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Cyclotomic polynomials with large coefficients

by

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Dedicated to Paul Erdős
on the occasion of his eightieth birthday

1. Introduction. Let $\Phi_n(z) = \sum_{m=0}^{\varphi(n)} a(m,n) z^m$ be the nth cyclotomic polynomial. Let

$$A(n) = \max_{0\le m\le\varphi(n)} |a(m,n)| \quad \text{and} \quad S(n) = \sum_{0\le m\le\varphi(n)} |a(m,n)| \,.$$

The coefficients $a(m,n)$ and especially $A(n)$ and $S(n)$ have been the subject of numerous investigations (see [1] and the references given there). Until recently all these investigations concerned very thin sets of integers n. In [3] the author could establish a property valid for a set of integers of asymptotic density 1. Let $\varepsilon(n)$ be any function defined for all positive integers such that $\lim_{n\to\infty}\varepsilon(n) = 0$. Then $S(n) \ge n^{1+\varepsilon(n)}$ for a set of integers of asymptotic density 1. Here we deal with properties valid for sequences of positive lower density.

Theorem. *For any $N > 0$, there are $c(N) > 0$ and $x_0(N) \ge 1$ such that*

$$\mathrm{card}\{n \le x : A(n) \ge n^N\} \ge c(N)x \,,$$

for all $x \ge x_0(N)$.

2. A certain set of candidates. Let $N > 0$ be given. In this section we identify a certain set of integers in which a large subset will later be shown to have $A(n) \ge n^N$. To describe the set, we fix a positive odd integer

$$K = K(N) \tag{2.1}$$

(to be determined later) and set

$$L = 20K \,, \qquad \delta = \frac{1}{100L} \,, \qquad \varepsilon = \frac{\delta}{L^2} \,.$$

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The letter p always denotes prime numbers and $\omega(m)$ denotes the number of distinct prime factors of m. Basic for our construction is the set S.

Let

$$S = S(N,x) = \{n = mp_1 \ldots p_L \le x : x^{\frac{1-\delta}{L}-\varepsilon} < p_i \le x^{\frac{1-\delta}{L}+\varepsilon}\,, \quad \mu(m) = \mu(n) = 1\,,\ \omega(m) \le (1+\varepsilon)(\log\log x)\}\,. \tag{2.2}$$

LEMMA. *For each N, there are effectively computable constants $c_0(N) > 0$ and $x_0(N) > 0$ such that for all $x \ge x_0(N)$,*

$$\operatorname{card} S \ge c_0(N)x\,.$$

Proof. Let $\mathcal{T} = \mathcal{T}(x)$ be the set of $n \le x$ which have all of the properties of elements of S, but the condition $\mu(n) = 1$ fails. This implies that at least two of the primes $p_1, \ldots, p_L$ are the same. We have

$$\operatorname{card} S \ge \frac{1}{L!} \sideset{}{'}\sum_{p_1} \ldots \sideset{}{'}\sum_{p_L} \sum_{\substack{m \le x/p_1 \ldots p_L \\ \mu(m)=1, \omega(m) \le (1+\varepsilon)\log\log x}} 1 - \operatorname{card} \mathcal{T} \tag{2.3}$$

where $\sum'$ denotes a sum over p_i with $x^{\frac{1-\delta}{L}-\varepsilon} < p_i \le x^{\frac{1-\delta}{L}+\varepsilon}$.

Clearly

$$\operatorname{card} \mathcal{T} \le x \sideset{}{'}\sum_{p} \frac{1}{p^2} = o(x) \quad \text{for } x \to \infty\,.$$

From [2] and [4] we know that the inner sum in (2.3) is at least $x/10p_1 \ldots p_L$ for all $x \ge x_1(N)$, where $x_1(N)$ is a constant depending only on the choice of N. Thus

$$\operatorname{card} S \ge \frac{x}{10L!} \left(\sideset{}{'}\sum \frac{1}{p} \right)^L - o(x) \quad \text{for } x \to \infty\,. \tag{2.4}$$

Now

$$\sideset{}{'}\sum \frac{1}{p} = \log\left(\frac{1-\delta}{L} + \varepsilon\right) - \log\left(\frac{1-\delta}{L} - \varepsilon\right) + O\left(\frac{1}{\log x}\right),$$

so there is some number $c_1(N)$ such that

$$\sideset{}{'}\sum \frac{1}{p} \ge c_1(N) > 0$$

for all $x \ge x_2(N)$, where $x_2(N)$ is a constant depending only on N. The lemma now follows from (2.4).

