\varphi^{(n)}(x)}{\varphi^{(n)}(x) + 1}, \qquad \sqrt{\frac{x-2}{x+2}} = \prod_{n=0}^{\infty} \frac{\gamma^{(n)}(x-1)}{\gamma^{(n)}(x-1) + 2}\,,$$

where $\varphi(x) = 2x^2 - 1$ and $\gamma(z) = z^3 + 3z^2 - 2$, both give product expansions for integer x (with $k = 1$ or $k = 2$). Some other formulas can be derived from the work of Ostrowski [Ost] (see also [MF-VP]). P. Stambul ([Sta]) points out the following Cantor product expansion

$$\sqrt{2} - 1 = \prod_{n=0}^{\infty} \frac{\varphi^{(n)}(1)}{\varphi^{(n)}(1) + 1}\,,$$

where $\varphi(x) = 4x^2 - 1 + 2x\sqrt{2x^2 - 1}$ is not a polynomial. Thus, quadratic irrationals in $[0,1[$ are not characterized by the fact that their sequence of digits for the Cantor product has ultimately polynomial growth (cf. [Eng]).

In Section 1 we give some preliminary notations for cylinder sets and describe admissible sequences of digits $r_n(x)$ which occur in the product formula (4).

Our purpose is to study, as has been done for several other fibered systems (e.g. continued fractions in [Khi]), the metric properties of the system $(T,[0,1[,B)$. The motivation for this is that in the case of continued fractions, the asymptotic behaviour for the relevant sequence of digits was deduced from the identification of the density $1/(\log 2\cdot(1+x))$ for a Lebesgue-continuous ergodic invariant measure on $[0,1]$, for the transformation

$$x\mapsto \frac{1}{x}-\left[\frac{1}{x}\right]\quad \text{if } x\neq 0,\qquad \text{and}\qquad 0\mapsto 0$$

(see [Khi] or [Sch]).

But it appears, in Section 2, that the only probability invariant measure for T is the Dirac measure at 0, and that all σ-finite λ-continuous invariant measures for T are determined by their restrictions to wandering sets for T. Therefore, it should be the case that T is not ergodic with respect to λ.

However, in Section 3, in analogy with what happens in the case of Sylvester's series (see [Ver], [Sch]), and in some sense quite in contrast to what occurs for continued fractions, it appears that the limit function

$$\beta(x)=\lim_{n\to\infty}\frac{\log r_n(x)}{2^n}$$

exists λ-a.e., which enables us to conclude the nonergodicity of T with respect to λ. The limit function β should be proved to have most of the properties the relevant one for Sylvester's series was proved to have in [Go-Sm], where it essentially was providing the first explicitly defined function having jointly continuous occupation density (see also [Gal]).

Finally, in Section 4, we introduce the sequence of random variables $(t_n(\cdot))_{n\geq 0}$ defined on $[0,1[$ by

$$t_n(x)=\frac{T^{(n+1)}(x)-b_{r_n(x)}}{1-b_{r_n(x)}},\qquad x\in[0,1[,\ n\geq 0\,.$$

We show, using a modified version of a theorem of W. Philipp ([Phi]) in [Sch], Chapter 11, that λ-a.e., the sequence $(t_n(x))_{n\geq 0}$ is completely uniformly distributed modulo 1 (see [Ku-Ni]). This generalizes some similar uniform distribution for Sylvester's series, or Engel's series, proved in [Sch-1].

The author would like to express his thanks to Professors J. P. Allouche, P. Liardet, F. Schweiger, B. Host, and to the referee, for valuable discussions or useful remarks.

1. Admissible sequences of digits. From [Sie-1] and the definition of T one has

$$x = \prod_{i=0}^{\infty} \frac{r_i(x)}{r_i(x)+k}, \qquad T^{(n+1)}(x) \in [b_{r_n}, 1[\,, \tag{5}$$

with

$$T^{(n)}(x) \in \left[\frac{r_n - 1}{r_n + k - 1}, \frac{r_n}{r_n + k}\right[\quad \text{and} \quad r_n = r_n(x)\,.$$

This will be called the T-*expansion of* x.

Take 1 as the value of the empty product, and let $n \geq 0$. One has

$$0 < \prod_{j=0}^{n} \frac{r_j(x)}{r_j(x)+k} - x < \left(\prod_{j=0}^{n-1} \frac{r_j(x)}{r_j(x)+k}\right)\left(\frac{r_n(x)}{r_n(x)+k} - \frac{r_n(x)-1}{r_n(x)+k}\right)$$
$$< \frac{k}{(r_n(x)+k)(r_n(x)+k-1)}\,.$$

Let n be an integer ≥ 1 and let $r := (r_0, \ldots, r_{n-1}) \in \mathbb{N}^{*n}$. The set

$$B(r) := J_{r_0} \cap T^{-1}(J_{r_1}) \cap \ldots \cap T^{(-n+1)}(J_{r_{n-1}})$$

is said to be a *cylinder set of rank* n if it is not empty. For $r = (r_0, \ldots, r_{n-1}) \in \mathbb{N}^n$ (respectively $p = (p_i)_{i \geq 0}$) and $j \in [0, n]$ (resp. $j \geq 0$), define

$$\Pi_j(r) := \prod_{i=0}^{j-1} \frac{r_i}{r_i + k} \quad \left(\text{resp. } \Pi_j(p) := \prod_{i=0}^{j-1} \frac{p_i}{p_i + k}\right). \tag{6}$$

If $B(r)$ is a cylinder set of rank n we easily get from (1), (2) and (5),

$$B(r) = [\Pi_n(r) b_{r_{n-1}}, \Pi_n(r)[\,. \tag{7}$$

Definition 1.1. An n-uple $r = (r_0, \ldots, r_{n-1})$ (resp. a sequence $p = (p_m)_{m \geq 0} \in \mathbb{N}^{\mathbb{N}}$) is said to be a T-*admissible* n-*uple* (resp. *sequence*) *of digits* if $B(r) \neq \emptyset$ (resp. $B(p_0, \ldots, p_{n-1}) \neq \emptyset$ for all $n \geq 1$). The set of T-admissible n-uples will be denoted by A_n.

