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An application of the Hooley–Huxley contour

by

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To the memory of Professor Helmut Hasse (1898–1979)

1. Introduction and statement of results. This paper is a continuation of our paper [1]. We begin by stating a special case of what we prove in the present paper.

Theorem 1. *Let k be any complex constant and $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n) n^{-s}$ in $\sigma \ge 2$. Then*

$$\int_1^T |(\zeta(1+it))^k|^2\,dt = T\sum_{n=1}^{\infty} |d_k(n)|^2 n^{-2} + O((\log T)^{|k^2|}), \tag{1}$$

$$\int_1^T \left|\frac{\zeta'(1+it)}{\zeta(1+it)}\right|^2 dt = T\sum_{m\ge 1}\sum_{p} (\log p)^2 p^{-2m} + O((\log T)^2), \tag{2}$$

and

$$\int_1^T |\log\zeta(1+it)|^2\,dt = T\sum_{m\ge 1}\sum_{p} (mp^m)^{-2} + O(\log\log T)\,. \tag{3}$$

Remark 1. In [1] we proved (1) with $k = 1$ and studied the error term in great detail.

Remark 2. The proof of this theorem and Theorem 3 to follow require the use of the Hooley–Huxley contour as modified by K. Ramachandra in [2] (for some explanations see [3]). We write $m(HH)$ for this contour.

Remark 3. We have an analogue of these results for ζ and L-functions of algebraic number fields. In fact, under somewhat general conditions on

$F(s)=\sum_{n=1}^{\infty} a_n n^{-s}$ (or even $\sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ and so on) we can show that

$$\int_1^T |F(1+it)|^2\,dt = T\sum_{n=1}^{\infty} |a_n|^2 n^{-2} + O\Big(\log\log T + \sum_{n\le T^C} |a_n|^2 n^{-1}\Big) \tag{4}$$

where C (>0) is a large constant.

The following theorem is fairly simple to prove.

Theorem 2. *Let* $1=\lambda_1<\lambda_2<\ldots$ *be a sequence of real numbers with* $C_0^{-1}\le \lambda_{n+1}-\lambda_n\le C_0$ *where* C_0 (≥ 1) *is a constant and let* $a_1,a_2,\ldots$ *be any sequence of complex numbers satisfying the following conditions*:

(i) $\sum_{n\le x}|a_n|n^{-1}=O_\varepsilon(x^\varepsilon)$ *for all* $\varepsilon>0$ *and* $x\ge 1$.

(ii) $\sum_{n=1}^{\infty}|a_n|^2 n^{\lambda-2}$ *converges for some constant* λ *with* $0<\lambda<1$.

(iii) $F(s)=\sum_{n=1}^{\infty} a_n\lambda_n^{-s}$ (*which converges in* $\sigma>1$) *is continuable analytically in* $(\sigma\ge 1-\delta,\ t\ge t_0)$ *and there* $|F(s)|<t^A$, *where* δ $(0<\delta<1/10)$, t_0 (≥ 100) *and* A (≥ 2) *are any constants.*

Then

$$\int_{t_0+C_1\log\log T}^{T} |F(1+it)|^2\,dt = T\sum_{n=1}^{\infty}|a_n|^2\lambda_n^{-2} + O\Big(\log\log T + \sum_{n\le T^{C_2}}|a_n|^2 n^{-1}\Big) \tag{5}$$

where C_1 *and* C_2 *are certain positive constants depending on other constants which occur in the definition of* $F(s)$.