3. Investigation of $\log|\Phi_n(z)|$ on the unit circle. We start with the well-known identity

$$\Phi_n(z) = \prod_{d \mid n} (1 - z^d)^{\mu(n/d)} \tag{3.1}$$

for all complex z for which both sides are defined. We write $e^{2\pi i\alpha} = e(\alpha)$ and obtain

$$\log|\Phi_n(e(\alpha))| = \sum_{d\,|\,n} \mu\Big(\frac{n}{d}\Big) \log|1 - e(\alpha d)|\,. \tag{3.2}$$

To show that $A(n)$ is large it would suffice to show there is some α with $\log|\Phi_n(e(\alpha))|$ large. The terms on the right of (3.2) will be large for $\mu(n/d) = -1$ and $\|\alpha d\|$ small. (Here $\|\cdot\|$ denotes the distance to the nearest integer.) Simple choices for the pair (α, d) however do not work because of a certain cancellation effect which has already been described in [3]. We repeat its description for the convenience of the reader.

Assume $r\,|\,n/d$, $\mu(n/d) = -1$, $\mu(n) \neq 0$, $\omega(r) \geq 2$ and $\alpha d = k + \varrho$ with k an integer, $|\varrho| \leq 1/2$. Thus $|\varrho| = \|\alpha d\|$. Also assume that $|r\varrho| \leq 1/2$. For $t\,|\,r$ we have $e(\alpha dt) = e(t\varrho) = 1 + 2\pi i\varrho t + O((\varrho t)^2)$. This implies $\log|1 - e(\alpha dt)| = \log(\varrho t) + O(1)$. Thus

$$\begin{aligned}\sum_{t\,|\,r} \mu\Big(\frac{n}{dt}\Big) &\log|1 - e(\alpha dt)| \\ &= \sum_{t\,|\,r} \mu\Big(\frac{n}{dt}\Big)(\log\varrho + \log t) + O\Big(\sum_{t\,|\,r} 1\Big) = O\Big(\sum_{t\,|\,r} 1\Big),\end{aligned}$$

where we use

$$\sum_{t\,|\,r} \mu(t) = 0 \quad \text{and (for } \omega(r) \geq 2) \quad \sum_{t\,|\,r} \mu(t)\log t = 0\,.$$

Thus the large contribution $\mu(n/d)\log|1-e(\alpha d)|$ is cancelled by other terms.

A method to avoid this cancellation effect is to choose α and d such that for $t < t_0$ we have $|\varrho t| \leq 1/2$ but for $t \geq t_0$, $|\varrho t| > 1/2$. This leads to estimates of incomplete convolutions

$$\sum_{\substack{t\,|\,r\\ t<t_0}} \mu\Big(\frac{n}{dt}\Big)(\log\varrho + \log t)$$

which can be made large by an appropriate choice of d, r, t_0 and α. For the remaining sum

$$\sum_{\substack{t\,|\,r\\ t\geq t_0}} \mu\Big(\frac{n}{dt}\Big) \log|1 - e(\alpha dt)|\,,$$

we have to show that the terms are small for appropriate choice of α. This will be done by showing that $\|\alpha dt\|$ is not too small.