From (5), p is a T-admissible sequence of digits if and only if for all $n \geq 0$, one has $[b_{p_n}, 1[\cap J_{p_{n+1}} \neq \emptyset$.

Proposition 1.1. *A sequence $p = (p_n)_{n \geq 0}$ of natural numbers is a T-admissible sequence of digits if and only if for all $n \geq 0$ one has*

$$p_{n+1} \geq p_n^2 + (p_n - 1)(k-1) \ (\geq p_n^2)\,.$$

Proof. Since b_r has the form $a_{(r-1)(r+k)+1}$, an admissible sequence $(p_n)_{n \geq 0}$ is characterized by the inequalities $b_{p_n} < a_{p_{n+1}+1}$, $n \geq 0$. In other words,

$$\frac{(p_n - 1)(p_n + k)}{(p_n - 1)(p_n + k) + k} < \frac{p_{n+1}}{p_{n+1} + k}\,.$$

After simplification, we get the desired inequality. ∎

Remark 1.1. Let $p(\cdot)$ be the polynomial $p(x) := x^2 + (x-1)(k-1)$. From (2) we have $a_n = a_{n+1}a_{p(n)} = a_{n+1}a_{p(n)+1}a_{p^2(n)}$. Hence by induction we obtain the following product formula:

$$\frac{n-1}{n-1+k} = \prod_{j=1}^{\infty} \frac{p^{(j)}(n)}{p^{(j)}(n)+k}\,. \tag{8}$$

According to Proposition 1.1, formula (8) gives the T-expansion of $(n-1)/(n-1+k)$ for $n \in \mathbb{N}$ (this was known from [Opp]). However, formula (8) holds for all real numbers $k \ge 1$ and $n \ge 1$.

2. Invariant measures. The transformation T is such that $T(0) = 0$ and if $x \in]0,1[$, the sequence $(T^{(n)}(x))_{n\ge 0}$ is strictly increasing to 1. Thus, from the Riesz representation theorem and the individual ergodic theorem, using Cesàro means, taking any generic point for μ if μ is an ergodic invariant probability measure, one can see that necessarily, for any $f \in \mathcal{C}(\mathbb{R}/\mathbb{Z})$, $\int f d\mu = \lim_{x\to 1^-} f(x)$: since $T(0) = 0$ is the only fixed point for T, one must have $\mu = \delta_0$, where δ_0 denotes the Dirac measure at point 0.

Remark 2.1. It is more interesting to consider probability measures μ which are quasi-invariant under T, that is to say, μ is equivalent to $\mu\circ T^{-1}$. We give an example of such a measure which is discrete. Let β_j, $j \in \mathbb{Z}$, be the points in $[0,1[$ (identified with $\mathbb{R}/\mathbb{Z}$) given by

$$\beta_n := \frac{p^{(n)}(2)-1}{p^{(n)}(2)-1+k} \quad \text{and} \quad \beta_{-n} = (k+1)^{-n-1}$$

for $n = 0, 1, 2, \ldots$ By (5) and (8) one has

$$T^{(n)}\left(\frac{1}{k+1}\right) = \prod_{j=0}^{\infty} \frac{p^{(j)}(p^{(n)}(2))}{p^{(j)}(p^{(n)}(2))+k} \quad \text{for } n \ge 0$$

and

$$T((k+1)^{-(m+1)}) = (k+1)^{-m} \quad \text{for } m \ge 1\,.$$

Hence $T(\beta_n) = \beta_{n+1}$ for all $n \in \mathbb{Z}$. Let δ_a denote the Dirac measure at a; then $\delta_{b_n} \circ T^{-1} = \delta_{b_{n+1}}$. This proves that the probability measure $\mu := \frac{1}{3}\sum_{n\in\mathbb{Z}} 2^{-|n|}\delta_{\beta_n}$ is quasi-invariant under T.

Now let us look at σ-finite λ-continuous invariant measures. Let U be any proper neighbourhood of 1, e.g. take $U = [a,1]$, $0 < a < 1$, and extend T from $[0,1[$ to the 1-torus $[0,1]$ setting $T(1) = 1 = 0$. Let $V = T^{-1}(U)\setminus U$. Then define $V_n = T^{(n)}(V)$, $n \in \mathbb{Z}$. It is a so called *wandering set*; indeed, using the fact that the sequence $(T^{(n)}(U))_{n\in\mathbb{Z}}$ is decreasing, one has

$$\bigcup_{n=-\infty}^{\infty} V_n = [0,1] \quad \text{and} \quad V_n \cap V_m = \emptyset \quad \text{for } m \ne n\,. \tag{9}$$

Now assume we want to determine the density for a σ-finite T-invariant λ-continuous measure. Then if we take any positive, measurable and σ-finite function on V, we can define it on any V_n, taking its image via $T^{(n)}$, and finally we obtain a σ-finite density for a T-invariant λ-continuous measure (use (9)). For example, take $a = (k+2)/2(k+1)$; then

$$V = \left[\frac{k+2}{2(k+1)^2}, \frac{k+2}{2(k+1)}\right[.$$

3. Nonergodicity of T with respect to λ, and asymptotic behaviour of $(r_n(x))_{n\geq 0}$

LEMMA 3.1. *There are two positive constants d_1 and d_2 such that for any nonempty cylinder set $B(r_0, \ldots, r_{n-1})$ of rank $n \geq 1$ and for any integers w, j ($w \geq j \geq 1$) such that $B(r_0, \ldots, r_{n-1}, j, w)$ is a nonempty cylinder set of rank $n+2$ one has*

$$d_1 \frac{j^2}{w^2} \leq \frac{\lambda(B(r_0, \ldots, r_{n-1}, j, w))}{\lambda(B(r_0, \ldots, r_{n-1}, j))} \leq d_2 \frac{j^2}{w^2}.$$

