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Monotone method for nonlinear second order periodic boundary value problems with Carathéodory functions

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Abstract. The purpose of this paper is to study the periodic boundary value problem $-u''(t) = f(t, u(t), u'(t))$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ when f satisfies the Carathéodory conditions. We show that a generalized upper and lower solution method is still valid, and develop a monotone iterative technique for finding minimal and maximal solutions.

1. Introduction. In this paper we consider the following periodic boundary value problem (PBVP for short) of second order:

$$\text{(P)} \quad \begin{cases} -u''(t) = f(t, u(t), u'(t)), & (1.1) \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). & (1.2) \end{cases}$$

As is well known, the method of upper and lower solutions has been successfully applied to study this PBVP when f is a continuous function (see [2–6, 12] and the monograph [9] and the references therein).

Here, we generalize the method of upper and lower solutions to the case when f is a Carathéodory function. We point out that for f continuous the classical arguments of [2–6, 9, 12] are no longer valid since the solutions are in the Sobolev space $W^{2,1}(I)$, $I = [0, 2\pi]$. Thus, if u is a solution, u'' is not necessarily continuous on I but only $u'' \in L^1(I)$.

Our ideas are in the spirit of [7, 10] where $f(t, u(t), u'(t)) = f(t, u(t))$. There, when u is bounded we deduce that u'' is bounded, and so is u'. In our situation, we have to find a bound for the derivative of a solution since the

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derivative of the modified problem relative to (P) may be unbounded. To this purpose we prove a new result (Theorem 1). Thus, we improve the results of [8] where we require f to be locally Lipschitzian or locally equicontinuous in some variables. The proof of some known results are included for the convenience of the reader: For instance, Lemma 4 is taken from [10]. Also we note that part (c) of Lemma 1 is proved in [4] and Theorem 2 is related to the results of Adje in [1] but our proof is simpler using a convenient modified problem.

When v and w are (generalized) lower and upper solutions relative to (P) and $v \le w$, we denote by $S[v,w]$ the set of solutions of (P) in the sector $[v,w] = \{u \in W^{2,1}(I) : v(t) \le u(t) \le w(t) \text{ for } t \in I\}$ (see [7, 10]). We generalize the monotone method [9] to obtain minimal and maximal solutions as limits of monotone iterates.

2. The method of upper and lower solutions. We shall suppose that $f : I \times \mathbb{R}^2 \to \mathbb{R}$, $I = [0, 2\pi]$, is a *Carathéodory function*, that is:

(i) for a.e. $t \in I$, the function $(u,s) \in \mathbb{R}^2 \to f(t,u,s) \in \mathbb{R}$ is continuous,

(ii) for every $(u,s) \in \mathbb{R}^2$, the function $t \in I \to f(t,u,s)$ is measurable,

(iii) for every $R > 0$, there exists a real-valued function $h(t) = h_R(t) \in L^1(I)$ such that

$$|f(t,u,s)| \le h(t) \tag{2.1}$$

for a.e. $t \in I$ and every $(u,s) \in \mathbb{R}^2$ satisfying $|u| \le R$, $|s| \le R$.

A function $u \in W^{2,1}(I)$ is a solution of (P) if (1.1) holds for a.e. $t \in I$, and u satisfies (1.2). When f is continuous, any solution of (P) is a classical solution, that is, a C^2-solution. If, in addition, f is 2π-periodic in t, then any solution can be extended by periodicity to $\mathbb{R}$, and then it is a periodic solution of (1.1).

Let us say that a function $v : I \to \mathbb{R}$ is a *lower solution* of (P) if $v \in W^{2,1}(I)$,

$$-v''(t) \le f(t, v(t), v'(t)) \quad \text{for a.e. } t \in I \tag{2.2}$$

and

$$v(0) = v(2\pi), \quad v'(0) \ge v'(2\pi)\,. \tag{2.3}$$

Similarly, $w : I \to \mathbb{R}$ is an *upper solution* of (P) if $w \in W^{2,1}(I)$,

$$-w''(t) \ge f(t, w(t), w'(t)) \quad \text{for a.e. } t \in I \tag{2.4}$$

and

$$w(0) = w(2\pi), \quad w'(0) \le w'(2\pi)\,. \tag{2.5}$$

Throughout we shall suppose that $v \le w$ on I. We shall consider the following condition:

(H1) There exists $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous such that

$$|f(t,u,s)| \le g(|s|)$$

for a.e. $t \in I$ with $v(t) \le u \le w(t)$, and $s \in \mathbb{R}$, satisfying

$$\int_\lambda^\infty \frac{s}{g(s)+C}\, ds = \infty \qquad \forall \lambda > 0 \text{ and } \forall C > 0\,. \tag{2.6}$$

Note that the usual Nagumo condition $\int_\lambda^\infty (s/g(s))\, ds = \infty$ implies (2.6) when either $\limsup_{s\to\infty} g(s) < \infty$ or $\liminf_{s\to\infty} g(s) > 0$.

Now, we give a priori estimates for the derivative of solutions of (P).

Lemma 1. *Let* $0 \le t_1 < t_2 \le 2\pi$, $u \in W^{2,1}([t_1,t_2])$ *and assume that* $v \le u \le w$ *on* $[t_1,t_2]$ *and* (1.1) *is satisfied for a.e.* $t \in [t_1,t_2]$. *If* (H1) *holds, then there exists a positive constant* N *which depends only on* v, w, g *and a constant* C, *such that*:

(a) $u'(t_1) \le C$ *or* $u'(t_2) \le C$ *implies* $u'(t) \le N$ *on* $[t_1,t_2]$.
(b) $u'(t_1) \ge C$ *or* $u'(t_2) \ge C$ *implies* $u'(t) \ge -N$ *on* $[t_1,t_2]$.
(c) $u(t_1) - u(t_2) = u'(t_1) - u'(t_2) = 0$ *implies* $|u'(t)| \le N$ *on* $[t_1,t_2]$.