DEFINITIONS. Let $S(m_0) = \{n \in S : n = m_0 p_1 \ldots p_L\}$, $y_0 = x^{-\frac{1}{L}(K+1-\delta)}$. For each m_0 we define an interval

$$I(m_0) = [m_0^{-1} + m_0^{-1} y_0, m_0^{-1} + 2m_0^{-1} y_0]\,.$$

For $n \in S$ with $n = mp_1 \ldots p_L$, set $\Pi(n) = p_1 p_2 \ldots p_L$. If $n \in S(m_0)$, we write

$$\log |\Phi_n(e(\alpha))| = \sum\nolimits_0 + \sum\nolimits_1 + \sum\nolimits_2\,, \tag{3.3}$$

where

$$\sum\nolimits_0 = \sum_{\substack{d = m_0 t,\, t \mid \Pi(n) \\ \omega(t) \le K}} \mu\Big(\frac{n}{d}\Big) \log |1 - e(\alpha d)|\,,$$

$$\sum\nolimits_1 = \sum_{\substack{m^* \mid m_0 \\ m^* \ne m_0}} \mu\Big(\frac{n}{m^*}\Big) \log |1 - e(\alpha m^*)|\,,$$

$$\sum\nolimits_2 = \sum_{\substack{\text{all other divisors} \\ d \mid n}} \mu\Big(\frac{n}{d}\Big) \log |1 - e(\alpha d)|\,.$$

We shall investigate these three sums for $\alpha \in I(m_0)$.

4. The main part $\sum_0$. Let $t \mid \Pi(n)$ with $\omega(t) = K - l$, $0 \le l \le K$. Then t is the product of $K - l$ distinct primes from $[x^{\frac{1-\delta}{L} - \varepsilon}, x^{\frac{1-\delta}{L} + \varepsilon}]$. Therefore

$$t \in [x^{\frac{K-l}{L}(1-\delta) - (K-l)\varepsilon}, x^{\frac{K-l}{L}(1-\delta) + (K-l)\varepsilon}]\,. \tag{4.1}$$

Moreover, if $\alpha \in I(m_0)$, then $\alpha m_0 t \in [t + ty_0, t + 2ty_0]$, so that $\{\alpha m_0 t\} \in [ty_0, 2ty_0]$, where $\{\cdot\}$ means fractional part. We write $\{\alpha m_0 t\} = \eta t y_0$ with $1 \le \eta \le 2$. We have $e(\alpha m_0 t) = e(\{\alpha m_0 t\}) = 1 + 2\pi i \eta t y_0 + O((ty_0)^2)$ and thus

$$\log |1 - e(\alpha m_0 t)| = \log t y_0 + O(1). \tag{4.2}$$

From (4.1) and the definition of y_0 we get

$$t y_0 \in [x^{-\frac{l+1}{L} - (K-l-1)\frac{\delta}{L} - (K-l)\varepsilon}, x^{-\frac{l+1}{L} - (K-l-1)\frac{\delta}{L} + (K-l)\varepsilon}]. \tag{4.3}$$

For $l = 0$, that is, for $\omega(t) = K$, we use the upper bound in (4.3) and together with (4.2) we get

$$\log |1 - e(\alpha m_0 t)| \le \Big(-\frac{1}{L} - (K-1)\frac{\delta}{L} + K\varepsilon\Big) \log x + O(1) \le -\frac{1}{L} \log x\,,$$

for x sufficiently large. There are $\binom{L}{K}$ divisors $t \mid \Pi(n)$ with $\omega(t) = K$ and for each we have $\mu(n/(m_0t)) = -1$. Thus we get

$$\text{(4.4)}\qquad \sum_{\substack{t\,|\,\Pi(n)\\ \omega(t)=K}} \mu\left(\frac{n}{m_0t}\right)\log|1-e(\alpha m_0 t)| \ge \binom{L}{K} L^{-1}\log x\,.$$

For $1\le l\le K$, that is, $\omega(t)=K-l$, from (4.2) and (4.3) we get

$$|\log|1-e(\alpha m_0t)||\le\left(\frac{l+1}{L}+(K-l-1)\frac{\delta}{L}+(K-l)\varepsilon\right)\log x+O(1)\le\frac{3l}{L}\log x\,,$$

for x sufficiently large. Since there are $\binom{L}{K-l}$ divisors $t\,|\,\Pi(n)$ with $\omega(t)=K-l$ we get

$$\text{(4.5)}\qquad \Bigg|\sum_{\substack{t\,|\,\Pi(n)\\ \omega(t)=K-l}} \mu\left(\frac{n}{m_0t}\right)\log|1-e(\alpha m_0 t)|\Bigg| \le \frac{3l}{L}\binom{L}{K-l}\log x\,.$$

