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On functions satisfying more than one equation of Schiffer type

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Abstract. The paper concerns properties of holomorphic functions satisfying more than one equation of Schiffer type (D_n-equation). Such equations are satisfied, in particular, by functions that are extremal (in various classes of univalent functions) with respect to functionals depending on a finite number of coefficients.

Introduction. Let S be the class of functions f holomorphic and univalent in the unit disk U with $f(0) = f'(0) - 1 = 0$, and let V_n be the subset of $\mathbb{C}^{n-1}$consisting of all points $\mathcal{A}_n = (a_2, \ldots, a_n)$ corresponding to the initial coefficients of some $f \in S$. It is known that for each $n \ge 2$ the coefficient region V_n is simply connected and compact, and it coincides with the closure of its interior. Furthermore, to each $\mathcal{A}_n \in \partial V_n$ there corresponds a function $f \in S$ which satisfies a differential equation of the form

$$(*) \qquad \left(\frac{zw'}{w}\right)^2 P(w) = Q(z), \qquad z \in U\,,$$

where

$$P(w) = \sum_{\nu=1}^{k-1} \frac{A_\nu}{w^\nu}\,, \qquad Q(z) = \sum_{\nu=-k+1}^{k-1} \frac{B_\nu}{z^\nu},$$

$A_{k-1} \ne 0$, $k \le n$, $B_0 > 0$, $B_{-\nu} = \overline{B}_\nu$, $\nu = 1, \ldots, k-1$, $Q(z) \ge 0$ everywhere on ∂U, and $Q(z) = 0$ somewhere on ∂U. It is clear from $(*)$ that $A_{k-1} = B_{k-1}$. An equation of the form $(*)$ where P and Q have all the indicated properties is called a D_n-*equation* of degree k. If f is holomorphic near the origin and satisfies a D_n-equation of degree k, it must have the properties $f(0) = 0$, $[f'(0)]^{k-1} = 1$. Any such function is called a D_n-*function* if it is holomorphic in U and $f'(0) = 1$. It is known that every D_n-function is univalent ([6], p. 103) and there is a one-one correspondence between

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∂V_n and the D_n-functions. Every D_n-function corresponds to some boundary point of V_n and to any given boundary point of V_n there corresponds a unique D_n-function. However, a D_n-function can satisfy more than one independent D_n-equation.

It seems natural to expect that points $\mathcal{A}_n$ for which the corresponding function f satisfies more than one D_n-equation lie on an edge or vertex of the boundary surface while those points for which f satisfies only one D_n-equation lie on a part of the surface which is in some sense more smooth. It is therefore of interest to investigate the class of functions corresponding to more than one D_n-equation. There are many surprisingly precise results about this class due to Schaeffer and Spencer [6], Kubota [2] and Bahtin [1].

Similar investigations can be carried out for other classes of univalent functions, e.g. for the class S_1 defined bellow.

Let S_1 consist of all functions of the form

$$f(z) = b_1 z + b_2 z^2 + \ldots, \qquad z \in U,$$

with $b_1 > 0$, univalent in U and such that $f(U) \subset U$. The set of all points $\mathcal{B}_n = (b_1, \ldots, b_n)$ corresponding to functions of class S_1 forms a region V_n in the $(2n-1)$-dimensional space. Since S_1 becomes compact upon addition of the function $f(z) = 0$, in the topology of uniform convergence on compact sets, the region V_n is compact. It is known that to every $\mathcal{B}_n = (b_1, \ldots, b_n) \in \partial V_n$ there corresponds an $f \in S_1$ which satisfies an equation of the form $(*)$ with

$$(**) \qquad P(w) = \sum_{\nu=-k+1}^{k-1} \frac{A_\nu}{w^\nu}, \qquad Q(z) = \sum_{\nu=-k+1}^{k-1} \frac{B_\nu}{z^\nu},$$

where $A_{k-1} \neq 0$, $k \leq n$, $A_{-\nu} = \overline{A}_\nu$, $\nu = 0, \ldots, k-1$, and $Q(z)$ has the same properties as in the case of S. It is also clear that $A_{k-1}/B_{k-1} = b_1^{k-1} > 0$. Also in this case, an equation of the form $(*)$, where $P(w)$ and $Q(z)$ have the properties indicated, is called a D_n-*equation* of degree k. If f is holomorphic near the origin and satisfies a D_n-equation of degree k, it must have the properties $f(0) = 0$, $[f'(0)]^{k-1} = A_{k-1}/B_{k-1}$. A function f which is holomorphic in U and which has a positive derivative at the origin will be called a D_n-*function* if it satisfies a D_n-equation. It is known that every D_n-function belongs to S_1, and that there is a one-one correspondence between ∂V_n without 0 and the D_n-functions [5].

Also in the case of the classes of Bieberbach–Eilenberg and Grunsky–Schah functions (the classes B and K) it can be proved without difficulty that to each point of ∂V_n, where V_n is the coefficient region constructed for B or K, there corresponds a D_n-function, where the D_n-equation is of the form $(*)$ with $P(w)$ and $Q(z)$ of the form $(**)$ with $A_{-\nu} = A_\nu$ for B and

$A_{-\nu} = (-1)^\nu \overline{A}_\nu$ for K, and $Q(z) \ge 0$ on ∂U. Just as for S, it is also here of interest to investigate the class of D_n-functions belonging to more than one D_n-equation.