Proof. Put $B = B(r_0, \ldots, r_{n-1}, j, w)$, $A = B(r_0, \ldots, r_{n-1}, j)$ and $P = \Pi_n(r)$ for short, where $r = (r_0, \ldots, r_{n-1})$ (cf. (6)). Then, with (7),

$$\lambda(A) = P\frac{k}{(j+k)(j+k-1)}, \qquad \lambda(B) = P\frac{jk}{(j+k)(w+k)(w+k-1)}.$$

Therefore,

$$\frac{\lambda(B)}{\lambda(A)} = \frac{j(j+k-1)}{(w+k)(w+k-1)},$$

and the inequalities of the lemma follow with constants (for example) $d_1 = (k^2+k)^{-1}$ and $d_2 = k$. ∎

LEMMA 3.2. *The limit function $\beta(x) := \lim_{n\to\infty}(\log r_n(x))/2^n$ exists λ-a.e. Moreover, $\beta(\cdot)$ is measurable and there exists a constant $\gamma > 0$ such that for all $j \geq 1$, $n \geq 0$ and all $\varepsilon > 0$ one has*

$$(10)\qquad \begin{cases} \lambda(\{x : r_n(x) = j \text{ and } 0 \leq \beta(x) - 2^{-n}\log j \leq \varepsilon\}) \\ \qquad\qquad \geq \left(1 - \dfrac{2}{e^{\gamma\varepsilon 2^n} - 1}\right)\lambda(\{r_n = j\}), \\ \beta(x) = \dfrac{1}{2}\left(\log r_1(x) + \displaystyle\sum_{n=0}^{\infty} \frac{\log(r_{n+1}(x)/r_n(x)^2)}{2^n}\right) \quad \lambda\text{-a.e.} \end{cases}$$

Proof. The second part of formula (10) is obvious, provided the λ-a.e. existence of the limit function β is known.

Let $\varepsilon > 0$ and for $x \in [0,1[$ define $\beta_n(x) := 2^{-n}\log r_n(x)$. Since $r_{n+1}(x) \geq r_n(x)^2$, the sequence $(\beta_n(x))_{n\geq 0}$ is not decreasing. Then $\beta_{n+1}(x)$

$-\beta_n(x) > \varepsilon$ is equivalent to $r_{n+1}(x) > \exp(\varepsilon 2^{n+1}) r_n(x)^2$. From Lemma 3.1, we get

$$\lambda\{r_n = j \text{ and } \beta_{n+1} - \beta_n > \varepsilon\} \le d_2 \Big(\sum_{\substack{w \\ w > j^2 \exp(\varepsilon 2^{n+1})}} \frac{j^2}{w^2} \Big) \lambda\{r_n = j\}\,. \tag{11}$$

But it follows from elementary calculus that for all $j \ge 1$,

$$\sum_{\substack{w \\ w > j^2 \exp(\varepsilon 2^{n+1})}} \frac{j^2}{w^2} \le \frac{2}{e^{\varepsilon 2^{n+1}}}\,. \tag{12}$$

Using (11) and (12), we obtain

$$\lambda(\{r_n = j \text{ and } \beta_{n+1} - \beta_n > \varepsilon\}) \le 2e^{-\varepsilon 2^{n+1}} \lambda(\{r_n = j\})\,.$$

Define $\eta_m = (\sqrt{2}-1)(\sqrt{2})^{-(m+1)}$, so that $\sum_{m \ge 1} \eta_m = 1$. Let $n \ge 0$, $m \ge 1$ be integers and assume $\beta_{n+s}(x) - \beta_{n+s-1}(x) \le \varepsilon \eta_s$ for all $s \in \{1, 2, \ldots, m\}$. Then $\beta_{n+m}(x) - \beta_n(x) \le \varepsilon$ so that for

$$X_n(j; \varepsilon) := \{x : r_n(x) = j \text{ and } \exists\, m \ge 1,\ \beta_{n+m}(x) - \beta_n(x) > \varepsilon\}$$

we obtain

$$\begin{aligned} \lambda(X_n(j;\varepsilon)) &\le \lambda(\{r_n = j \text{ and } \exists\, m \ge 1,\ \beta_{n+m} - \beta_{n+m-1} > \varepsilon \eta_m\}) \\ &\le 2\Big(\sum_{m \ge 1} e^{-\varepsilon \eta_m 2^{n+m+1}}\Big) \lambda(\{r_n = j\}) \le \frac{2}{e^{\gamma \varepsilon 2^n} - 1} \lambda(\{r_n = j\}) \end{aligned} \tag{13}$$

where $\gamma = \sqrt{2} - 1$. But (13) is nothing but inequality (10) of Lemma 3.2. If we sum over j all inequalities (10) (n fixed) we also get

$$\lambda(\{\beta - \beta_n \le \varepsilon\}) \ge 1 - \frac{2}{e^{\gamma \varepsilon 2^n} - 1}\,.$$

Now it is quite clear that the sequence $(\beta_n(x))_{n \ge 0}$ converges (in $[0, \infty[$) for almost all $x \in [0,1[$. Since β_n is measurable, so is β. ■

Remark 3.1. Notice that β satisfies the following functional equations:

$$\beta(Tx) = 2\beta(x) \quad \text{ and } \quad \beta\Big(\frac{1}{k+1} x\Big) = \frac{1}{2}\beta(x)\,.$$

As in the case of Sylvester's series (see [Go-Sm]), it can be proved that β is dense in its epigraph and has local minima at rational points exactly. In [Go-Sm] it was first proved that the β function for Sylvester's series has a $\mathcal{C}^\infty$ density. In [Gal], it was proved that for the Cantor product, β has a $\mathcal{C}^1$ density. This last result at least should hold for the generalized Cantor products we are dealing with here.