We study the ratio of these upper bounds for consecutive l-values. For $l\ge 1$,

$$\begin{aligned}\frac{3(l+1)}{L}\binom{L}{K-l-1}\Big/\frac{3l}{L}\binom{L}{K-l} &= \left(1+\frac{1}{l}\right)\frac{K-l}{L-(K-l)+1}\\ &< \frac{2K}{L-K}=\frac{2}{19}\,.\end{aligned}$$

From this, (4.4) and (4.5) we obtain for x sufficiently large

$$\begin{aligned}\text{(4.6)}\qquad \sum\nolimits_0 &\ge \frac{1}{L}\binom{L}{K}\log x-\frac{3}{L}\binom{L}{K-1}\log x\sum_{i=0}^{\infty}\left(\frac{2}{19}\right)^i\\ &= \frac{1}{L}\binom{L}{K}\log x-\frac{57}{17}\cdot\frac{1}{L}\binom{L}{K-1}\log x\\ &= \left(1-\frac{57}{17}\frac{K}{L-K+1}\right)\frac{1}{L}\binom{L}{K}\log x > \frac{14}{17L}\binom{L}{K}\log x\,.\end{aligned}$$

5. The divisors of m_0. Our aim now is to show that $\sum_1$ is small for $n\in S$, $\alpha\in I(m_0)$. By definition we have

$$\sum\nolimits_1 = \sum_{\substack{m^*\,|\,m_0\\ m^*\ne m_0}} \mu\left(\frac{n}{m^*}\right)\log|1-e(\alpha m^*)|\,.$$

Note that for $\alpha\in I(m_0)$ and $m^*\,|\,m_0$, $m^*<m_0$ we have $0<\alpha m^*<1$. Thus

$$e(\alpha m^*) = 1+2\pi i\alpha m^* + O((\alpha m^*)^2)\,.$$

From this we get

$$|1-e(\alpha m^*)| = 2\pi\alpha m^*(1+O(\alpha m^*))$$

and so

$$\log|1-e(\alpha m^*)| = \log m^* + \log\alpha + O(1)\,.$$

Thus for all $n\in S$,

$$\sum\nolimits_1 = \mu\left(\frac{n}{m_0}\right)\sum_{\substack{m^*\,|\,m_0\\ m^*\neq m_0}}\mu\left(\frac{m_0}{m^*}\right)(\log m^* + \log\alpha) + O((\log x)^{(1+\varepsilon)\log 2})\,,$$

since $\omega(m_0)\le(1+\varepsilon)\log\log x$.

We have (since $\mu(m_0)=1$ implies m_0 is not a prime or prime power)

$$\sum_{\substack{m^*\,|\,m_0\\ m^*\neq m_0}}\mu\left(\frac{m_0}{m^*}\right)\log m^* = -\log m_0$$

and

$$\sum_{\substack{m^*\,|\,m_0\\ m^*\neq m_0}}\mu\left(\frac{m_0}{m^*}\right)\log\alpha = -\log\alpha\,.$$

Since $\log(\alpha m_0)\ll 1$, this yields

$$\sum\nolimits_1 \ll (\log x)^{(1+\varepsilon)\log 2}\quad \text{for } n\in S\,,\ \alpha\in I(m_0)\,. \tag{5.1}$$

6. The divisors $d=m^*t$ with $\omega(t)\le K$. The remaining divisors in $\sum_2$ are of two kinds. The first kind are of the form m^*t with $m^*\,|\,m_0$, $m^*<m_0$, $t\,|\,\Pi(n)$ and $1\le\omega(t)\le K$. We treat the contribution of these divisors in this section, leaving the treatment for the remaining divisors, which are of the form m^*t with $\omega(t)>K$, for the final section.