The same problems for the classes S_1 and B were investigated by one of the present authors in [7] and [4], by Jakubowski and Majchrzak in [3] and by Starkov in [8]. It turned out that not all the properties of the functions in $(*)$ were in fact used in the derivation of the majority of the results of [7] and [4]. The aim of this paper is to investigate functions satisfying two independent equations of the form $(*)$ under weaker assumptions on $P(w)$ and $Q(z)$. These assumptions are always satisfied for equations of the form $(*)$ constructed for S_1, B and K, and maybe for other classes of univalent functions.

I. Properties of functions satisfying more than one D_n-equation. Consider a differential equation of the form

$$\left(\frac{zw'}{w}\right)^2 P(w) = Q(z)\,, \tag{1}$$

where

$$P(w) = \sum_{\nu=-k+1}^{k-1} \frac{A_\nu}{w^\nu}\,, \quad Q(z) = \sum_{\nu=-k+1}^{k-1} \frac{B_\nu}{z^\nu}\,,$$
$$k \ge 2,\ z \in U = \{z : |z| < 1\}\,,$$

$|A_{-k+1}| = |A_{k-1}| \ne 0$, $B_{-\nu} = \overline{B}_\nu$, $\nu = 1, \ldots, k-1$, $Q(z)$ is real and nonnegative on the circle $\partial U = \{z : |z| = 1\}$ and $Q(z) = 0$ somewhere on ∂U.

Let

$$f(z) = b_1 z + b_2 z^2 + \ldots, \qquad 1 \ne b_1 > 0,\ z \in U\,, \tag{2}$$

satisfy (1) in U. It follows from Theorem 4.1 of [9] that f is bounded and univalent in U. Furthermore, it has an analytic continuation to ∂U except for a finite number of points, where this continuation is only continuous. At these points

$$f(z) = \sum_{k=m}^{\infty} c_k (z - z_0)^{k/n}, \qquad n, m \in \mathbb{N}\,.$$

The obvious necessary condition for the function (2) to satisfy (1) is that $A_{k-1} = b_1^{k-1} B_{k-1}$. The equation (1), by analogy to equations satisfied by extremal functions in various classes of univalent functions, will be called an *equation of Schiffer type*. In the case when $A_{-\nu} = \overline{A}_\nu$, $A_{-\nu} = A_\nu$ or $A_{-\nu} = (-1)^\nu \overline{A}_\nu$, $\nu = 1, \ldots, k-1$, it can be proved ([5], [9], Th. 4.4) that

f is respectively a bounded function ($f(U) \subset U$), a Bieberbach–Eilenberg function or a Grunsky–Schah function.

Suppose now that f, apart from (1), satisfies another equation of the same type,

$$\left(\frac{zw'}{w}\right)^2 P_1(w) = Q_1(z)\,, \tag{3}$$

where

$$P_1(w) = \sum_{\nu=-l+1}^{l-1} \frac{C_\nu}{w^\nu}\,, \qquad Q_1(z) = \sum_{\nu=-l+1}^{l-1} \frac{D_\nu}{z^\nu}\,, \qquad l \geq 2\,,$$

$|C_{-l+1}| = |C_{l-1}| \neq 0$, $D_{-\nu} = \overline{D}_\nu$, $\nu = 1, \ldots, l-1$, $Q_1(z)$ is real and nonnegative on ∂U. Suppose that $l > k$. The properties of f are then given in Theorems 1–6.

Theorem 1. *The function f can be continued to the entire plane as an algebraic function.*

Proof. This follows by division of (1) by (3).

Let F denote the algebraic function obtained in this way.

Theorem 2. *Both at* 0 *and at* ∞, *all analytic elements of F only assume values* 0 *and* ∞.

Proof. Suppose that an element of F assumes a value $w_0 \neq 0, \infty$ at 0. Then this element has the form

$$w = w(z) = w_0 + \sum_{j=q}^{\infty} c_j z^{j/m}\,, \tag{4}$$

where $c_q \neq 0$, and $q \geq 1$, $m \geq 1$ are integers. Putting (4) into (1) or (3) and letting $z \to 0$, since $zw'(z)/w(z) \to 0$, we obtain $0 = \infty$ in both cases. In a similar way we obtain a contradiction when the element has centre ∞.

Theorem 3. *All elements of F with centre* 0 *and* ∞ *are smooth and invertible. Furthermore*, *the number of elements with centre* 0 *which assume value* 0 *at* 0 *and value* ∞ *at* 0, *and the number of elements with centre* ∞ *which are* 0 *at* ∞ *and* ∞ *at* ∞ *do not exceed*

$$\min(k-1, l-k)\,. \tag{5}$$

Proof. Let $w = w(z)$, $w(0) = 0$, be an element of F. Then

$$w^{l-k} \frac{A_{k-1} + \ldots + A_{-k+1} w^{2k-2}}{C_{l-1} + \ldots + C_{-l+1} w^{2l-2}} = z^{l-k} \frac{B_{k-1} + \ldots + \overline{B}_{k-1} z^{2k-2}}{D_{l-1} + \ldots + \overline{D}_{l-1} z^{2l-2}}\,, \tag{6}$$

where $A_{k-1} = b_1^{k-1} B_{k-1}$ and $C_{l-1} = b_1^{l-1} D_{l-1}$. Taking the $(l-k)$th roots we obtain

$$w(1 + \lambda_1 w + \ldots) = b_1 \varepsilon_j z(1 + \mu_1 z + \ldots)\,,$$

where $\varepsilon_j = \exp\{i2\pi j/(l-k)\}$, $j = 0, \ldots, l-k-1$. By the implicit function theorem, $w = w(z)$ is a smooth invertible element of the form