Theorem 3.1. *T is not ergodic with respect to λ, i.e. there exist two disjoint T-invariant subsets of* $[0,1[$ *with positive Lebesgue measure.*

Proof. Let J be a nonempty open subinterval of $]0,\infty[$. Choose $\varepsilon>0$ such that there exist integers $p\ge 1$ and $m\ge 1$ satisfying

$$\left[\frac{\log p}{2^m}-\varepsilon,\frac{\log p}{2^m}+\varepsilon\right]\subset J\,.$$

Let N_ε be an integer such that $1-2/(e^{\gamma\varepsilon 2^n}-1)>0$ for all $n\ge N_\varepsilon$. We can easily choose integers $d\ge 2$ and $n\ge N_\varepsilon$ in order to have $2^{-n}\log d$ close enough to $2^{-m}\log p$ such that we still have

$$\left[\frac{\log d}{2^n}-\varepsilon,\frac{\log d}{2^n}+\varepsilon\right]\subset J\,.$$

Since $\lambda(\{r_n=d\})>0$ for any integer $d\ge 1$, inequality (10) implies $\lambda(\{x:\beta(x)\in J\})>0$ and the set

$$E(J):=\Big\{x:\beta(x)\in\bigcup_{m\in\mathbb{Z}}2^mJ\Big\}$$

is measurable and T-invariant with $\lambda(E(J))>0$. Let J and J' be two nonempty open intervals such that $J\subset[\frac12,\frac34[$ and $J'\subset[\frac34,1[$. Then the sets $E(J)$ and $E(J')$ are disjoint, T-invariant and $\mu(E(J))>0$ and $\mu(E(J'))>0$. This ends the proof. ∎

4. Uniform distribution. In this section we study the distribution of $T^{(n)}(x)$ in the interval $[a_{r_n(x)},a_{r_n(x)+1}[$. More precisely, let $(t_n(\cdot))_{n\ge 0}$ be the sequence of random variables defined on $[0,1[$ by

$$t_n(x):=\frac{T^{(n)}(x)-a_{r_n}}{a_{r_n+1}-a_{r_n}}=\frac{T^{(n+1)}(x)-b_{r_n(x)}}{1-b_{r_n(x)}},\qquad x\in[0,1[,\ n\ge 0\,.$$

Let $\Phi_n(\cdot)$ denote the distribution function of $t_n(\cdot)$, and define

$$W_n(d):=\{x:0\le t_n(x)<d\},\qquad d\in[0,1]\,.$$

Theorem 4.1. *The sequence of random variables $(t_n(\cdot))_{n\ge0}$ is identically and uniformly distributed* (*i.e.*, $\Phi_n(d)=d$ *for* $0\le d\le 1$, $n\ge 0$).

Proof. For $d\in[0,1]$ we have $\Phi_n(d)=\lambda(\{x:0\le t_n(x)<d\})$. Let $r=(r_0,\dots,r_n)\in A_{n+1}$ (see Definition 1.1). Since $T^{(n+1)}(x)=\Pi_{n+1}^{-1}(r)x$ on $B(r_0,\dots,r_n)$ and $T^{(n+1)}(B(r))=[b_{r_n},1[$, the set $W_n(d)$ is the union of the following pairwise disjoint sets:

$$B(r)\cap W_n(d)=\{x:\ b_{r_n}\Pi_{n+1}(r)\le x<\Pi_{n+1}(r)(b_{r_n}+d(1-b_{r_n}))\}\,.$$

But $\lambda(B(r)\cap W_n(d))=d\lambda(B(r))$ so

$$\lambda(W_n(d))=\sum_{r\in A_{n+1}}d\lambda(B(r))=d\,. \ ∎$$

With a view to the study of the λ-a.e. complete uniform distribution of the sequence $(t_n(x))_{n\geq 0}$, let us introduce

DEFINITION 4.1. Let $p \in \mathbb{N}$ and $(d_0,\dots,d_p),(d'_0,\dots,d'_p) \in [0,1]^{p+1}$. Then, for any $n \geq 0$, let

$$E_n(d_0,\dots,d_p) = W_n(d_0)\cap\dots\cap W_{n+p}(d_p)\,.$$

If $m \geq 1$, let

$$(d_0,\dots,d_p,1^m,d'_0,\dots,d'_p) = (d_0,\dots,d_p,\underbrace{1,\dots,1}_{m\ \text{times}},d'_0,\dots,d'_p)\,.$$

Let $d_{-1} = 1$ and $E_n(\emptyset) = [0,1]$.

With the above notations, we have

THEOREM 4.2. *For any integer $p \geq 0$, for any integer $n \geq 1$, any integer $m \geq 0$, any $(d_0,\dots,d_p,d'_0,\dots,d'_p) \in [0,1]^{2(p+1)}$,*

$$(\alpha)\qquad \begin{aligned}&|\lambda(E_n(d_0,\dots,d_p,1^m,d'_0,\dots,d'_p)) - d_0\dots d_p d'_0\dots d'_p|\\ &\qquad\qquad\leq 20(p+1)^2k^2(k+1)^2(\tfrac12)^n\,,\end{aligned}$$

$$(\beta)\qquad |\lambda(E_n(d_0,1^m,d'_0)) - d_0d'_0| \leq \tfrac52 k^2(k+1)^2(\tfrac12)^{n+m}\,.$$

Proof. Step 1. We need several lemmas and definitions.