Let $C>0$ be a constant that we will soon choose as a large absolute constant. If we have $\|m_0^{-1}m^*t\|>2(\log x)^{-C}$ for $\omega(t)\le K$, then we also have

$$\|\alpha m^*t\|\ge(\log x)^{-C} \tag{6.1}$$

for all $\alpha\in I(m_0)$. Indeed,

$$|\alpha m^*t - m_0^{-1}m^*t|\le 2m_0^{-1}y_0m^*t\le y_0t = o((\log x)^{-C})$$

for any C.

We study the exceptional set

$$\begin{aligned}S_E(m_0)=\{n\in S(m_0):\|m_0^{-1}m^*t\|\le 2(\log x)^{-C}\\ \text{for some } m^*\,|\,m_0\,,\ m^*<m_0\,,\ t\,|\,\Pi(n)\,,\ 1\le\omega(t)\le K\}\,.\end{aligned} \tag{6.2}$$

We shall replace the inequality with a congruence. Let $n\in S_E$ and suppose $\|m_0^{-1}m^*t\|\le 2(\log x)^{-C}$. Let $m_0^{-1}m^*t=k+\varrho$ where k is an integer and

$|\varrho| \le 2(\log x)^{-C}$. Then

$$t = \frac{m_0}{m^*}k + \frac{m_0}{m^*}\varrho\,.$$

Note that $r := (m_0/m^*)\varrho$ is an integer. Thus

$$t \equiv r \bmod \frac{m_0}{m^*}\,, \qquad |r| \le 2\frac{m_0}{m^*}(\log x)^{-C}\,. \tag{6.3}$$

We estimate the cardinality of $S_E(m_0)$ by writing $S_E(m_0)$ as a union of subsets. For a given $m^* \mid m_0$ with $m^* < m_0$ and a given integer g with $1 \le g \le K$, let

$$S_E(m_0, m^*, g) = \{n \in S(m_0) : \|m_0^{-1}m^*t\| \le 2(\log x)^{-C} \text{ for some } t \mid \Pi(n) \text{ with } \omega(t) = g\}\,.$$

Note that if $n = m_0tu \in S_E(m_0, m^*, g)$ then $u \le x/(m_0t)$ and

$$x^{g(\frac{1-\delta}{L}-\varepsilon)} < t \le x^{g(\frac{1-\delta}{L}+\varepsilon)} \tag{6.4}$$

and (6.3) holds for some integer r. Thus

$$|S_E(m_0, m^*, g)| \le \sum_t{}^{*} \sum_{u \le x/(m_0t)} 1 \le \frac{x}{m_0} \sum_t{}^{*}\frac{1}{t}$$

where $\sum^*$ denotes a sum over t satisfying (6.4) and satisfying (6.3) for some integer r. Since $t \mid \Pi(n)$ and $\omega(t) \ge 1$ one has $t > m_0/m^*$, so that possible solutions of (6.3) with $t \le m_0/m^*$ do not occur in the sum $\sum^*$. Thus for a fixed r, we have

$$\sum_t{}^{*(r)}\,\frac{1}{t} \ll \frac{g\varepsilon\log x}{m_0/m^*}$$

uniformly in r. Since each prime divisor of t exceeds m_0/m^*, we see that $r = 0$ is not a possibility in (6.3) and so the set of possible values of r is empty when $2(m_0/m^*)(\log x)^{-C} < 1$. Therefore

$$|S_E(m_0, m^*, g)| \ll g\varepsilon\frac{m^*}{m_0^2}x\log x \sum_{|r| \le 2|m_0/m^*|(\log x)^{-C}} 1\,.$$

Thus

$$\begin{aligned}\sum_{m_0}|S_E(m_0)| &\le \sum_{m_0}\sum_{m^* \mid m_0}\sum_{g=1}^{K}|S_E(m_0, m^*, g)| \\ &\ll \sum_{m_0}\sum_{m^* \mid m_0}\frac{K^2\varepsilon}{m_0}x(\log x)^{1-C} \\ &< \sum_{m_0}\frac{\tau(m_0)}{m_0}x(\log x)^{1-C} \ll x(\log x)^{3-C}\,.\end{aligned}$$

7. The larger divisors. Here we study the divisors $d = m^*t$ with $m^* \mid m_0$, $t \mid \Pi(n)$, $\omega(t) > K$. In contrast to the last section we here have to remove exceptional α-values from $I(m_0)$.