Proof. (a) Suppose that $u'(t_1) \le C$ and that

$$\forall n \in \mathbb{N},\ \exists T_n \in [t_1,t_2] \text{ such that } u'(T_n) = n\,. \tag{2.7}$$

Let $n_0 \in \mathbb{N}$ be such that

$$\int_{|C|}^{n_0} \frac{s}{g(s)}\, ds > \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.$$

By (2.7) there exists $\bar t \in [t_1, T_{n_0}]$ such that $u'(\bar t) = |C|$ and $0 \le |C| \le u'(t) \le n_0$ for all $t \in [\bar t, T_{n_0}]$. In this interval we obtain

$$|-u''(t)| = |f(t,u(t),u'(t))| \le g(|u'(t)|)$$

and

$$\frac{u''u'}{g(u')} \le \frac{|-u''|u'}{g(|u'|)} \le u'\,.$$

Thus,

$$\begin{aligned}\int_{\bar t}^{T_{n_0}} \frac{u'(t)u''(t)}{g(u'(t))}\, dt &\le \int_{\bar t}^{T_{n_0}} u'(t)\, dt = u(T_{n_0}) - u(\bar t)\\ &\le w(T_{n_0}) - v(\bar t) \le \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.\end{aligned}$$

On the other hand,

$$\int_{\bar t}^{T_{n_0}} \frac{u'(t)u''(t)}{g(u'(t))}\, dt = \int_{|C|}^{n_0} \frac{s}{g(s)}\, ds > \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.$$

As a consequence, there exists $N > 0$ such that $u'(t) \le N$ on $[t_1, t_2]$.

If $u'(t_2) \le C$ and the assertion of (a) is not satisfied, then we deduce that property (2.7) holds.

Let $n_1 \in \mathbb{N}$ be such that

$$\int_{|C|}^{n_1} \frac{s}{g(s)}\, ds > \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.$$

By (2.7) there exists $T \in [T_{n_1}, t_2]$ such that $u'(T) = |C|$ and $0 \le |C| \le u'(t) \le n_1$ for all $t \in [T_{n_1}, T]$. Thus,

$$-\frac{u''u'}{g(u')} \le \frac{|-u''|u'}{g(|u'|)} \le u' \qquad \text{on } [T_{n_1}, T]$$

and

$$-\int_{T_{n_1}}^{T} \frac{u'(t)u''(t)}{g(u'(t))}\, dt \le \int_{T_{n_1}}^{T} u'(t)\, dt \le \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.$$

On the other hand,

$$-\int_{T_{n_1}}^{T} \frac{u'(t)u''(t)}{g(u'(t))}\, dt = -\int_{n_1}^{|C|} \frac{s}{g(s)}\, ds = \int_{|C|}^{n_1} \frac{s}{g(s)}\, ds > \max_{t\in I} w(t) - \min_{t\in I} v(t)\,.$$

Therefore there exists $N > 0$ such that $u'(t) \le N$ on $[t_1, t_2]$.

Analogously we prove (b). The proof of (c) is given in Lemma 3.2 of [4]. ■

For any $u \in X = C^1(I)$, we define

$$p(t,u) = \begin{cases} v(t), & u < v(t),\\ u, & v(t) \le u \le w(t),\\ w(t), & u > w(t). \end{cases}$$

We obtain the following series of results:

Lemma 2. *For $u \in X$, the following two properties hold*:

(a) $\frac{d}{dt}p(t,u(t))$ *exists for a.e.* $t \in I$.

(b) *If* $u, u_m \in X$ *and* $u_m \xrightarrow{X} u$, *then*

$$\left\{\frac{d}{dt}p(t,u_m(t))\right\} \to \frac{d}{dt}p(t,u(t)) \qquad \text{for a.e. } t \in I\,.$$

Proof. Note that if $u \in X$ then $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ are absolutely continuous. We rewrite $p(t,u) = [u - v(t)]^- - [u - w(t)]^+ + u$. Because $u, v, w \in X$, it is enough to prove that if $u, u_m \in X$ and $u_m \xrightarrow{X} u$, then

$$\left\{\frac{d}{dt}p(t,u_m^{\pm}(t))\right\} \to \frac{d}{dt}p(t,u^{\pm}(t)) \qquad \text{for a.e. } t \in I\,.$$

Since $\frac{d}{dt}u_m^+(t)$, $\frac{d}{dt}u^+(t)$ exist for a.e. $t \in I$, suppose that $t_0 \in I$ is such that $\frac{d}{dt}u_m^+(t_0)$ and $\frac{d}{dt}u^+(t_0)$ exist for all $m = 1, 2, \ldots$

If $u(t_0) > 0$, then $u(t_0) = u^+(t_0) > 0$. Therefore $\frac{d}{dt}u^+(t_0) = \frac{d}{dt}u(t_0)$ and there exists $M \in \mathbb{N}$ such that $u_m(t_0) = u_m^+(t_0) > 0$ for all $m \geq M$. Thus

$$\frac{d}{dt}u_m^+(t_0) = \frac{d}{dt}u_m(t_0) \to \frac{d}{dt}u(t_0)\,.$$

If $u(t_0) < 0$, then there exists $M > 0$ such that $u_m(t_0) < 0$ for all $m \geq M$. Therefore $u_m(t) < 0$ on $(t_0 - \delta_m, t_0 + \delta_m)$ for some $\delta_m > 0$ and then $u_m^+(t) = 0$ on $(t_0 - \delta_m, t_0 + \delta_m)$. Hence $\frac{d}{dt}u_m^+(t_0) = \frac{d}{dt}u^+(t_0) = 0$ and then obviously

$$\frac{d}{dt}u_m^+(t_0) \to \frac{d}{dt}u^+(t_0) \quad \text{as } m \to \infty\,.$$

If $u(t_0) = 0$, then $u^+(t_0) = 0$. Since $\frac{d}{dt}u^+(t_0)$ exists, we have $\frac{d}{dt}u^+(t_0) = 0$. It is easy to prove that $\frac{d}{dt}u(t_0) = 0$.