LEMMA 4.1. *For any $n \in \mathbb{N}$, $m \geq 1$, $r = (r_0,r_1,\dots,r_{n+m}) \in A_{n+m+1}$, one has*

$$\frac{r_n^2\lambda(B(r_{n+1},\dots,r_{n+m}))}{k} \leq \frac{\lambda(B(r))}{\lambda(B(r_0,\dots,r_n))} \tag{14}$$

$$\leq (k+1)r_n^2\lambda(B(r_{n+1},\dots,r_{n+m}))$$

$$\leq \frac{k(k+1)}{2^m}\,. \tag{15}$$

Moreover,

$$\lambda(B(r_0,\dots,r_n)) \leq \min\left\{2^{-(n+1)},\frac{k}{(r_n+k)(r_n+k-1)}\right\}. \tag{16}$$

Proof. Notice that

$$\begin{aligned}\lambda(B(r)) &= \left(\frac{r_0}{r_0+k}\cdots\frac{r_{n+m-1}}{r_{n+m-1}+k}\right)\frac{k}{(r_{n+m}+k)(r_{n+m}+k-1)}\\ &= \lambda(B(r_0,\dots,r_n))\frac{(r_n+k)(r_n+k-1)}{k}\lambda(B(r_{n+1},\dots,r_{n+m}))\end{aligned}$$

and then inequality (14) follows from

$$\frac{x^2}{k} \leq \frac{(x+k)(x+k-1)}{k} \leq (k+1)x^2 \quad \text{for } x \geq 1\,.$$

On the other hand, put $p(x) = x^2 + (x-1)(k-1)$ and assume that $r_{s-1} = 1$ $(\neq r_s)$ for a digit with $0 < s \leq n$. Proposition 1.1 and (7) imply

$$\lambda(B(r_0, \ldots, r_n)) \leq (k+1)^{-s} \frac{k}{(p^{(n-s)}(r_s) + k - 1)(p^{(n-s)}(r_s) + k)}\,.$$

If $r_s = 1 = r_n$ the inequality (15) is evident. Otherwise $r_s \geq 2$ but $p^{(n-s)}(2) \geq 2^{2^{n-s}}$ and therefore (16) is still true. It remains to prove (15). If $r_n = 1$, the inequality follows from (16), otherwise we have

$$\lambda(B(r_{n+1}, \ldots, r_{n+m})) \leq k(p^{(m)}(r_n))^{-2} \leq k r_n^{-2^{m+1}} \leq k r_n^{-2} 2^{-m}\,. \ \blacksquare$$

LEMMA 4.2. *For positive natural numbers n and m let*

$$F_n(m) = \#\{(r_0, \ldots, r_{n-2}) \in \mathbb{N}^{n-1} : \ (r_0, \ldots, r_{n-2}, m) \in A_n\}\,.$$

Then $F_n(m) \leq m$.

Proof. We use induction on n. It is clear that $F_1(m) \leq m$. Now, let $n \geq 1$ be given and assume $F_n(m) \leq m$ for all $m \geq 1$. Proposition 1.1 implies that for any $(r_0, \ldots, r_{n-1}, m) \in A_{n+1}$ one has $r_{n-1} \leq \sqrt{m}$. Therefore

$$F_{n+1}(m) \leq \sum_{1 \leq j \leq \sqrt{m}} j \leq m\,. \ \blacksquare$$

LEMMA 4.3. *For any positive natural numbers n, m and for any map $s : A_n \to \mathbb{N}^m$ satisfying $((r_0, \ldots, r_{n-1}), s(r_0, \ldots, r_{n-1})) \in A_{n+m}$, one has*

$$\sum_{r \in A_n} \lambda(B(r, s(r))) \leq \frac{5k^3(k+1)^3}{2^{n+m}}$$

(*we identify $\mathbb{N}^{n+m}$ with $\mathbb{N}^n \times \mathbb{N}^m$*).

Proof. We first study the case $m = 1$. If $n = 1$, first notice that for any map $s_1 : A_1 = \mathbb{N}^* \to \mathbb{N}^*$ such that for any $r \in \mathbb{N}^*$, $(r, s_1(r)) \in A_2$, from (7) and Proposition 1.1,

$$\sum_{r \in \mathbb{N}^*} \lambda(B(r, s_1(r))) \leq \sum_{r \geq 1} \frac{kr}{(r+k)(s_1(r)+k)(s_1(r)+k-1)} \leq \sum_{r \geq 1} \frac{kr}{(r+k)(r^2+(k-1)r+1)(r^2+(k-1)r)}\,.$$

But since $k \geq 1$,

$$\sum_{r \geq 1} \frac{k}{(r+k)(r^2+(k-1)r+1)(r+k-1)} \leq \sum_{r \geq 1} \frac{1}{(r+1)(r^2+1)} \leq \frac{1}{2}\,,$$

and indeed $2 \leq 5k^3(k+1)^3$.