We sketch a proof of this theorem. We put $s=1+it,\ t\ge t_0$,

$$R(w)=\exp\left(\left(\sin\frac{w}{100}\right)^2\right), \tag{6}$$

$$\Delta(u)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} u^w R(w)\frac{dw}{w}\qquad (u>0), \tag{7}$$

and

$$\sum_{n=1}^{\infty} a_n\lambda_n^{-s}\Delta\left(\frac{X}{\lambda_n}\right)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} F(s+w)X^w R(w)\frac{dw}{w}\qquad (X=T^{C_3}), \tag{8}$$

C_3 (>0) being a large constant. In the integral just mentioned we cut off the portion $|\mathrm{Im}\,w|\ge C_4\log\log T$ where C_4 (>0) is a large constant and in the remaining part we move the line of integration to $\mathrm{Re}\,w=-\delta$. Observe that in $|\mathrm{Re}\, w| \le 3$ we have

$$R(w) = O\biggl(\biggl(\exp\exp\biggl(\biggl|\mathrm{Im}\,\frac{w}{100}\biggr|\biggr)\biggr)^{-1}\biggr).$$

Without much difficulty we obtain

$$F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s} \varDelta\biggl(\frac{X}{\lambda_n}\biggr) + O(T^{-2}) = A(s) + E(s) \quad \text{say.} \tag{9}$$

Using a well-known theorem of H. L. Montgomery and R. C. Vaughan we have

$$\begin{aligned} \int\limits_{t_0 + C_1 \log\log T}^{T} & |A(1+it)|^2\, dt \\ &= \sum_{n=1}^{\infty} |a_n|^2 \lambda_n^{-2} \biggl|\varDelta\biggl(\frac{X}{\lambda_n}\biggr)\biggr|^2 (T - C_1 \log\log T + O(n)). \end{aligned} \tag{10}$$

Now $\varDelta(u) = O(u^2)$ always but it is also $1 + O(u^{-2})$ and using these we are led to the theorem.

However, the proof of Theorem 1 (and also that of Theorem 3) is not simple. It has to use the density results $N(\sigma, T) = O(T^{B(1-\sigma)}(\log T)^B)$ and $N(\sigma, T) = O(T^{B'(1-\sigma)^{3/2}}(\log T)^{B'})$ (the former is a consequence of the latter if we are not particular to have a small value of B) where B (> 0) and B' (> 0) are constants and $1 - \delta \le \sigma \le 1$. Also it has to use the zero free region $\sigma \ge 1 - C_3(\log t)^{-2/3}(\log\log t)^{-1/3}$ ($t \ge t_0$) for the Riemann zeta function (and more general functions). Since the constant B is unimportant in our proof, Remark 3 below Theorem 1 holds. (In fact, as will be clear from our proof, only the portion $\sigma \ge 1-\delta$ of the $m(HH)$ contour will be enough for our purposes.) Also if only the density result $N(\sigma, T) = O(T^{B(1-\sigma)}(\log T)^B)$ and the zero free region $\sigma \ge 1 - C_5(\log T)^{-1}$ are available then we end up with

$$O\Bigl(\log\log T + \sum_{n \le \exp((\log T)^3)} |a_n|^2 n^{-1}\Bigr)$$

for the error term and it is not hard to improve this to some extent. We now proceed to state our general result.

Consider the set S_1 of all abelian L-series of all algebraic number fields. We can define $\log L(s,\chi)$ in the half plane $\mathrm{Re}\, s > 1$ by the series

$$\sum_m \sum_p \chi(p^m)(mp^{ms})^{-1} \tag{11}$$

where the sum is over all positive integers $m \ge 1$ and p runs over all primes (in the case of algebraic number fields p runs over the norm of all prime ideals). More generally, we can (by analytic continuation) define $\log L(s,\chi)$

in any simply connected domain containing $\operatorname{Re} s > 1$ which does not contain any zero or pole of $L(s,\chi)$. For any complex constant z we can define $(L(s,\chi))^z$ as $\exp(z\log L(s,\chi))$. Let S_2 consist of the derivatives of $L(s,\chi)$ for all L-series and let S_3 consist of the logarithms as defined above for all L-series.