Because $\frac{d}{dt}u_m^+(t_0)$ exists, we find that

$$\frac{d}{dt}u_m^+(t_0) = \begin{cases} u_m'(t_0), & u_m(t_0) > 0,\\ 0, & u_m(t_0) \leq 0. \end{cases}$$

Therefore

$$\left|\frac{d}{dt}u_m^+(t_0)\right| \leq \left|\frac{d}{dt}u_m(t_0)\right| \to \left|\frac{d}{dt}u(t_0)\right| = 0 = \frac{d}{dt}u^+(t_0)\,.$$

Similarly, we can prove the conclusions about $u^-(t)$, and thus the proof of Lemma 2 is complete. ■

Now, consider the following modified problem:

$$\begin{cases} -u'' + u = f^*(t,u,u') + p(t,u),\\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \end{cases} \tag{2.8}$$

where $f^*(t,u(t),u'(t)) = f(t,p(t,u(t)),\frac{d}{dt}p(t,u(t)))$.

Since $u \in X$, $\frac{d}{dt}p(t,u(t))$ exists for a.e. $t \in I$. If $t_0 \in I$ is such that $\frac{d}{dt}p(t_0,u(t_0))$ does not exist, then it is easy to prove that the left and right derivatives of $p(t,u(t))$ at t_0 must exist and both values depend only on the X-norms of u, v and w. Therefore we can complement the values of $\frac{d}{dt}p(t,u(t))$ in such a way that it is bounded and the bound depends only on the X-norm of u, v and w. For any $z \in X$, the linear problem

$$\begin{cases} -u'' + u = f^*(t,z(t),z'(t)) + p(t,z(t)) \equiv \sigma(t),\\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \end{cases} \tag{2.9}$$

has a unique solution u given by the formula

$$u(t) = C_1 e^t + C_2 e^{-t} - \frac{e^t}{2} \int_0^t \sigma(s) e^{-s}\, ds + \frac{e^{-t}}{2} \int_0^t \sigma(s) e^s\, ds \tag{2.10}$$

where

$$C_1 = \frac{1}{2(e^{2\pi} - 1)} \int_0^{2\pi} \sigma(s) e^{2\pi - s}\, ds\,,$$

$$C_2 = \frac{1}{2(e^{2\pi} - 1)} \int_0^{2\pi} \sigma(s) e^{s}\, ds\,.$$

Note that $\sigma(t) = f^*(t, z(t), z'(t)) + p(t, z(t))$ is measurable, $|p(t, z(t))| \le R$ and $\left|\frac{d}{dt} p(t, z(t))\right| \le R$, which implies that $|f^*(t, z(t), z'(t))| \le h(t) \in L^1(I)$ and $\sigma \in L^1(I)$.

From (2.10) and the formula

$$u'(t) = C_1 e^t - C_2 e^{-t} - \frac{e^t}{2} \int_0^t \sigma(s) e^{-s}\, ds + \frac{e^{-t}}{2} \int_0^t \sigma(s) e^s\, ds \tag{2.11}$$

it is clear that $u \in X$.

Define the operator $T : X \to X$, where $T(z) = u$, with u defined by (2.10). For this operator we obtain the following result

LEMMA 3. *$T : X \to X$ is compact.*

Proof. Let $z_m \in X$, $m \in \mathbb{N}$, $z_m \xrightarrow{X} z$, $T(z_m) = u_m$, $T(z) = u$. We have $\|z_m\|_X \le M$ for some $M > 0$. Then $p(t, z_m(t)) \to p(t, z(t))$ for a.e. $t \in I$ and $|p(t, z_m(t))| \le M$, $\left|\frac{d}{dt} p(t, z_m(t))\right| \le N$ for a.e. $t \in I$ and for some N depending only on M, v and w.

Now, $|f^*(t, z(t), z'(t))| \le h(t) \in L^1(I)$ follows from (2.1). By the hypothesis on f and Lemma 2 we know that $f^*(t, z_m(t), z'_m(t))$ converges to $f^*(t, z(t), z'(t))$ in measure. Hence, by the Lebesgue dominated convergence theorem,

$$\lim_{m\to\infty} \int_0^t \sigma_m(s) e^{\pm s}\, ds = \int_0^t \sigma(s) e^{\pm s}\, ds$$

where $\sigma_m(t) = f^*(t, z_m(t), z'_m(t)) + p(t, z_m(t))$.

By (2.10) and (2.11) we have $\lim_{m\to\infty}(u_m(t), u'_m(t)) = (u(t), u'(t))$ for every $t \in I$. Because $|\sigma_m(s)| \le h(s) + |v(s)| + |w(s)| \in L^1(I)$, the sequence $\{g_m(t)\} = \{\int_0^t \sigma_m(s) e^{\pm s}\, ds\}$ is equicontinuous, and so are $\{u_m(t)\}$ and $\{u'_m(t)\}$.

It is obvious that $\{u_m(t), u'_m(t)\}$ is uniformly bounded. Therefore, by the Ascoli theorem $u_m \xrightarrow{X} u$. Hence T is continuous.

Similarly, for any bounded set $B \subset X$, let $B_1 = \{u : u = T(z)$ for some $z \in B\}$ and $B_2 = \{u' : u \in B_1\}$. Then B_1 and B_2 are equicontinuous and uniformly bounded. Thus, there exist subsequences $\{u_m\} = \{Tz_m\} \subset B_1$ and $\{u'_m\} \subset B_2$ such that $u_m \to u$ and $u'_m \to \overline{u}$ uniformly on I. Using (2.10) and (2.11) it is easy to prove that $\overline{u}(t) = u'(t)$. In consequence, $u_m \xrightarrow{X} u$, and this shows that T is compact. ∎

LEMMA 4. *Let $y \in W^{2,1}(I)$ and suppose that there exists $M \in L^1(I)$ such that $M(t) > 0$ for a.e. $t \in I$ and $y''(t) \ge M(t)y(t)$ for a.e. $t \in I$, $y(0) = y(2\pi)$, $y'(0) \ge y'(2\pi)$. Then $y(t) \le 0$ for every $t \in I$.*

Proof. The proof can be found in [10, Lemma 3.1] and we present it for the sake of completeness. If $X \subset I$ is such that $y(t) > 0$ for a.e. $t \in X$, then $y''(t) > 0$ for a.e. $t \in X$. In consequence, there exists at least one $\tau \in I$ with $y(\tau) \le 0$. If $y(0) > 0$, then there exist $0 \le s_1 \le s_2 \le 2\pi$ with $y(s_1) = y(s_2) = 0$ and $y(s) > 0$ for $s \in J = [0, s_1) \cup (s_2, 2\pi] \subset X$. Thus, y' is nondecreasing on J and we get a contradiction since $y'(0) \ge y'(2\pi)$. Hence, $y(0) \le 0$.