Assume now that $n \geq 2$. Then from Lemma 4.1, it follows that for any $r \in A_n$, $r = (r_0, \ldots, r_{n-1})$,

$$\lambda(B(r, s(r))) \leq k(k+1)\lambda(B(r))\frac{r_{n-1}^2}{p(r_{n-1})^2} \leq k(k+1)\frac{\lambda(B(r))}{r_{n-1}^2} \leq \frac{k^2(k+1)}{r_{n-1}^4}\,.$$

Then, for any $N \geq 1$,

$$\begin{aligned}\sum_{r\in A_n} \lambda(B(r, s(r))) &\leq k^2(k+1) \sum_{\substack{r\in A_n\\ r_{n-1}>N}} \frac{1}{r_{n-1}^4} + \sum_{\substack{r\in A_n\\ r_{n-1}\leq N}} \lambda(B(r, s(r)))\\ &\leq k^2(k+1)\sum_{t>N}\frac{1}{t^3} + k(k+1)\sum_{\substack{r\in A_n\\ r_{n-1}\leq N}}\frac{\lambda(B(r))}{r_{n-1}^2}\\ &\leq \frac{k^2(k+1)}{2N^2} + k^2(k+1)^2 \sum_{\substack{(r_0,\ldots,r_{n-1})\in A_n\\ r_{n-1}\leq N}} \lambda(B(r_0, \ldots, r_{n-2}))\frac{r_{n-2}^2}{r_{n-1}^4}\,.\end{aligned}$$

But $r_{n-1} \geq r_{n-2}^2$ and therefore with $g = 4k^2(k+1)^2$ and (16),

$$\begin{aligned}\sum_{r\in A_n} \lambda(B(r, s(r))) &\leq \frac{k^2(k+1)}{2N^2} + \frac{g}{2^{n+1}} \sum_{\substack{(r_0,\ldots,r_{n-2})\in A_{n-1}\\ r_{n-2}\leq\sqrt{N}}} r_{n-2}^{-6}\\ &\leq \frac{k^2(k+1)}{2N^2} + \frac{g}{2^{n+1}} \sum_{1\leq k\leq\sqrt{N}} k^{-5}\,.\end{aligned}$$

Passing to the limit as N tends to infinity, we get the case $m = 1$ with $\frac{5}{4}g$. The general case follows from (15) which gives

$$\lambda(B(r, s(r))) \leq \lambda(B(r, s_1(r)))\frac{k(k+1)}{2^{m-1}}\,. \ \blacksquare$$

Definition 4.2. Let $n \geq 1$ be an integer. Let $r = (r_0, \ldots, r_{n-1}) \in A_n$. Let $d \in [0, 1[$. Then define $r'(d, r)$ to be the unique integer such that, if $r'' = (r_0, \ldots, r_{n-1}, r'(d, r))$, we have

$$\Pi_n(r)(b_{r_{n-1}} + d(1 - b_{r_{n-1}})) \in B(r'')\,.$$

Denote the above admissible $(n + 1)$-uple r'' by $rr'(d, r)$ (as a concatenation). If $(r, r') \in \mathbb{N}^n \times \mathbb{N}^m$, let rr' be the $(n + m)$-uple defined by $rr' = (r_0, \ldots, r_{n-1}, r'_0, \ldots, r'_{m-1})$. Endow the sets A_n with the lexicographic order. If $d = 1$ and $r \in A_n$, let $r'(1, r) = +\infty$, and $B(r, +\infty) = \emptyset$.

Let $n \geq 0$ and $m \geq 1$. Let $r \in A_{n+1}$, $r = (r_0, \ldots, r_n)$, and define

$$A_{n+1,m}(r) := \{r' = (r'_{n+1}, \ldots, r'_{n+m}) \in \mathbb{N}^m :\ rr' \in A_{n+m+1}\}\,.$$

LEMMA 4.4. *For any $q \ge 1$ and any $k \ge 1$,*

$$\frac{1}{(q+k)(q+k-1)} > 2\Big(\sum_{m\ge 0}\frac{1}{(q+m+k)((q+m)^2+(q+m)(k-1)+1)(q+m+k-1)}\Big).$$

Proof. The sum of the series is clearly bounded by

$$\frac{1}{(q+k)(q+k-1)(q^2+q(k-1)+1)} + \Big(\sum_{t\ge q+1}\frac{1}{(t+k)(t+k-1)}\Big)\frac{1}{(q+1)^2+(q+1)(k-1)+1}$$

$$\le \frac{1}{(q+k)(q+k-1)}\Big(\frac{1}{(q+1)(q+k)-2q-k+1}+\frac{1}{q+1}-\frac{1}{(q+1)(q+k)}\Big)$$

$$\le \Big(\frac{1}{q+1}\Big)\frac{1}{(q+k)(q+k-1)},$$

and $q \ge 1$. ∎

Step 2. Let $p' \ge 1$. Using refining partitions of cylinders on $[0,1[$, one can see quite easily, with the use of Theorem 4.1 and Definition 4.2, that, given $(d_0,\ldots,d_{p'}) \in [0,1]^{p'+1}$, $n \ge 1$ and $r=(r_0,\ldots,r_n) \in A_{n+1}$,

$$\begin{aligned}(17)\quad & \lambda(E_n(d_0,\ldots,d_{p'})\cap B(r)) = \\ & \sum_{\substack{r_{n+1}\in A_{n+1,1}(r)\\ r_{n+1}<r'(d_0,r)}}\Big(\sum_{\substack{r_{n+2}\in A_{n+2,1}(rr_{n+1})\\ r_{n+2}<r'(d_1,rr_{n+1})}}\cdots\Big(\sum_{\substack{r_{n+p'}\in A_{n+p',1}(rr_{n+1}\ldots r_{n+p'-1})\\ r_{n+p'}<r'(d_{p'-1},rr_{n+1}\ldots r_{n+p'-1})}} d_{p'}\lambda(B(rr_{n+1}\ldots r_{n+p'}))\Big)\Big) \\ & + \sum_{\substack{r_{n+1}\in A_{n+1,1}(r)\\ r_{n+1}<r'(d_0,r)}}\Big(\cdots\Big(\sum_{\substack{r_{n+p'-1}\in A_{n+p'-1,1}(r\ldots r_{n+p'-2})\\ r_{n+p'-1}<r'(d_{p'-2},r\ldots r_{n+p'-2})}} \lambda(B(r\ldots r_{n+p'-1}r'(d_{p'-1},r\ldots r_{n+p'-1}))\cap E_n(d_0,\ldots,d_{p'}))\Big)\Big) \\ & + \ldots \\ & + \sum_{\substack{r_{n+1}\in A_{n+1,1}(r)\\ r_{n+1}<r'(d_0,r)}} \lambda(B(rr_{n+1}r'(d_1,rr_{n+1}))\cap E_n(d_0,\ldots,d_{p'})) \\ & + \lambda(B(rr'(d_0,r))\cap E_n(d_0,\ldots,d_{p'})).\end{aligned}$$