$$w = b_1 \varepsilon_j (z + c_2^{(j)} z^2 + \ldots) \tag{7}$$

and the number of different elements does not exceed $l-k$ (each determined by an $(l-k)$th root of 1). Likewise, an element with centre 0 such that $w(0) = \infty$ and

$$\frac{1}{w^{l-k}} \frac{A_{-k+1} + \ldots + A_{k-1} w^{-2k+2}}{C_{-l+1} + \ldots + C_{l-1} w^{-2l+2}} = z^{l-k} \frac{B_{k-1} + \ldots + \overline{B}_{k-1} z^{2k-2}}{D_{l-1} + \ldots + \overline{D}_{l-1} z^{2l-2}}\,,$$

where $|A_{-k+1}| = |A_{k-1}| = b_1^{k-1}|B_{k-1}|$ and $|C_{-l+1}| = |C_{l-1}| = b_1^{l-1}|D_{l-1}|$, has the form

$$w = b_1^{-1} \varepsilon_j \eta_1 (z^{-1} + d_0^{(j)} + d_1^{(j)} z + \ldots)\,, \tag{8}$$

where ε_j is as above and $|\eta_1| = 1$. Considering elements with centre ∞, we conclude in a similar way that they have either the form

$$w = b_1^{-1} \varepsilon_j \eta_2 (z + e_0^{(j)} + e_1^{(j)} z^{-1} + \ldots)\,, \qquad |\eta_2| = 1\,, \tag{9}$$

or

$$w = b_1 \varepsilon_j \eta_3 (z^{-1} + f_2^{(j)} z^{-2} + \ldots), \qquad |\eta_3| = 1\,, \tag{10}$$

where each of the elements (8), (9) and (10) is determined by an $(l-k)$th root of 1. So the number of elements of each of these three forms does not exceed $l-k$. The same bound has been obtained in the case of elements of the form (7). On the other hand, each of the elements (7)–(10) satisfies both (1) and (3). In particular, it follows that ε_j in (7)–(10) must be both a $(k-1)$th and an $(l-1)$th root of 1. Hence we obtain (5).

COROLLARY 1. *The number of elements of F with centre* 0 *is equal to the rang of multivalency of this function.*

Remark 1. If q denotes the greatest common divisor of $k-1$ and $l-1$, then the number of elements of each of the forms (7)–(10) does not exceed q.

COROLLARY 2. *The algebraic function F can be at most $2q$-valued, where q is the greatest common divisor of $k-1$ and $l-1$.*

THEOREM 4. *If p denotes the number of elements of F with centre* 0 *then F satisfies an algebraic equation of the form*

$$P(z, w) = b_p(z) w^p + b_{p-1}(z) w^{p-1} + \ldots + b_0(z) = 0\,, \tag{11}$$

$b_p(z) \not\equiv 0$, where $b_p(z), \ldots, b_0(z)$ are polynomials in z of degree at most p, at least one of them having degree p. These polynomials have no common factor of positive degree and $P(z,w)$ is irreducible as a polynomial in w, i.e. it cannot be represented as a product of two polynomials in w of positive degrees whose coefficients are polynomials in z.

Proof. F is an algebraic p-valued function, because all its elements with centre 0 are smooth, and hence it satisfies an equation of the form (11), where $P(z,w)$ is irreducible as a polynomial in w and $b_j(z)$ have no common factor of positive degree. It remains to prove that the $b_j(z)$ are polynomials of degree at most p and at least one of them has degree p. Let F^{-1} denote the inverse function to F. All elements with centre 0 and ∞ are smooth and invertible, so their inverses are smooth elements of F^{-1} with centres 0 and ∞. We shall prove that these are the only elements of F^{-1} with centres 0 and ∞. Indeed, F^{-1} is also an algebraic function and the element $z = f^{-1}(w)$ satisfying two equations of the form

$$\left(\frac{wz'}{z}\right)^2 \sum_{j=-k+1}^{k-1} \frac{B_j}{z^j} = \sum_{j=-k+1}^{k-1} \frac{A_j}{w^j}$$

and

$$\left(\frac{wz'}{z}\right)^2 \sum_{j=-l+1}^{l-1} \frac{D_j}{z^j} = \sum_{j=-l+1}^{l-1} \frac{C_j}{w^j}$$

belongs to this function. As in the case of F, it can be proved that elements with centre 0 can only assume values 0 and ∞ at 0 and, analogously, elements with centre ∞ can only assume values 0 and ∞ at ∞, and apart from this, they are invertible, so their inverses are elements with centres 0 and ∞ of the function F (and these are all such elements). The number of the latter elements is $2p$, so the number of elements with centres 0 and ∞ of F^{-1} is $2p$. Hence we conclude that there are p elements with centre 0 and p elements with centre ∞. Thus F^{-1} is a p-valued function. Therefore $P(z,w)$ must be a polynomial of degree p with respect to z, and this gives the assertion.

COROLLARY 3. *The number of elements of F with centre 0 assuming value 0 at 0 is equal to the number of elements with centre ∞ assuming value ∞ at ∞. Hence the number of elements with centre 0 assuming value ∞ at 0 equals the number of elements with centre ∞ assuming value 0 at ∞.*

Proof. Let $(a \to b)$ denote an element with centre a assuming value b at a. Let μ be the number of elements $(0 \to 0)$ and μ' the number of elements $(\infty \to \infty)$, and suppose $\mu' > \mu$. The number of elements $(0 \to \infty)$ is $p - \mu$, the number of elements $(\infty \to 0)$ is $p - \mu'$. Therefore the number of elements of F^{-1} with centre 0 is $\mu+p-\mu' < \mu+p-\mu = p$, which is not the case. The proof in the case $\mu' < \mu$ is analogous.