Now, if $\max\{y(s) : s \in I\} = y(t_0) > 0$, then there exist $t_1, t_2 \in (0, 2\pi)$ such that $t_1 < t_0 < t_2$, $y(t_1) = y(t_2) = 0$, and $y(s) > 0$ for $s \in (t_1, t_2)$. In consequence, y' is nondecreasing on (t_1, t_2), and this is not possible since $y(t_1) = y(t_2) = 0$ and $y(t_0) > 0$. ∎

LEMMA 5. *Let $u \in W^{2,1}([t_1, t_2])$, $h \in L^1([t_1, t_2])$ and c be constant, $-u''(t) = f(t)$, with $|f(t)| \le h(t)$ for a.e. $t \in [t_1, t_2]$. Then there exists a constant $N > 0$ depending only on c and h such that*:

(a) *$u'(t_1) \le c$ or $u'(t_2) \le c$ implies $u'(t) \le N$ on $[t_1, t_2]$.*

(b) *$u'(t_1) \ge c$ or $u'(t_2) \ge c$ implies $u'(t) \ge -N$ on $[t_1, t_2]$.*

Proof. (a) If $u'(t_1) \le c$, taking into account that $u''(t) \le |-u''(t)| = |f(t)| \le h(t)$, we obtain

$$u'(t) \le u'(t_1) + \int_{t_1}^{t} h(s)\,ds \le c + \|h\|_1 \qquad \text{on } [t_1, t_2]\,.$$

If $u'(t_2) \le c$, then from $-u''(t) \le |-u''(t)| = |f(t)| \le h(t)$ we get

$$u'(t) \le u'(t_2) + \int_{t_2}^{t} h(s)\,ds \le c + \|h\|_1 \qquad \text{on } [t_1, t_2]\,.$$

(b) If $u'(t_1) \ge c$, then $-u'(t) \le -u'(t_1) + \|h\|_1 \le -c + \|h\|_1$ on $[t_1, t_2]$, that is, $u'(t) \ge -N$ on $[t_1, t_2]$.

If $u'(t_2) \ge c$, then $-u'(t) \le -u'(t_2) + \|h\|_1 \le -c + \|h\|_1$, i.e. $u'(t) \ge -N$ on $[t_1, t_2]$. ∎

Using the previous lemmas we obtain the following a priori estimate for the solutions of problem (2.8).

THEOREM 1. *There exists a constant $M > 0$ such that if $\lambda \in [0,1]$, $u \in X$ and $u = \lambda Tu$, then $\|u\|_X \le M$.*

Proof. The equation $u = \lambda Tu$ is equivalent to

$$\begin{cases} -u'' + u = \lambda f^*(t, u(t), u'(t)) + \lambda p(t, u(t)), \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \end{cases} \tag{2.12}$$

We divide the proof into two parts:

Step 1: *Estimate for $u(t)$*. Let $I^0 = (0, 2\pi)$, $A_1 = \{t \in I^0 : u(t) > w(t)\}$. We distinguish two cases:

(1.a) $I^0 = A_1$. Then, for a.e. $t \in I^0$ we have

$$-u''(t) + u(t) = \lambda f(t, w(t), w'(t)) + \lambda w(t) \le -\lambda w''(t) + \lambda w(t)\,.$$

Hence, $y(t) = u(t) - \lambda w(t)$ satisfies

$$\begin{cases} y''(t) \ge y(t) \quad \text{for a.e. } t \in I^0, \\ y(0) = y(2\pi),\ y'(0) \ge y'(2\pi). \end{cases}$$

From Lemma 4 we conclude that $y \le 0$, that is, $u \le \lambda w \le C$ on I.

(1.b) $I^0 \ne A_1$. Thus, there exists $s_1 \in I^0$ such that $u(s_1) \le w(s_1)$. We first prove that there exists a positive constant C depending only on w such that $u(0) \le C$. Obviously this is true if $u(0) \le w(0)$.

In case $u(0) > w(0)$, let $y(t) = u(t) - \lambda w(t)$. We suppose that $y(0) > 0$ since $y(0) \le 0$ implies that $u(0) \le \lambda w(0)$.

For $y'(0) \ge 0$, let $t_0 = \sup\{t \in I : y(s) > 0 \text{ for } s \in [0, t)\}$ and

$$t^* = \sup\{t \in [0, s_1) : u(s) > w(s) \text{ for } s \in [0, t)\}\,.$$

Then $t^* \le s_1 < 2\pi$, $u(t^*) = w(t^*)$ and $u > w$ on $[0, t^*)$.

We shall prove that $t_0 > t^*$. If not, $y''(t) \ge y(t) > 0$ for a.e. $t \in [0, t_0)$ and $y'(t) > y'(0) \ge 0$. Hence, $y'(t_0) > y'(0) > 0$. By the definition of t_0 we see that $t_0 = 2\pi$, and $y'(2\pi) > y'(0)$. This implies that $w'(2\pi) < w'(0)$, a contradiction with (2.5). This shows that $t_0 > t^*$.

Therefore $y''(t) \ge y(t) > 0$ for a.e. $t \in [0, t^*)$ and thus $y'(t) > y'(0) \ge 0$. This implies that $y(0) \le y(t^*) = u(t^*) - \lambda\, w(t^*) = (1-\lambda) w(t^*)$ and that $u(0) \le \lambda\, w(0) + (1-\lambda) w(t^*) \le C$.