Let, for $i \in [1,p]$,

$$(18)\qquad X_i(d_0,\ldots,d_p,n) = |\lambda(E_n(d_0,\ldots,d_i)) - d_i\lambda(E_n(d_0,\ldots,d_{i-1}))|.$$

Notice that $X_i(d_0,\dots,d_p,n)=0$ if $p=0$ or $d_i\in\{0,1\}$. Let, for $i\in[1,p]$,

$$(19)\quad Y_i(d_0,\dots,d_p,n)=\sum_{r\in A_{n+1}}\Big(\dots\Big(\sum_{\substack{r_{n+i}\in A_{n+i,1}(rr_{n+1}\dots r_{n+i-1})\\ r_{n+i}<r'(d_{i-1},rr_{n+1}\dots r_{n+i-1})}}\lambda(B(rr_{n+1}\dots r_{n+i}r'(d_i,r\dots r_{n+i}))\cap E_n(d_0,\dots,d_p))\Big)\dots\Big),$$

and

$$(19)'\qquad Y_0(d_0,\dots,d_p,n)=\sum_{r\in A_{n+1}}\lambda(B(rr'(d_0,r))\cap E_n(d_0,\dots,d_p)).$$

Definition 4.3. Let $r'(r)$ denote the smallest element of $A_{n,1}(r)$ for $r\in A_n$.

Let, for $i\in\mathbb{N}^*$, with Definitions 4.2 and 4.3,

$$(20)\quad R_i(n)=\sum_{r\in A_{n+1}}\Big(\dots\Big(\sum_{r_{n+i}\in A_{n+i,1}(rr_{n+1}\dots r_{n+i-1})}\lambda(B(rr_{n+1}\dots r_{n+i}r'(r\dots r_{n+i})))\Big)\dots\Big),$$

and

$$(20)'\qquad R_0(n)=\sum_{r\in A_{n+1}}\lambda(B(rr'(d_0,r))).$$

Define, for $i\in[1,p]$,

$$(21)\quad Z_i(d_0,\dots,d_p,n)=\sum_{r\in A_{n+1}}\Big(\dots\Big(\sum_{\substack{r_{n+i}\in A_{n+i,1}(rr_n\dots r_{n+i-1})\\ r_{n+i}<r'(d_{i-1},rr_n\dots r_{n+i-1})}}\lambda(B(rr_n\dots r_{n+i}r'(d_i,r\dots r_{n+i})))\Big)\dots\Big),$$

and

$$(21)'\qquad Z_0(d_0,\dots,d_p)=\sum_{r\in A_{n+1}}\lambda(B(rr'(d_0,r))).$$

Observe that if $p>0$,

$$(22)\quad \Big|\sum_{r\in A_{n+1}}\Big(\sum_{\substack{r_{n+1}\in A_{n+1,1}(r)\\ r_{n+1}<r'(d_0,r)}}\Big(\dots\sum_{\substack{r_{n+p}\in A_{n+p,1}(rr_{n+1}\dots r_{n+p-1})\\ r_{n+p}<r'(d_{p-1},rr_{n+1}\dots r_{n+p-1})}}\lambda(B(rr_{n+1}\dots r_{n+p}))\Big)\Big)$$
$$-d_{p-1}\Big(\sum_{r\in A_{n+1}}\Big(\sum_{\substack{r_{n+1}\in A_{n+1,1}(r)\\ r_{n+1}<r'(d_0,r)}}\Big(\dots\sum_{\substack{r_{n+p-1}\in A_{n+p-1,1}(rr_{n+1}\dots r_{n+p-2})\\ r_{n+p}<r'(d_{p-2},rr_{n+1}\dots r_{n+p-2})}}\lambda(B(rr_{n+1}\dots r_{n+p-1}))\Big)\Big)\Big)\Big|\le Z_{p-1}(d_0,\dots,d_p,n).$$

Then, from relations (17) to (22), if we put $Z_{-1}(d_0,n)=0$,

$$(23)\quad |\lambda(E_n(d_0,\ldots,d_p))-d_p\lambda(E_n(d_0,\ldots,d_{p-1}))| \leq \delta_{p\neq 0}\delta_{d_p\notin\{0,1\}}\Big(2\Big(\sum_{i=0}^{p-1}Y_i(d_0,\ldots,d_p,n)\Big)+Z_{p-1}(d_0,\ldots,d_p,n)\Big) \leq \underbrace{\delta_{p\neq 0}\delta_{d_p\notin\{0,1\}}\Big(2\Big(\sum_{i=0}^{p-1}R_i(n)\Big)+Z_{p-1}(d_0,\ldots,d_p,n)\Big)}_{W(d_0,\ldots,d_p,n)},$$

where if P is a proposition, $\delta_P=0$ if P is false, 1 otherwise. Let

$$(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p)=(a_0,\ldots,a_{2p+m+1}).$$

From (17), (18), Definition 4.2 and repeated application of the triangle inequality,

$$(24)\quad |\lambda(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p))-d_0\ldots d_pd'_0\ldots d'_p| \leq \sum_{i=1}^{p}X_i(d_0,\ldots,d_p,n)+\sum_{i=p+m+1}^{2p+m+1}X_i(a_0,\ldots,a_{2p+m+1},n).$$