THEOREM 5. *The function F is not odd-valued unless it is single-valued.*

Proof. It is sufficient to prove that the number of elements with centre 0 is even. So it is sufficient to prove that the number of elements assuming value 0 at 0 is the same as the number of elements assuming value ∞ at 0. Let the two numbers be μ and ν respectively. Suppose first that $\nu > 0$ (we always have $\mu \geq 1$). The elements of the first group have the form

$$w = b_1 \varepsilon z + O(z^2)\,, \tag{α}$$

and those of the second are

$$w = b_1^{-1} \varepsilon z^{-1} + O(1)\,, \tag{β}$$

where $|\varepsilon| = 1$. Each of them satisfies an equation of the form

$$P(z,w) = b_p(z) w^p + \ldots + b_0(z) = 0\,, \qquad \mu+\nu = p\,,$$

where $b_j(z) = a_0^{(j)} + \ldots + a_p^{(j)} z^p$. From the Viète formulas we see that

$$\frac{b_\nu(z)}{b_p(z)} = b_1^{-\nu} \eta z^{-\nu} + O(z^{-\nu+1})\,, \qquad |\eta| = 1\,,$$

in the neighbourhood of 0, that is,

$$a_0^{(\nu)} z^\nu + \ldots + a_p^{(\nu)} z^{p+\nu} = (b_1^{-\nu}\eta + O(z))(a_0^{(p)} + \ldots + a_p^{(p)} z^p)\,.$$

Hence

$$a_0^{(p)} = \ldots = a_{\nu-1}^{(p)} = 0\,. \tag{12}$$

Consider now the elements with centre ∞. The number of elements with value ∞ at ∞ must be μ, and the number of those with value 0 at ∞ must be ν. These elements have the form

$$w = b_1^{-1} \varepsilon z + O(1), \qquad |\varepsilon| = 1\,, \tag{γ}$$

and

$$w = b_1 \varepsilon z^{-1} + O(z^{-2})\,, \qquad |\varepsilon| = 1\,. \tag{δ}$$

Using again the Viète formulas we obtain

$$\frac{b_\mu(z)}{b_p(z)} = b_1^{-\mu} \eta z^{\mu} + O(z^{\mu-1})\,, \qquad |\eta| = 1\,,$$

in the neighbourhood of ∞, that is,

$$a_0^{(\mu)} z^{-\mu} + \ldots + a_p^{(\mu)} z^{p-\mu} = (a_\nu^{(p)} z^\nu + \ldots + a_p^{(p)} z^p)(b_1^{-\nu}\eta + O(z^{-1}))\,.$$

Hence

$$a_p^{(p)} = a_{p-1}^{(p)} = \ldots = a_{\nu+1}^{(p)} = 0\,. \tag{13}$$

From (12) and (13) we conclude that

$$b_p(z) = a_\nu^{(p)} z^\nu . \tag{14}$$

Using once more the Viète formulas we have

$$\frac{b_0(z)}{b_p(z)} = b_1^{\mu-\nu} \varepsilon z^{\mu-\nu} + O(z^{\mu-\nu+1}), \qquad |\varepsilon| = 1 , \tag{15}$$

in the neighbourhood of 0, that is,

$$a_0^{(0)} + \ldots + a_p^{(0)} z^p = a_\nu^{(p)} b_1^{\mu-\nu} \varepsilon z^\mu + O(z^{\mu+1}) . \tag{16}$$

Similarly,

$$\frac{b_0(z)}{b_p(z)} = b_1^{\nu-\mu} \eta z^{\mu-\nu} + O(z^{\mu-\nu-1}) \tag{17}$$

in the neighbourhood of ∞, giving

$$a_0^{(0)} + \ldots + a_p^{(0)} z^p = a_\nu^{(p)} b_1^{\nu-\mu} \eta z^\mu + O(z^{\mu-1}) . \tag{18}$$

From (16) and (18) it follows that $a_0^{(0)} = \ldots = a_{\mu-1}^{(0)} = 0$ and $a_p^{(0)} = \ldots = a_{\mu+1}^{(0)} = 0$, and hence

$$b_0(z) = a_\mu^{(0)} z^\mu . \tag{19}$$

By (14), (19), (15) and (17),

$$\frac{a_\mu^{(0)}}{a_\nu^{(p)}} = b_1^{\mu-\nu} \varepsilon = b_1^{\nu-\mu} \eta .$$

Since $b_1 \neq 1$, it follows that $\mu = \nu$.

Suppose now that $\nu = 0$, i.e. the only elements with centre 0 are of the type $(0 \to 0)$, so they have the form (α). In this case $\mu = p \geq 1$. It follows from Corollary 3 that the only elements with centre ∞ are of the type $(\infty \to \infty)$, i.e. have the form (γ). The number of these is, of course, also p. It follows from the Viète formulas that

$$\frac{b_0(z)}{b_p(z)} = b_1^p \eta z^p + O(z^{p+1}) , \qquad |\eta| = 1 ,$$

in the neighbourhood of 0, that is,

$$a_0^{(0)} + \ldots + a_p^{(0)} z^p = z^p (b_1^p \eta + O(z))(a_0^{(p)} + \ldots + a_p^{(p)} z^p) .$$