For $y'(0) < 0$, we have $y'(2\pi) \le y'(0) < 0$, $y(2\pi) = y(0) > 0$, $u(2\pi) - w(2\pi) = u(0) - w(0) > 0$. Choosing $t_1 = \inf\{t \in I : y(s) > 0 \text{ for } s \in (t, 2\pi]\}$ and $\overline{t} = \inf\{t \in (s_1, 2\pi] : u(s) > w(s) \text{ for } s \in (t, 2\pi]\}$ and reasoning as in the previous case we again obtain $u(0) \le C$.

We decompose $A_1 = \bigcup (a_i, b_i)$ so that $u(t) > w(t)$ for $t \in (a_i, b_i)$ and

$$-y''(t) + y(t) \le 0 \qquad \text{for a.e. } t \in (a_i, b_i)\,. \tag{2.13}$$

By the definition of a_i and b_i we obtain $y(a_i) = (1-\lambda)w(a_i)$ and $y(b_i) = (1-\lambda)w(b_i)$. In consequence, there exists $C \in \mathbb{R}$ such that

$$y(a_i) \le C \quad \text{and} \quad y(b_i) \le C \,. \tag{2.14}$$

Now, (2.13) and (2.14) imply that $y(t) \le C+1$ for $t \in (a_i, b_i)$. Therefore, $u(t) \le C + 1 + \lambda w(t) \le M$ on $\overline{A}_1$. Obviously, $u \le M$ on $I \setminus \overline{A}_1$ and thus $u \le M$ on I.

Similarly, we can prove that $u \ge -M$ on I. Hence $|u(t)| \le M$ for any $t \in I$.

Step 2: *Estimate for* $u'(t)$. Let $B = \{t \in I : v(t) < u(t) < w(t)\}$. Suppose that $B \ne \emptyset$. Then $p(t, u(t)) = u(t)$ for $t \in B$ and $u(t) \le v(t)$ or $u(t) \ge w(t)$ for $t \in I \setminus B$. We write $B = \bigcup (a_i, b_i)$ since B is an open set. For (a_i, b_i), only one of the following situations hold:

(2.i) $0 < a_i < b_i < 2\pi$, $[u(a_i) - v(a_i)] \cdot [w(a_i) - u(a_i)] = 0$, $[u(b_i) - v(b_i)] \cdot [w(b_i) - u(b_i)] = 0$ and $v(t) < u(t) < w(t)$ for $t \in (a_i, b_i)$.

(2.ii) $a_i = 0$ or $b_i = 2\pi$.

In the first situation we have $p(t, u(t)) = u(t)$ and $\frac{d}{dt} p(t, u(t)) = u'(t)$.

Now, consider the following four cases:

(2.i.I) $u(a_i) = v(a_i)$ and $u(b_i) = v(b_i)$. Then $u'(a_i) \ge v'(a_i)$ and $u'(b_i) \le v'(b_i)$. Thus,

$$-u'' = \lambda f(t, u, u') + (\lambda - 1)u \equiv \widetilde{f}(t, u, u') \,,$$
$$|\widetilde{f}(t, u, u')| \le g(|u'|) + C \equiv \widetilde{g}(|u'|)$$

and, by the hypothesis (H1),

$$\int_{\lambda}^{\infty} \frac{s}{\tilde{g}(s) + K} \, ds = \infty \qquad \forall \lambda > 0 \text{ and } \forall K > 0 \,.$$

By Lemma 1 we know that there exists a constant N depending only on g, v and w such that $|u'| \le N$ on $[a_i, b_i]$.

(2.i.II) $u(a_i) = w(a_i)$ and $u(b_i) = w(b_i)$. Then $|u'| \le N$ on $[a_i, b_i]$.

(2.i.III) $u(a_i) = v(a_i)$ and $u(b_i) = w(b_i)$. Then $u'(a_i) \ge v'(a_i)$ and $u'(b_i) \ge w'(b_i)$. By Lemma 1, $u'(t) \ge -N$ on $[a_i, b_i]$.

If $u'(a_i) = v'(a_i)$ or $u'(b_i) = w'(b_i)$, then by Lemma 1, $u' \le N$ on $[a_i, b_i]$. Otherwise $u'(a_i) > v'(a_i)$ and $u'(b_i) > w'(b_i)$. Let $a = \inf\{t : u'(s) > v'(s)$ for $s \in (t, a_i)\}$ and $b = \sup\{t : u'(s) > w'(s)$ for $s \in (b_i, t)\}$. Then $a < a_i < b_i < b$, $u'(a) \ge v'(a)$ and $u'(b) \ge w'(b)$. Moreover, $u' > v'$ on (a, a_i) and $u' > w'$ on (b_i, b).

Now, $u(a_i) = v(a_i)$ and $u(b_i) = w(b_i)$ imply that $u > w$ on $(b_i, b]$ and $u < v$ on $[a, a_i)$. We conclude that $(u'(a) - v'(a)) \cdot (u'(b) - w'(b)) = 0$. Otherwise, $u'(a) > v'(a)$ and $u'(b) > w'(b)$. Therefore $a = 0$ and $b = 2\pi$ by the definitions of a and b. Thus $u(0) < v(0) \le w(0) = w(2\pi) < u(2\pi)$, and this is a contradiction.

If $u'(b) = w'(b)$, then $-u'' = \lambda f(t, w, w') + \lambda w - u \le \lambda h(t) + c$ for a.e. $t \in (b_i, b]$. By integration,

$$\begin{aligned} u'(t) &\le \lambda \int_t^b h(s)\,ds + 2\pi c + u'(b) \\ &= w'(b) + \lambda \int_t^b h(s)\,ds + 2\pi c \le C \qquad \text{on } (b_i, b]\,. \end{aligned}$$

Hence $u'(b_i) \le C$. Using again Lemma 1 we have $u'(t) \le N$ on $[a_i, b_i]$. If $u'(a) = v'(a)$, then similarly we see that $u'(t) \le N$ on $[a_i, b_i]$. Hence $|u'| \le N$ on $[a_i, b_i]$.