From Proposition 1.1, Definition 4.3, for any integer $m\geq 1$ and any $r\in A_m$,

$$\sum_{p\geq r'(r)}\lambda(B(rpr'(rp))) \leq \sum_{p\geq r'(r)}\Big(\prod_{i=0}^{m-1}\frac{r_i}{r_i+k}\Big)\frac{kp}{(p+k)(p^2+(k-1)p+1)(p^2+(k-1)p)},$$

and from Lemma 4.4, with $q=r'(r)$, we deduce from the above inequality that

$$\sum_{p\geq r'(r)}\lambda(B(rpr'(rp)))\leq \tfrac{1}{2}\lambda(B(rr'(r))).$$

Then, from definitions (20), (20)$'$ and the above,

$$(25)\quad R_i(n)\leq \tfrac12 R_{i-1}(n)\leq\ldots\leq(\tfrac12)^iR_0(n).$$

It follows from (20), (21) and (23)–(25) that

$$(26)\quad |\lambda(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p))-d_0\ldots d_pd'_0\ldots d'_p| \leq \sum_{i=1}^{p}W(d_0,\ldots,d_i,n)+\sum_{i=p+m+1}^{2p+m+1}W(a_0,\ldots,a_i,n) \leq 4p(p+1)R_0(n)+2(p+1)^2R_{p+m+1}(n).$$

From Lemma 4.3, we have

$$R_0(n) \le \frac{5k^2(k+1)^2}{2^{n+1}}\,. \tag{27}$$

Thus, from (25), (26), (27), we obtain

$$\begin{aligned}(28)\quad & |\lambda(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p)) - d_0\ldots d_p d'_0\ldots d'_p| \\ &\qquad \le 10(p+1)k^2(k+1)^2\bigg(\frac{p}{2^n} + (p+1)\bigg(\frac{1}{2}\bigg)^{p+2}\bigg(\frac{1}{2}\bigg)^{n+m}\bigg),\end{aligned}$$

hence

$$\begin{aligned}(28)'\quad & |\lambda(E_n(d_0,\ldots,d_p,1^m,d'_0,\ldots,d'_p)) - d_0\ldots d_p d'_0\ldots d'_p| \\ &\qquad \le 20(p+1)^2k^2(k+1)^2(\tfrac{1}{2})^n\,.\end{aligned}$$

Now formula (α) of Theorem 4.2 is given in $(28)'$ above, and (β) comes from (28) in the case $p = 0$. This ends the proof of Theorem 4.2. ∎

Theorem 4.3. *For almost all* x, *the sequence* $(t_n(x))_{n\ge0}$ *is completely uniformly distributed in* $[0,1]$, *e.g., for almost all* $x\in[0,1[$ *and every* $p\ge 0$, *the sequence* $(t_n(x),\ldots,t_{n+p}(x))_{n\ge0}$ *is uniformly distributed in* $[0,1]^{p+1}$. *More precisely, for all* $\varepsilon>0$ *and all* $(d_0,d_1,\ldots,d_p)\in[0,1]^{p+1}$, *one has*

$$\begin{aligned}&\frac{1}{N}\sum_{n<N}\mathbf{1}_{[0,d_0[\times\ldots\times[0,d_p[}(t_n(x),\ldots,t_{n+p}(x)) \\ &\qquad = d_0d_1\ldots d_p + O\bigg(\frac{(\log N)^{3/2+\varepsilon}}{\sqrt{N}}\bigg), \qquad \lambda\text{-}a.e.\end{aligned}$$

Proof. It is a direct application of Theorem 4.2 (α) and Theorem 11.3 from [Sch]. Indeed, given $p\ge0$ and $(d_0,\ldots,d_p)\in[0,1]^{p+1}$ from (α), one has, if we let $E_n := E_n(d_0,\ldots,d_p)$,

$$\lambda(E_n) = d_0\ldots d_p + O(1/2^n)\,,$$

where the constant in the O is bounded when $(d_0,\ldots,d_p)$ is fixed, and $E_n(d_0,\ldots,d_p)\cap E_{n+m+p+1}(d_0,\ldots,d_p) = E_n(d_0,\ldots,d_p,1^m,d_0,\ldots,d_p)$, for m large enough. Thus, we can find a convergent series of nonnegative numbers $(\gamma_k)_{k\ge0}$ such that $\gamma_k = O'(1/2^k)$, and for any $n\ge0$ and $t\ge0$,

$$\lambda(E_n\cap E_{n+t}) \le \lambda(E_n)\lambda(E_{n+t}) + (\lambda(E_n)+\lambda(E_{n+t}))\gamma_t + \lambda(E_{n+t})\gamma_n\,. \ ∎$$

However, using only (β), we have

Corollary 4.1. *For* λ*-a.e.* $x\in[0,1[$, *the sequence* $(t_n(x))_{n\ge0}$ *is uniformly distributed in* $[0,1]$ *and for all* $\varepsilon>0$, $d\in[0,1]$, *and* $N\in\mathbb{N}^*$,

$$A(N,x,d) := \#\{0\le n<N : 0\le t_n(x)<d\} = Nd + O(\sqrt{N}(\log N)^{3/2+\varepsilon})\,.$$

Proof. A straightforward computation gives

$$\int_0^1 \Big| \sum_{n=M+1}^{M+N} (\mathbf{1}_{[0,d[}(t_n(x)) - d)\Big|^2 \lambda(dx) = O(N)\,,$$

and the corollary results from [Ga-Ko]. ■

Remark 4.1. In a forthcoming paper with A. Thomas, we shall give, as an application, an alternative proof of this fact ([La-Th]). However, the present proof has the advantage that it presents materials that can be quite directly used for proving the nonindependence, or stochasticity, of the sequence $(t_n(\cdot))_{n\geq 0}$.

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Received on 27.2.1992
and in revised form on 3.6.1992 (2233)