Given a fixed triplet (m_0, m^*, t) with $m^* \mid m_0$, $t \mid \Pi(n)$, $\omega(t) > K$, we want an estimate for $\lambda E(m_0, m^*, t)$, where $E(m_0, m^*, t) = \{\alpha \in I(m_0) : \|m^*t\alpha\| \le (\log x)^{-C}\}$, and λ denotes the Lebesgue measure. We have

$$\lambda E(m_0, m^*, t) = \lambda\biggl\{\alpha \in I(m_0) : \\ \alpha \in \left[\frac{k}{m^*t} - \frac{(\log x)^{-C}}{m^*t}, \frac{k}{m^*t} + \frac{(\log x)^{-C}}{m^*t}\right] \text{ for some } k \in \mathbb{Z}\biggr\}.$$

We determine the number of integers k for which

$$\left[\frac{k}{m^*t} - \frac{(\log x)^{-C}}{m^*t}, \frac{k}{m^*t} + \frac{(\log x)^{-C}}{m^*t}\right] \cap I(m_0) \ne \emptyset\,. \tag{7.1}$$

From the definition of $I(m_0)$, such k-values satisfy

$$\frac{m^*t}{m_0} + \frac{y_0 m^*t}{m_0} - (\log x)^{-C} \le k \le \frac{m^*t}{m_0} + \frac{2y_0 m^*t}{m_0} + (\log x)^{-C}\,.$$

Since $\omega(t) > K$, we have $y_0 m^*t/m_0 \to \infty$ as $x \to \infty$. Thus for large x, the number of integers k satisfying (7.1) is at most $2y_0 m^*t/m_0$. Thus for large x, we have

$$\lambda E(m_0, m^*, t) \le \frac{2y_0}{m_0(\log x)^C}. \tag{7.2}$$

For $n \in S$, let

$$J(n) = I(m_0) - \bigcup_{\substack{m^* \mid m_0 \\ t \mid \Pi(n), \omega(t) > K}} E(m_0, m^*, t)\,.$$

Thus from (7.2) we have

$$\begin{aligned}\lambda J(n) &\ge \frac{y_0}{m_0} - \sum_{\substack{m^* \mid m_0 \\ t \mid \Pi(n), \omega(t) > K}} \frac{2y_0}{m_0(\log x)^C} \\ &\ge \frac{y_0}{m_0}\biggl(1 - \frac{2\tau(m_0)2^L}{(\log x)^C}\biggr) \ge \frac{y_0}{m_0}(1 - 2^{L+1}(\log x)^{1-C})\end{aligned}$$

from the definition of S. Thus for x large, we have $J(n) \ne \emptyset$.

We now use the results of this section and the previous section to estimate $\sum_2$. Let $n \in S - S_E$ and let $\alpha \in J(n)$. Then from (6.1) and the definition of $J(n)$, we have for each divisor d of n in the sum $\sum_2$ that

$$\|\alpha d\| \ge (\log x)^{-C}\,.$$

Thus for these values of d and α we have

$$|\log|1-e(\alpha d)|| \ll \log\log x\,.$$

Recalling the definition of $\sum_2$ in (3.3) we conclude that

$$\sum\nolimits_2 \ll \tau(m_0)2^L \log\log x \ll (\log x)^{(1+2\varepsilon)\log 2}\,.$$

Combining this estimate with our estimates (4.6) and (5.1) for $\sum_0$ and $\sum_1$, we have for x sufficiently large, $n \in S - S_E$, and $\alpha \in J(n)$,

$$\log(nA(n)) \ge \log S(n) \ge \log|\Phi_n(e(\alpha))| > \frac{1}{2}\binom{L}{K}L^{-1}\log x\,.$$

From Sections 2 and 6 we have $|S - S_E| \gg_K x$ for any fixed K. Thus by choosing K sufficiently large, we have the Theorem.

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