(2.i.IV) If $u(b_i) = v(b_i)$ and $u(a_i) = w(a_i)$, then analogously to (2.i.III), $|u'| \le N$ on $[a_i, b_i]$.

To show (2.ii), suppose $a_i = 0$; the boundary conditions for v, u and w imply that $b_i = 2\pi$.

Let $a = \sup\{t \in I : v(s) < u(s) < w(s) \text{ for } s \in [0, t)\}$. Then $u(a) = v(a)$ or $u(a) = w(a)$.

If $u(a) = v(a)$, then it is clear that $u'(a) \le v'(a)$. Lemma 1 implies $u'(t) \le N$ for a.e. $t \in [0, a]$. If $u'(a) = v'(a)$ we obtain $u'(t) \ge -N$; therefore $u'(a) < v'(a)$.

Now, let $t_0 = \sup\{t \in I : u'(s) < v'(s) \text{ for } s \in (a, t)\}$.

If $u'(t_0) < v'(t_0)$ we obtain $t_0 = 2\pi$ and $u(2\pi) < v(2\pi)$, which is a contradiction. In consequence, $u'(t_0) = v'(t_0)$ and $t_0 < 2\pi$. In the interval (a, t_0) we have

$$-u'' = \lambda f(t, v, v') + \lambda v - u \ge -\lambda h + C\,.$$

Thus

$$-\int_t^{t_0} u''(s)\,ds \ge K$$

and $u'(t) \ge K + v'(t_0) = K_1$ on (a, t_0). By continuity $u'(a) \ge K_1$, and Lemma 1 implies $|u'| \le N$ on $[0, a]$.

If $u(a) = w(a)$, the reasoning is analogous.

If $b_i = 2\pi$, we obtain $|u'| \le N$ on $[b, 2\pi]$ for

$$b = \inf\{t \in I : v(s) < u(s) < w(s) \text{ for } s \in (t, 2\pi]\}\,.$$

Thus, we obtain $|u'(t)| \le N$ for all $t \in B \cup D$, with

$$\begin{aligned} D = \{a_i, b_i \in (0, 2\pi) :& \\ &\text{either } (a_i, b_i) \in B;\ \text{or } [0, b_i) \in B;\ \text{or } (a_i, 2\pi] \in B\}\,. \end{aligned}$$

If $B \neq I^0$, let $B_1 = \{t \in I : u(t) < v(t)\}$, $B_2 = \{t \in I : u(t) > w(t)\}$. Then $B_1 \neq I$ and $B_2 \neq I$.

First we suppose that $B_1 \neq \emptyset$ and $B_2 \neq \emptyset$. Decompose $B_1 = \bigcup(a_i, b_i)$ and $B_2 = \bigcup(c_i, d_i)$.

For (a_i, b_i), we have one of the following possibilities:

(2.A) $0 < a_i < b_i < 2\pi$.
(2.B) $a_i = 0$ or $b_i = 2\pi$.

In the first case $u'(a_i) \leq v'(a_i)$ and $u'(b_i) \geq v'(b_i)$. Since $-u'' = \lambda f(t, v, v') + \lambda v - u$, Lemma 5 implies $|u'| \leq N$ on $[a_i, b_i]$.

In the second situation, we first consider $a_i = 0$. Then $u(0) < v(0)$ and $u(2\pi) < v(2\pi)$, that is, $b_i = 2\pi$. In consequence, there exists $a \in (0, 2\pi)$ such that $u(a) = v(a)$ and $u(t) < v(t)$ on $[0, a)$. Thus, without loss of generality, we can assume $u'(a) > v'(a)$ (otherwise, Lemma 5 implies $|u'| \leq N$ on $[0, a]$).

Now, if $v(a) < w(a)$, let $b = \sup\{t \in I : v(s) < w(s) \text{ for } s \in [a, t)\}$. Hence, there exists $\overline{t} \leq b$ such that $u'(t) > v'(t)$ on $[a, \overline{t})$. Therefore $v(t) < u(t) < w(t)$ on $[a, \overline{t})$, and consequently $a \in D$. Thus $|u'(a)| \leq N$ and Lemma 5 assures that $|u'| \leq N$ on $[0, a]$.

On the other hand, if $v(a) = w(a)$ and $v'(a) < u'(a) < w'(a)$ there exists a subinterval $(a, a+\delta) \subset (0, 2\pi)$ such that $v < u < w$ on $(a, a+\delta)$; then $a \in D$ and $|u'(a)| \leq N$. Again, Lemma 5 implies $|u'| \leq N$ on $[0, a]$.

If $v(a) = w(a)$ and $u'(a) = w'(a)$, Lemma 5 implies $|u'| \leq N$ on $[0, a]$.

Finally, if $v(a) = w(a)$ and $u'(a) > w'(a)$ there exists $t_0 \in (0, 2\pi)$ such that $u > w$ on (a, t_0) with $u'(t_0) = w'(t_0)$. Therefore $-u'' = \lambda f(t, v, v') + \lambda v - u$ on (a, t_0) and $u'(t) \leq w'(t_0) + c$ for all $t \in (a, t_0)$. The continuity of u' and Lemma 5 imply $|u'| \leq N$ on $[0, a]$.

If $b_i = 2\pi$ the proof is analogous.

For the set B_2 the reasoning is similar.

Thus, we obtain $|u'(t)| \leq N$ for all $t \in E \cup F \equiv S$, where $E = B \cup B_1 \cup B_2$ and

$$F = \{a_i, b_i \in (0, 2\pi) : \text{either } (a_i, b_i) \in E;\ \text{or } [0, b_i) \in E;\ \text{or } (a_i, 2\pi] \in E\}.$$

If $t \in I \backslash S$, then obviously either $u(t) = v(t)$ or $u(t) = w(t)$. Also there exists $\{x_n\} \subset F$, $x_n \neq t$ for all $n \in \mathbb{N}$, such that $t = \lim_{n\to\infty} x_n$ because if there exists $\delta > 0$ such that $I \cap (t - \delta, t + \delta) \cap F = \emptyset$ then $t \in S$. Since $|u'(x_n)| \leq N$ for all $\{x_n\} \subset F$ we obtain $|u'(t)| = |\lim_{n\to\infty} u'(x_n)| \leq N$ for all $t \in I \backslash S$.