Hence

$$a_0^{(0)} = \ldots = a_{p-1}^{(0)} = 0 \quad \text{or} \quad b_0(z) = a_p^{(0)} z^p ,\ a_p^{(0)} \neq 0 .$$

Using once more the Viète formulas, we have in the neighbourhood of ∞,

$$\frac{b_0(z)}{b_p(z)} = b_1^{-p} \eta z^p + O(z^{p-1}) , \qquad |\eta| = 1 ,$$

that is,

$$a_0^{(p)} z^p = z^p (b_1^{-p}\eta + O(z^{-1}))(a_0^{(p)} + \ldots + a_p^{(p)} z^p)\,.$$

Hence

$$a_1^{(p)} = \ldots = a_p^{(p)} = 0 \quad \text{or} \quad b_p(z) = a_0^{(p)} \neq 0\,.$$

Let now $0 < l \le p$. Using again the Viète formulas, we have in the neighbourhood of 0,

$$\frac{b_{p-l}(z)}{a_0^{(p)}(z)} = cz^l + O(z^{l+1})\,,$$

and hence $a_0^{(p-l)} = \ldots = a_{l-1}^{(p-l)} = 0$, and in the neighbourhood of ∞,

$$\frac{b_{p-l}(z)}{a_0^{(p)}(z)} = dz^l + O(z^{l-1})\,,$$

and hence $a_{l+1}^{(p-l)} = \ldots = a_p^{(p-l)} = 0$. We have thus proved that

$$b_{p-l}(z) = a_l^{(p-l)} z^l \qquad \text{for } l = 0, 1, \ldots, p\,,$$

and the polynomial $P(z,w)$ has the form

$$P(z,w) = a_0^{(p)} w^p + a_1^{(p-1)} z w^{p-1} + \ldots + a_l^{(p-l)} z^l w^{p-l} + \ldots + a_p^{(0)} z^p\,,$$

and, in the case $p > 1$, it is reducible, which is a contradiction. If $p = 1$ then f can be continued as a single-valued function. So the theorem has been proved.

COROLLARY 4. *If F is a single-valued function it has the form*

$$F(z) = b_1 z\,.$$

Proof. In this case the equation (11) takes the form

$$b_1(z)w + b_0(z) = 0\,,$$

where $b_1(z) = a_0^{(1)}$, $b_0(z) = a_1^{(0)} z$. From the fact that $w = f(z)$ satisfies this equation, we have $-a_1^{(0)}/a_0^{(1)} = b_1$ and $w = F(z) = b_1 z$.

Remark 2. It has been proved by the way that

$$b_p(z) = a_\mu^{(p)} z^\mu \quad \text{and} \quad b_0(z) = a_\mu^{(0)} z^\mu\,, \tag{20}$$

where $\mu = p/2$, p is even, and furthermore, $|a_\mu^{(p)}| = |a_\mu^{(0)}|$.

THEOREM 6. *If F is double-valued then each of its elements $w = w(z)$ satisfies the equation*

$$e^{i\alpha} w + e^{-i\alpha} w^{-1} = b_1^{-1}\left(e^{i\varphi} z - \frac{b_2}{b_1} e^{-i\alpha} + e^{-i\alpha} z^{-1}\right). \tag{21}$$

Proof. This follows from Remark 2 and from the fact that $w = f(z)$ is an element of F.

Remark 3. The conditions $Q(z) \geq 0$, $Q_1(z) \geq 0$ on ∂U were not used in the proofs of Theorems 1–6.

II. The case when one of the equations is of degree 3. We are now concerned with the case when the equation (1) is of degree 3. So it has the form

$$\left(\frac{zw'}{w}\right)^2\left(\frac{A_2}{w^2}+\frac{A_1}{w}+A_0+A_{-1}w+A_{-2}w^2\right) = \frac{B_2}{z^2}+\frac{B_1}{z}+B_0+\overline{B}_1 z+\overline{B}_2 z^2\,, \tag{22}$$

where the right-hand side is nonnegative on ∂U.

We assume additionally that the right-hand side of (22) has at least one zero on ∂U.

The number l in (3) must of course be greater than 3.

Theorem 7. *If a function f of the form* (2) *satisfies* (22) *and* (3) *then it can be continued as an algebraic single-valued or double-valued function F.*

Proof. It follows from Corollary 1 and Theorem 5 that F can only be single-valued, double-valued or four-valued. We now exclude this last possibility.

Suppose that F is four-valued. Then, by Theorem 4 and Remark 2, each element $w = w(z)$ of F satisfies an equation of the form

$$b_4(z)w^4 + \ldots + b_0(z) = 0\,, \tag{23}$$

where

$$b_\nu(z) = a_4^{(\nu)} z^4 + \ldots + a_0^{(\nu)}\,, \qquad \nu = 0, \ldots, 4\,,$$
$$b_4(z) = a_2^{(4)} z^2\,, \qquad b_0(z) = a_2^{(0)} z^2\,,$$
$$a_2^{(4)} \neq 0\,, \qquad a_2^{(0)} \neq 0\,, \qquad |a_2^{(4)}| = |a_2^{(0)}|\,,$$

and $b_1(z)$, $b_2(z)$, $b_3(z)$ are polynomials of degree at most 4, at least one of them having degree 4. By (7)–(10) and since $\varepsilon_0 = 1$ and $\varepsilon_1 = -1$, elements of F with centre 0 have the form