Let $P_1(s)$ be any finite power product (with complex exponents) of functions in S_1. Let $P_2(s)$ be any finite power product (with non-negative integral exponents) of functions in S_2. Also let $P_3(s)$ be any finite power product (with non-negative integral exponents) of functions in S_3. Let b_n ($n = 1,2,3,\ldots$) be complex numbers which are $O_\varepsilon(\exp((\log n)^\varepsilon))$ for every fixed $\varepsilon > 0$ and suppose that $F_0(s) = \sum_{n=1}^\infty b_n n^{-s}$ is absolutely convergent in $\operatorname{Re} s \ge 1-\delta$ where δ ($0 < \delta < 1/10$) is a positive constant. Finally, put

$$F(s) = P_1(s)P_2(s)P_3(s)F_0(s) = \sum_{n=1}^\infty a_n n^{-s}\,. \tag{12}$$

Then we have

Theorem 3. *We have*

$$\int_1^T |F(1+it)|^2\,dt = T\sum_{n=1}^\infty |a_n|^2 n^{-2} + O\Big(\log\log T + \sum_{n\le T^{C_6}} |a_n|^2 n^{-1}\Big) \tag{13}$$

where C_6 (>0) is a large constant.

Remark 1. It is possible to have a more general result. For example we can replace $F(s)$ in (12) and (13) by $F(s) + \sum_{n=1}^\infty d_m(n)(n+\alpha)^{-s}$ where m is a positive integer constant and α is any constant with $0 < \alpha < 1$. Then the right hand side of (13) has to be replaced by

$$T\sum_{n=1}^\infty |a_n|^2 n^{-2} + T\sum_{n=1}^\infty (d_m(n))^2(n+\alpha)^{-2} + O(\log\log T) + O\Big(\sum_{n\le T^{C_6}} (|a_n|^2 + (d_m(n))^2)n^{-1}\Big)\,.$$

2. Proof of Theorem 3. We form the $m(HH)$ contour (associated with L-functions occurring in $F(s)$) as in [2]. But we select a small constant δ ($0 < \delta < 1$) and treat the points $1-\delta+i\nu$ ($\nu = 0,\pm1,\pm2,\ldots$) as though they were zeros associated with L-functions occurring in $F(s)$. We recall

$R(w)=\exp((\sin(w/100))^2)$. Put $s=1+it,\ T_0=C_7\log\log T\le t\le T$,

$$A(s)=\sum_{n=1}^{\infty}a_n n^{-s}\Delta\left(\frac{X}{n}\right) \tag{14}$$

where $\Delta(u)$ and X are as in (8). Then

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}F(s+w)X^wR(w)\frac{dw}{w}=A(s)\,. \tag{15}$$

We write $w=u+iv$ and truncate the portion $|v|\ge\frac12T_0$ and move the w-line of integration so that $s+w$ lies in the portion of the $m(HH)$ contour pertaining to $|v|\le\frac12T_0$. We obtain

$$F(s)=A(s)+E(s) \tag{16}$$

where for fixed t in $(T_0\le t\le T)$,

$$E(s)=-\frac{1}{2\pi i}\int_P F(s+w)X^wR(w)\,\frac{dw}{w} \tag{17}$$

where P is the path consisting of the $m(HH)$ contour in $(u\ge-\delta,\ |v|\le\frac12T_0)$ and the lines connecting it to $\sigma=1$ by lines perpendicular to it at the ends. Notice that to the right of the $m(HH)$ we have (by Lemma 5 of [2])

$$|F(s+w)|\le\exp((\log t)^{\psi}) \tag{18}$$

with a certain constant ψ (satisfying $0<\psi<1$) for $s+w$ on $M_{1,1}$ and $M_{1,2}$ (we adopt the notation of [2]). Also