This completes the proof of Theorem 1. ∎

THEOREM 2. *Suppose that* $v(t) \leq w(t)$ *are lower and upper solutions of* (P), *respectively. If* (H1) *holds, then there exists a solution* u *of* (P) *such that* $u \in [v, w]$.

Proof. Let $X = C^1(I)$. By Lemma 2, $\frac{d}{dt}p(t,u(t))$ exists for a.e. $t \in I$. Problem (2.8) is equivalent to the functional equation $u = Tu$, with T defined as in Lemma 3. By Theorem 1 we know that every solution of $u = \lambda Tu$ satisfies $\|u\|_X \le M$ for some constant $M > 0$. In consequence, the Shaefer theorem [11] implies that there exists a solution u of problem (2.8).

Finally, we prove that every solution u of (2.8) is such that $u \in [v,w]$, that is, u is a solution in $[v,w]$ of problem (P). Indeed, suppose that $u > w$ on $[0,2\pi]$. Then

$$-u'' + u = f(t,w,w') + w \le -w'' + w\,.$$

Since $(u-w)(0) = (u-w)(2\pi)$ and $(u-w)'(0) \ge (u-w)'(2\pi)$, Lemma 4 implies that $u \le w$ on $[0,2\pi]$, which is a contradiction. Consequently, there exists $s \in [0,2\pi]$ such that $u(s) \le w(s)$. If there exists $s_1 \in [0,2\pi]$ with $u(s_1) > w(s_1)$, and there exists $t_1 < t_2$ in $(0,2\pi)$ such that $u > w$ on (t_1,t_2), with $(u-w)(t_1) = (u-w)(t_2) = 0$, then in the interval (t_1,t_2) we have

$$-u'' + u = f(t,w,w') + w \le -w'' + w\,.$$

This, together with the boundary conditions, implies that $u \le w$ on (t_1,t_2), which is a contradiction.

Therefore, suppose that there exist $t_1 \le t_2$ in $(0,2\pi)$ such that $u > w$ on $[0,t_1) \cup (t_2,2\pi]$ with $(u-w)(t_1) = (u-w)(t_2) = 0$. In both intervals we have $(u-w)'' \ge u - w > 0$.

If $(u-w)'(0) \ge 0$ then $(u-w)'(t) > 0$ for any $t \in [0,t_1)$ and $(u-w)(t_1) > (u-w)(0) > 0$, which is not possible.

On the other hand, if $(u-w)'(0) < 0$, we obtain $(u-w)'(2\pi) < 0$. In consequence, $(u-w)' < 0$ on $(t_2,2\pi]$ and $(u-w)(t_2) > (u-w)(2\pi) > 0$.

Therefore $u \le w$ on the interval I. Analogously we can prove that $u \ge v$ on I. Hence, every solution of (2.8) is a solution of problem (P) in the sector $[v,w]$.

This completes the proof of Theorem 2. ■

3. Monotone iterative technique. Throughout this section we suppose that $v \le w$ are lower and upper solutions of (P), respectively. We introduce the following hypotheses:

(H2) There exists $M \in L^1(I)$ such that $M(t) > 0$ for a.e. $t \in I$ and

$$f(t,\phi,s) - f(t,\varphi,s) \ge -M(t)(\phi - \varphi) \tag{3.1}$$

for a.e. $t \in I$ and every $v(t) \le \varphi \le \phi \le w(t)$, $s \in \mathbb{R}$.

(H3) There exists $N \in L^1(I)$ such that $N(t) \ge 0$ for a.e. $t \in I$ and

$$f(t,u,s) - f(t,u,y) \ge -N(t)(s - y) \tag{3.2}$$

for a.e. $t \in I$ and every $v(t) \le u \le w(t)$, $s \ge y$, $s,y \in \mathbb{R}$.

THEOREM 3. *Suppose that* (H1)–(H3) *hold. Then there exist monotone sequences* $v_n \nearrow x$ *and* $w_n \searrow z$ *as* $n \to \infty$, *uniformly on* I, *with* $v_0 = v$ *and* $w_0 = w$. *Here*, x *and* z *are the minimal and maximal solutions of* (P) *respectively on* $[v, w]$, *that is*, *if* $u \in [v, w]$ *is a solution of* (P), *then* $u \in [x, z]$. *Moreover*, *the sequences* $\{v_n\}$ *and* $\{w_n\}$ *satisfy* $v = v_0 \le \ldots \le v_n \le \ldots \ldots \le w_n \le \ldots \le w_0 = w$.

Proof. For any $q \in [v, w] \cap X$, consider the following quasilinear periodic boundary value problem:

$$\begin{cases} -u''(t) = f(t, q(t), \frac{d}{dt} p(t, u(t))) + M(t)[q(t) - u(t)], \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \end{cases} \tag{3.3}$$

Using (3.1), we deduce that if u is a solution of (3.3), then

$$\begin{aligned} f\left(t, v(t), \frac{d}{dt} p(t, u(t))\right) + Mv(t) &\le -u''(t) + Mu(t) \\ &\le f\left(t, w(t), \frac{d}{dt} p(t, u(t))\right) + Mw(t)\,. \end{aligned} \tag{3.4}$$

Using (2.1), (H1) and (3.4), and reasoning as in the proof of Theorem 1, we can say that (3.3) has a solution $u \in X$. It is not difficult (using Lemma 4) to prove that this solution is unique. Using the same arguments as in the proof of Theorem 2.1 of [10], it can be proved that $v \le u \le w$. Hence (3.3) is equivalent to

$$\begin{cases} -u''(t) = f(t, q(t), u'(t)) + M(t)(q(t) - u(t)), \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi). \end{cases} \tag{3.5}$$

Now, define the operator $T : X \to X$, $T(q) = u$, where u is the solution of (3.3).