$$w = \pm b_1(z + O(z^2))\,,$$

and

$$w = \pm b_1^{-1}\eta_1(z^{-1} + O(1))\,, \qquad |\eta_1| = 1\,,$$

and elements with centre ∞ have the form

$$w = \pm b_1^{-1}\eta_2(z + O(1)), \qquad |\eta_2| = 1\,,$$

and

$$w = \pm b_1\eta_3(z^{-1} + O(z^{-2}))\,, \qquad |\eta_3| = 1\,.$$

From the Viète formulas we now obtain $b_3(z) = a_2^{(3)} z^2$ and $b_1(z) = a_2^{(1)} z^2$, and so (23) takes the form

$$Aw^2 + Bw + Cw^{-1} + Dw^{-2} = Pz^2 + Qz + R + Sz^{-1} + Tz^{-2}\,, \tag{24}$$

where $A \neq 0$, $D \neq 0$, $P \neq 0$, $T \neq 0$, $|A| = |D|$. This equation is satisfied by each element of F. Denoting the left-hand side of (24) by $M(w)$ and the right-hand side by $N(z)$ we obtain

$$M(w) = N(z)\,. \tag{25}$$

If $w = w(z)$ is an arbitrary element of F then

$$M(w(z)) = N(z) \tag{26}$$

in the circle of this element. Differentiating (26) with respect to z we have

$$M'(w(z))w'(z) = N'(z)\,. \tag{27}$$

Relations (22) and (26) give

$$\left(\frac{zN'(z)}{wM'(w)}\right)^2 \left(\frac{A_2}{w^2} + \ldots + A_{-2}w^2\right) = \frac{B_2}{z^2} + \ldots + \overline{B}_2 z^2\,. \tag{28}$$

Furthermore,

$$wM'(w) = 2Aw^2 + Bw - Cw^{-1} - 2Dw^{-2}\,,$$

and by (25),

$$\begin{aligned}(wM'(w))^2 = {} & \left(4N^2(z) + \frac{B^2}{A}N(z) - 2BC - 16AD\right) \\ & - \left(4BN(z) + \frac{B^3}{A} + 8AC\right)w \\ & - \left(4CN(z) + 8BD + \frac{B^2C}{A}\right)\frac{1}{w} \\ & + \left(C^2 - \frac{B^2D}{A}\right)\frac{1}{w^2}\,.\end{aligned} \tag{29}$$

Analogously, from (25) we have

$$A_{-2}w^2 = \frac{A_{-2}}{A}\left(N(z) - Bw - C\frac{1}{w} - D\frac{1}{w^2}\right). \tag{30}$$

Putting (29) and (30) into (28) we conclude that each element of F satisfies

$$\begin{aligned}(zN'(z))^2 \bigg[& \left(A_2 - \frac{A_{-2}D}{A}\right) + \left(A_1 - \frac{A_{-2}C}{A}\right)w \\ & + \left(A_0 + \frac{A_{-2}}{A}N(z)\right)w^2 + \left(A_{-1} - \frac{A_{-2}B}{A}\right)w^3\bigg]\end{aligned}$$

$$
\begin{aligned}
&= \frac{S(z)}{z^2}\biggl[\biggl(C^2 - \frac{B^2D}{A}\biggr) - \biggl(4CN(z) + 8BD + \frac{B^2C}{A}\biggr)w \\
&\quad + \biggl(4N^2(z) + \frac{B^2}{A}N(z) - 2BC - 16AD\biggr)w^2 \\
&\quad - \biggl(4BN(z) + \frac{B^3}{A} + 8AC\biggr)w^3\biggr],
\end{aligned}
$$

with $S(z) = B_2 + B_1 z + B_0 z^2 + \overline{B}_1 z^3 + \overline{B}_2 z^4$, which is an equation of degree at most 3 with respect to w. In order that it could be an equation of the four-valued function F, the coefficients of all powers must vanish identically. This leads to the identities

$$
\text{(a)}\qquad (zN'(z))^2\biggl(A_2 - \frac{A_{-2}D}{A}\biggr) = \frac{S(z)}{z^2}\biggl(C^2 - \frac{B^2D}{A}\biggr),
$$

$$
\text{(b)}\qquad (zN'(z))^2\biggl(A_1 - \frac{A_{-2}C}{A}\biggr) = -\frac{S(z)}{z^2}\biggl(4CN(z) + 8BD + \frac{B^2C}{A}\biggr),
$$

$$
\begin{aligned}
\text{(c)}\qquad & (zN'(z))^2\biggl(A_0 + \frac{A_{-2}}{A}N(z)\biggr) \\
&= \frac{S(z)}{z^2}\biggl(4N^2(z) + \frac{B^2}{A}N(z) - 2BC - 16AD\biggr),
\end{aligned}
$$

$$
\text{(d)}\qquad (zN'(z))^2\biggl(A_{-1} - \frac{A_{-2}B}{A}\biggr) = -\frac{S(z)}{z^2}\biggl(4BN(z) + \frac{B^3}{A} + 8AC\biggr).
$$

Dividing now (b) and (d) by (c) we obtain

$$
\frac{A_1 - A_{-2}CA^{-1}}{A_0 + A_{-2}A^{-1}N(z)} = -\frac{4CN(z) + 8BD + B^2CA^{-1}}{4N^2(z) + B^2A^{-1}N(z) - 2BC - 16AD}
$$

and

$$
\frac{A_{-1} - A_{-2}BA^{-1}}{A_0 + A_{-2}A^{-1}N(z)} = -\frac{4BN(z) + 8AC + B^3A^{-1}}{4N^2(z) + B^2A^{-1}N(z) - 2BC - 16AD},
$$

and hence $A_1 = A_{-1} = 0$ and

$$
\text{(31)}\qquad CA_0 + 2A_{-2}BDA^{-1} = 0, \qquad BA_0 + 2A_{-2}C = 0.
$$

Then we have

$$
\text{(32)}\qquad B^2D = AC^2.
$$

Suppose first that $B \neq 0$. From (32) and (31) we obtain $C \neq 0$ and $A_0 = -2A_{-2}CB^{-1}$, and from (32) and (a),

$$
A_2 = A_{-2}DA^{-1} = A_{-2}C^2B^{-2},
$$

where $|C| = |B|$. In this case (22) takes the form

$$A_{-2}\left(w - \frac{e^{i\varphi}}{w}\right)^2 \left(\frac{zw'}{w}\right)^2 = \frac{S(z)}{z^2}\,, \tag{33}$$

where the right-hand side is nonnegative on ∂U.