$$|F(s+w)|\le\exp((\log T)^{\psi'}) \tag{19}$$

with a small constant ψ' $(0<\psi'<1/5)$ for $s+w$ on $M_{1,3}$. With these we have the following contributions to $\int_{T_0/2}^{T+T_0/2}|E(s)|\,dt$ and $\int_{T_0/2}^{T+T_0/2}|E(s)|^2\,dt$. We handle the first integral and the treatment of the second is similar. We have (denoting by P_1 the contour P with the horizontal lines connecting P to $\sigma=1$ omitted)

$$\begin{aligned}\int_{T_0}^{T}|E(s)|\,dt&\le(\log T)^2\int_{T_0}^{T}\int_{P_1}|F(s+w)|X^u|dw|\,dt+T^{-10}\\&\le(\log T)^3\int_Q|F(s)|X^{\sigma-1}|ds|+T^{-10}\end{aligned} \tag{20}$$

where Q is the portion of the $m(HH)$ in $(\sigma\ge1-\delta, T_0/2\le t\le T+T_0/2)$. (Note that s is used as a variable on the $m(HH)$ in the integral in (20).)

(In the case of $\int_{T_0}^T |E(s)|^2\,dt$ we majorise it by

$$(\log T)^4 \int_{T_0}^{T} \Big(\int_{P_1} |F(s+w)|X^u|dw|\Big)^2\,dt + T^{-10}$$
$$\le (\log T)^5 \int_{T_0}^{T} \int_{P_1} |F(s+w)|^2 X^{2u}|dw|\,dt + T^{-10}$$

by Hölder's inequality.)

The contribution to (20) from $M_{1,1}$ is

$$O((\log T)^{20} \max_{1-\delta\le\sigma\le 1-\tau_1} (N(\sigma,T)X^{-(1-\sigma)})\exp((\log T)^{\psi}))$$

and that from $M_{1,2}$ is

$$O((\log T)^{20} \max_{1-\tau_1\le\sigma\le 1-\tau_2} (N(\sigma,T)X^{-(1-\sigma)})\exp((\log T)^{\psi'}))$$

and that from $M_{1,3}$ is

$$O((\log T)^{D}\exp((\log T)^{\psi'})X^{-\tau_3})$$

where τ_1 and τ_2 are determined by $M_{1,1}$, $M_{1,2}$ and $M_{1,3}$ and $\tau_3 = C_3(\log T)^{-2/3}(\log\log T)^{-1/3}$. Here D (>0) is some constant. (Note that X is a large positive constant power of T.) Using the standard estimates (for some details which are very much similar to what we need, see equations (1)–(3) of [3]) we obtain

Lemma 1. *Both* $\int_{T_0}^T |E(s)|\,dt$ *and* $\int_{T_0}^T |E(s)|^2\,dt$ *are* $O(\exp(-(\log T)^{0.1}))$.

Lemma 2. *We have* $A(s) = O(\exp((\log T)^{\varepsilon}))$.

Proof. Follows from the fact that

$$|A(s)| \le \sum_{n=1}^{\infty} |a_n|n^{-1}\left|\Delta\left(\frac{X}{n}\right)\right|.$$

Lemma 3. *The integral* $\int_{T_0}^T |A(s)E(s)|\,dt$ *is* $O(\exp(-\frac12(\log T)^{0.1}))$.

Proof. Follows from Lemmas 1 and 2.

Lemma 4. *We have*

$$\int_{T_0}^{T} |F(s)|^2\,dt = \int_{T_0}^{T} |A(s)|^2\,dt + O(\exp(-\tfrac12(\log T)^{0.1}))\,. \tag{21}$$

Proof. Follows from Lemmas 2 and 3. Now the integral on the right hand side of (21) is

$$\sum_{n=1}^{\infty}(T-T_0+O(n))|a_n|^2 n^{-2}\left|\Delta\left(\frac{X}{n}\right)\right|^2$$

by a well-known theorem of H. L. Montgomery and R. C. Vaughan, and so Theorem 3 follows by a slight further work since $a_n = O_\varepsilon(n^\varepsilon)$ for all $\varepsilon > 0$.

References

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