We shall prove that if $v \le q_1 \le q_2 \le w$, $q_1, q_2 \in X$, then $T(q_1) \le T(q_2)$. Indeed, let $u_i = T(q_i)$, $i = 1, 2$. Then

$$\begin{cases} -u_i''(t) = f(t, q_i(t), u_i'(t)) + M(t)(q_i(t) - u_i(t)), \\ u_i(0) = u_i(2\pi),\ u_i'(0) = u_i'(2\pi). \end{cases} \tag{3.6}$$

If $u_1 \le u_2$ is not true, then there exist $\varepsilon > 0$ and $t_0 \in I$ such that $u_1(t_0) = u_2(t_0) + \varepsilon$ and $u_1 \le u_2 + \varepsilon$ on I.

First, we shall prove that there exists $(t_1, t_2) \subset I^0$ such that $u_1 > u_2$ and $u_1' \le u_2'$ on (t_1, t_2), $u_1'(t_1) = u_2'(t_1)$ and $u_1(t_1) - u_2(t_1) \ge u_1(t_2) - u_2(t_2)$.

Indeed, let $y(t) = u_1(t) - u_2(t)$. If there exists $[t_1, t_2]$ such that $y(t) = \varepsilon$ on $[t_1, t_2]$, then the conclusion holds. Suppose that for any subinterval $(a, b) \subset I^0$, there exists $t \in (a, b)$ such that $y(t) < \varepsilon$. If $t_0 = 2\pi$, then $t_0 = 0$. Thus $y(0) = y(2\pi) = \varepsilon$ and $0 \le y'(2\pi) = y'(0) \le 0$. If $t_0 \in I^0$, then $y'(t_0) = 0$. Hence we always have $y'(t_0) = 0$.

Since $y(0) = y(2\pi)$, we can take $t_0 < 2\pi$. Because $y(t_0) = \varepsilon \geq y(t)$ and $y(t) \not\equiv \varepsilon$ in any right neighborhood of t_0, there exists $t_2 \in (t_0, 2\pi)$ such that $y'(t_2) < 0$ and $y(t_2) > 0$. Hence, there exists $t_1 \in [t_0, t_2)$ such that $y'(t_1) = 0$ and $y'(t) < 0$ for $t \in (t_1, t_2]$. Consequently, (t_1, t_2) satisfies our requirements.

We consider (3.6) in (t_1, t_2). Since $y' \leq 0$ on (t_1, t_2), (H2) and (H3) imply that

$$\begin{aligned} -u_1''(t) + u_2''(t) &= f(t, q_1(t), u_1'(t)) - f(t, q_2(t), u_2'(t)) + M(t)[q_1(t) - q_2(t)], \\ -M(t)[u_1(t) - u_2(t)] &\leq -N(t)[u_1'(t) - u_2'(t)] - M(t)[u_1(t) - u_2(t)] \end{aligned}$$

for a.e. (t_1, t_2).

The function $y = u_1 - u_2$ satisfies

$$\begin{cases} y''(t) \geq M(t)y(t) + N(t)y'(t) > N(t)y'(t), \\ y(t_1) \geq y(t_2),\ 0 = y'(t_1) \geq y'(t_2), \end{cases}$$

for a.e. (t_1, t_2).

Solving the differential inequality, we obtain

$$y'(t_2) \exp\Big(- \int_{t_1}^{t_2} N(s)\, ds \Big) > y'(t_1) = 0\,.$$

This is a contradiction with $y'(t_2) \leq 0$. Therefore, $u_1 \leq u_2$ on I.

Now, define sequences $v_0 = v$, $v_n = T(v_{n-1})$, $w_0 = w$ and $w_n = T(w_{n-1})$. Because the solution u of (3.3) satisfies $v \leq u \leq w$ on I, using the monotonicity of T we see that $v = v_0 \leq v_1 \leq \ldots \leq v_n \leq \ldots \leq w_n \leq \ldots \leq w_1 \leq w_0 = w$. Hence, the limits $\lim_{n\to\infty} v_n(t) = x(t)$ and $\lim_{n\to\infty} w_n(t) = z(t)$ exist. Note that v_n satisfies

$$\begin{cases} -v_n''(t) = f(t, v_{n-1}(t), v_n'(t)) + M(t)[v_{n-1}(t) - v_n(t)] \equiv \widetilde{f}(t, v_n(t), v_n'(t)), \\ v_n(0) = v_n(2\pi),\ v_n'(0) = v_n'(2\pi),\ v(t) \leq v_n(t) \leq w(t), \end{cases}$$

and

$$|\widetilde{f}(t, v_n(t), v_n'(t))| \leq g(|v_n'(t)|) + C \equiv \widetilde{g}(|v_n'(t)|)$$

and

$$\int_{\lambda}^{\infty} \frac{s}{\widetilde{g}(s) + K}\, ds = \infty\,.$$

By Lemma 1, there exists a constant N depending only on g, v and w such that $|v_n'| \leq N$ on I for any $n = 1, 2, \ldots$, that is, $\{v_n\}$ is a bounded set of X.

Similarly, $\{w_n\}$ is a bounded set of X. Using the same arguments as in Lemma 3, it follows that $v_n \xrightarrow{X} x$ and $w_n \xrightarrow{X} z$, that is,

$$\lim_{n\to\infty} (v_n(t), v_n'(t), w_n(t), w_n'(t)) = (x(t), x'(t), z(t), z'(t)) \quad \text{uniformly on } I\,.$$

Writing the integral equations of v_n and w_n respectively and using standard arguments, we deduce that x and z satisfy (P) and $v \le x \le z \le w$ on I. Now, we know that if $u \in X$, $v \le u \le w$ and u solves (P), then $Tu = u$, so that $v_n \le u \le w_n$ for any $n = 1, 2, \ldots$ and thus $x \le u \le z$ on I.

This completes the proof of Theorem 3. ∎

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