Let now $B = 0$. It follows from (32) that also $C = 0$, and

$$M(w) = Aw^2 + \frac{D}{w^2}\,.$$

Putting $A/D = e^{-2i\varphi}$ and $L(z) = A^{-1}N(z)$ in (c) we obtain

$$A^2(zL'(z))^2(A_0 + A_{-2}L(z)) = \frac{S(z)}{z^2}(4A^2L^2(z) - 16A^2e^{2i\varphi})\,,$$

or

$$\frac{(zL'(z))^2(A_0 + A_{-2}L(z))}{4(L(z) - 2e^{i\varphi})(L(z) + 2e^{i\varphi})} = \frac{S(z)}{z^2}\,, \tag{34}$$

where the only poles of the right-hand side are 0 and ∞.

Let $L(z_j) = 2e^{i\varphi}$, $j = 1,2,3,4$, and $L(\zeta_j) = -2e^{i\varphi}$, $j = 1,2,3,4$, $z_j, \zeta_j \neq 0$. At these points either

$$A_0 + A_{-2}L(z) = 0 \quad \text{or} \quad L'(z) = 0\,.$$

If $A_0 + A_{-2}L(z) = 0$ for at least one of these points then $A_0 = \pm 2A_{-2}e^{i\varphi}$ and by (a), $A_2 = A_{-2}DA^{-1} = A_{-2}e^{2i\varphi}$, so also in this case (22) takes the form (33):

$$A_{-2}\left(w \pm \frac{e^{i\varphi}}{w}\right)^2 \left(\frac{zw'}{w}\right)^2 = \frac{S(z)}{z^2}\,.$$

In the opposite case each of these points satisfies $L'(z) = 0$, so all roots of $L(z) - 2e^{i\varphi}$ and of $L(z) + 2e^{i\varphi}$ are at least double (because the first derivative at these roots is 0):

$$\begin{aligned} L(z) - 2e^{i\varphi} &= \frac{\lambda(z - z_1)^2(z - z_2)^2}{z^2}\,, \\ L(z) + 2e^{i\varphi} &= \frac{\mu(z - \zeta_1)^2(z - \zeta_2)^2}{z^2}\,. \end{aligned} \tag{35}$$

Differentiating (35) we obtain

$$\begin{aligned} zL'(z) &= \frac{2\lambda}{z^2}(z - z_1)(z - z_2)(z^2 - z_1z_2) \\ &= \frac{2\mu}{z^2}(z - \zeta_1)(z - \zeta_2)(z^2 - \zeta_1\zeta_2)\,. \end{aligned} \tag{36}$$

From (35) and (36) we obtain

$$
\begin{aligned}
zL'(z) &= \frac{2\lambda}{z^2}(z^4 - z_1^4)\,,\\
L(z) \pm 2e^{i\varphi} &= \frac{\lambda}{z^2}(z^2 \pm z_1^2)^2\,.
\end{aligned}
\tag{37}
$$

After substituting (37) into (34) we have

$$
A_0 + A_{-2}L(z) = \frac{S(z)}{z^2}\,. \tag{38}
$$

But, by the assumption, $S(z)$ has at least one root on the circle ∂U, and it is at least double. Denote this root by z_0: $S(z_0) = 0$, $S'(z_0) = 0$. From (38) we have

$$
A_0 + A_{-2}L(z_0) = 0 \quad \text{and} \quad A_{-2}L'(z_0) = 0\,.
$$

Therefore z_0 must be one of the points z_1, z_2, ζ_1, ζ_2, hence $L(z_0) = \pm 2e^{i\varphi}$ and thus $A_0 = \pm 2A_{-2}e^{i\varphi}$, and we again obtain (33).

Consider now the consequences of (33) for $w = f(z)$. Put $w = f(z)$ in (33), where $z = e^{it}$, $t \in [0, 2\pi)$. Since the right-hand side of (33) is nonnegative on ∂U we have

$$
e^{2i\alpha}\left(f(e^{it}) - \frac{e^{i\varphi}}{f(e^{it})}\right)^2 \left(\frac{e^{it}f'(e^{it})}{f(e^{it})}\right)^2 \ge 0\,,
$$

where $2\alpha = \arg A_{-2}$. Hence taking square roots gives

$$
\operatorname{Re}\left\{e^{i\alpha}\frac{ie^{it}f'(e^{it})}{f(e^{it})}\left(f(e^{it}) - \frac{e^{i\varphi}}{f(e^{it})}\right)\right\} = 0\,,
$$

and integration shows that

$$
\operatorname{Re}\left\{e^{i\alpha}\left(w + \frac{e^{i\varphi}}{w}\right)\right\} = c\,, \qquad c = \text{const.}\,,\ w = f(e^{it})\,. \tag{39}
$$

We now investigate the set of points for which (39) is satisfied. Putting $w = e^{i\varphi/2}\omega$ in (39) gives

$$
\operatorname{Re}\left\{e^{i\beta}\left(\omega + \frac{1}{\omega}\right)\right\} = c\,, \tag{40}
$$

where $\alpha + \varphi/2 = \beta$. Putting $\omega = u + iv$ in (40) we obtain

$$
u\cos\beta((u^2+v^2)^{-2}+1) - v\sin\beta(1-(u^2+v^2)^{-2}) = c\,. \tag{41}
$$

Let $f_1(z) = e^{-i\varphi/2}f(z)$. Then f_1 is also univalent and bounded and $\partial f_1(U)$ is the rotation of $\partial f(U)$ through the angle $-\varphi/2$. So $\partial f_1(U)$, and likewise $\partial f(U)$, must intersect both coordinate axes in at least two points, and there must exist at least two points of intersection of opposite signs on each axis. If $(u, 0) \in \partial f_1(U)$ then

$$
u\cos\beta(u^{-2}+1) = c\,.
$$

If $\cos\beta \neq 0$ then

$$u^{-1} + u = \frac{c}{\cos\beta},$$

so

$$u^2 - \frac{c}{\cos\beta} u + 1 = 0.$$

This is impossible because this equation has no solutions of opposite signs. Therefore $\cos\beta = 0$, and hence $\sin\beta = \pm 1$ and (41) takes the form

$$v(1 - (u^2 + v^2)^{-2}) = 0.$$

Then $\partial f_1(U)$ consists of the circle ∂U and of one or two slits extending from ± 1 and lying on the u axis, and $\partial f(U)$ consists of ∂U and of slits extending from $\pm e^{i\varphi/2}$ and lying on the line through 0.

We now prove that f can be continued as a double-valued function. In fact, if $f(U) = U$ then $f(z) = z$, which is not true because $b_1 \neq 1$. Suppose now that $f(U) = U - L$ where L is a segment extending from one of the points $\pm e^{i\varphi/2}$ and lying on the line through 0: $L = \{w : w = \pm e^{i\varphi/2}(1-t) + td\}$, $0 \le t \le 1$. Let $f(e^{it_0}) = d$; hence $f'(e^{it_0}) = 0$ and by (33), e^{it_0} is an at least double root of $S(z)$. Thus $S(z)$ has the form

$$S(z) = \overline{B}_2 (z - e^{it_0})^2 (z - z_0)(z - \overline{z}_0^{-1}),$$

where $|z_0| \le 1$. If $|z_0| < 1$ then $f(z_0) = \pm e^{i\varphi/2}$, which is impossible because no point of U is mapped to $\partial f(U)$. So $|z_0| = 1$. Let $z_0 = e^{it_1}$ and

$$S(z) = \overline{B}_2 (z - e^{it_0})^2 (z - e^{it_1})^2. \tag{42}$$

In the case when $\partial f(U)$ consists of ∂U and two slits, an analogous reasoning leads to the same result. Putting now (42) into (33) and taking square roots we obtain

$$\sqrt{A_{-2}}\left(\frac{e^{i\varphi}}{w^2} - 1\right) w' = \sqrt{\overline{B}_{-2}}\left(1 - (e^{it_0} + e^{it_1})\frac{1}{z} + e^{i(t_0+t_1)}\frac{1}{z^2}\right).$$

Integration yields

$$-\sqrt{A_{-2}}\left(\frac{e^{i\varphi}}{w} + w\right) = \sqrt{\overline{B}_{-2}}\left(z - (e^{it_0} + e^{it_1})\log z - e^{i(t_0+t_1)}\frac{1}{z}\right) + c. \tag{43}$$

Of course, $e^{it_0} + e^{it_1} = 0$, and (43) is an equation of a double-valued function, which was to be proved.

References

[1] A. K. Bahtin, *Some properties of functions of class S*, Ukrain. Mat. Zh. 33 (1981), 154–159 (in Russian); English transl.: Ukrainian Math. J. 33 (1981), 122–126.

[2] Y. Kubota, *On extremal problems which correspond to algebraic univalent functions*, Kodai Math. Sem. Rep. 25 (1973), 412–428.

[3] Z. J. Jakubowski and W. Majchrzak, *On functions realizing the maximum of two functionals at a time*, Serdica 10 (1984), 337–343.
[4] A. Rost and J. Śladkowska, *Sur les fonctions de Bieberbach–Eilenberg satisfaisant à plus qu'une équation de type de Schiffer*, Demonstratio Math., to appear.
[5] H. L. Royden, *The coefficient problem for bounded schlicht functions*, Proc. Nat. Acad. Sci. U.S.A. 35 (1949), 637–662.
[6] A. C. Schaeffer and D. C. Spencer, *Coefficient Regions for Schlicht Functions*, Amer. Math. Soc., 1950.
[7] J. Śladkowska, *Sur les fonctions univalentes, bornées, satisfaisant deux au moins D_n-équations*, Demonstratio Math. 11 (1978), 1–28.
[8] V. V. Starkov, *Bounded univalent functions realizing the extrema of two coefficients functionals at a time*, in: Proc. XIth Instructional Conf. on the Theory of Extremal Problems, Łódź 1990, 31–33.
[9] K. Tochowicz, *Functions which satisfy the differential equation of Schiffer type*, Zeszyty Nauk. Politechn. Rzeszowskiej Mat. Fiz. 38 (6) (1987), 103–113.

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