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ON CONJUGATE POISSON INTEGRALS AND RIESZ TRANSFORMS FOR THE HERMITE EXPANSIONS

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1. Introduction. Analogues of the classical conjugate harmonic functions and conjugate mappings have been studied for a variety of classical expansions such as ultraspherical expansions by Muckenhoupt and Stein in [6]. Later similar objects were studied by Muckenhoupt [5] for the Hermite and Laguerre expansions on the real line. The conjugate mapping of Muckenhoupt may be termed as the Hilbert transform for the Hermite series. In the higher dimensional case analogues of the classical Riesz transforms have been studied by several authors (see Meyer [4], Urbina [13] and Pisier [7]). All these authors have considered expansions in terms of Hermite polynomials. They have proved that when $1 < p < \infty$, the Riesz transforms are bounded on $L^p(\mathbb{R}^n, d\mu)$ where μ is the Gaussian measure. In the one-dimensional case Muckenhoupt has proved that the conjugate mapping is of weak type $(1,1)$. But a weak type result is not known for the higher dimensional case.

Our point of departure from the above mentioned works lies in the fact that we consider expansions in terms of Hermite functions rather than Hermite polynomials. In [11] we have defined Riesz transforms for the Hermite operator and proved that they are bounded on $L^p(\mathbb{R}^n,\, dx)$. The Riesz transforms of Meyer–Pisier–Urbina are defined using the Ornstein–Uhlenbeck semigroup whereas our Riesz transforms are related to the Hermite semigroup e^{-tH}. Hermite function expansions are better suited for L^p harmonic analysis than Hermite polynomial expansions. This remark is justified for example by the fact that in the one-dimensional case the Hermite polynomial expansion of an L^p function does not converge in the norm unless $p = 2$ whereas Hermite function expansion converges as long as $4/3 < p < 4$. For various summability results for the Hermite function expansions see [1], [9], [10] and [12].

The aim of this paper is to study conjugate Poisson integrals associated with Hermite function expansions. Using them we are able to define the Riesz transforms even for L^1 functions. The Riesz transforms are then shown to be of weak type $(1,1)$.

2. Main results. Let Φ_α be the normalized Hermite functions on $\mathbb{R}^n$ which are eigenfunctions of the *Hermite operator* $H = -\Delta + |x|^2$ with eigenvalue $2|\alpha| + n$. If P_k is the projection $P_k f = \sum_{|\alpha|=k}(f, \Phi_\alpha)\Phi_\alpha$ then the spectral decomposition of H is written as $H = \sum_{k=0}^{\infty}(2k+n)P_k$. Consider the *annihilation* and *creation operators* $A_j^* = \partial/\partial x_j + x_j$ and $A_j = -\partial/\partial x_j + x_j$. The action of these operators on Φ_α is given by

$$A_j\Phi_\alpha = (2\alpha_j + 2)^{1/2}\Phi_{\alpha+e_j}, \qquad A_j^*\Phi_\alpha = (2\alpha_j)^{1/2}\Phi_{\alpha-e_j} \tag{2.1}$$

where e_j are the coordinate vectors. The Hermite operator H can be written in terms of A_j and A_j^* as $H = \frac{1}{2}\sum_{j=1}^{n}(A_jA_j^* + A_j^*A_j)$. In analogy with the classical Riesz transforms for the Laplacian we define operators R_j and R_j^* by $R_j = A_jH^{-1/2}$ and $R_j^* = A_j^*H^{-1/2}$. The *Hermite expansions* of R_jf and R_j^*f are given by

$$R_jf = \sum_{\alpha=0}^{\infty}(2\alpha_j + 2)^{1/2}(2|\alpha| + n)^{-1/2}(f, \Phi_\alpha)\Phi_{\alpha+e_j}, \tag{2.2}$$

$$R_j^*f = \sum_{\alpha=0}^{\infty}(2\alpha_j)^{1/2}(2|\alpha| + n)^{-1/2}(f, \Phi_\alpha)\Phi_{\alpha-e_j}. \tag{2.3}$$

Here $\sum_{\alpha=0}^{\infty}$ stands for the sum extended over all multiindices. R_j and R_j^* may be called the *Riesz transforms for the Hermite operator*.

Observe that the series defining R_jf and R_j^*f converge whenever f is in the Schwartz class. It is also clear that R_j and R_j^* are bounded on $L^2(\mathbb{R}^n)$. If we can show that for f in the Schwartz class

$$\|R_jf\|_p \le C\|f\|_p, \qquad \|R_j^*f\|_p \le C\|f\|_p \tag{2.4}$$

then we can define R_jf and R_j^*f for L^p functions using a density argument. The estimates (2.4) were proved in [11] for $1 < p < \infty$. But for $p = 1$ we do not have such an estimate and a priori it is not clear how to define R_j and R_j^* on L^1.

To define the Riesz transforms for L^1 functions we introduce the conjugate Poisson integrals. Let $e^{-tH^{1/2}}$ be the *Poisson–Hermite semigroup*. The *Poisson integral* of a function f in $L^p(\mathbb{R}^n)$ is then the function

$$u(x,t) = e^{-tH^{1/2}}f(x). \tag{2.5}$$

This function satisfies the differential equation

$$\partial_t^2 u - Hu = 0, \qquad t > 0,\ x \in \mathbb{R}^n. \tag{2.6}$$

The *conjugate Poisson integrals* are then defined by

$$\begin{aligned} u_j(x,t) &= R_ju(x,t) = A_jH^{-1/2}e^{-tH^{1/2}}f(x),\\ u_j^*(x,t) &= R_j^*u(x,t) = A_j^*H^{-1/2}e^{-tH^{1/2}}f(x). \end{aligned} \tag{2.7}$$

We also consider the *maximal conjugate Poisson integrals*

$$U_jf(x) = \sup_{0<t<1} |u_j(x,t)|, \qquad U_j^*f(x) = \sup_{0<t<1} |u_j^*(x,t)|. \tag{2.8}$$

The basic results on U_j and U_j^* are given in the following theorem.

THEOREM 1. *Assume that $n \geq 2$. Then we have*

(i) $\|U_jf\|_p \leq C\|f\|_p$, $1 < p < \infty$,
(ii) $|\{x : U_jf(x) > \lambda\}| \leq C\|f\|_1/\lambda$.

Similar results also hold for U_j^*.

When f is a finite linear combination of Φ_α's it is clear that $u_j(x,t)$ converges to R_jf pointwise. If $1 < p < \infty$ then Theorem 1(i) implies that $u_j(x,t)$ converges to R_jf in the norm and hence the R_j are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. When $p = 1$, the weak type inequality (ii) implies that $u_j(x,t)$ converges almost everywhere to a function which we call R_jf. Thus for f in $L^1(\mathbb{R}^n)$, R_jf is defined by

$$R_jf(x) = \lim_{t\to 0} u_j(x,t). \tag{2.9}$$

Clearly, this R_j is of weak type $(1,1)$.

COROLLARY 1. $|\{x : |R_jf(x)| > \lambda\}| \leq C\|f\|_1/\lambda$.

We will also prove the following theorem on the boundedness of R_j and R_j^* on the local Hardy spaces. First let us briefly recall the definition of the local Hardy spaces $h^1(\mathbb{R}^n)$. Given a nonnegative Schwartz class function φ with $\int \varphi\, dx = 1$ let $\varphi_t(x) = t^{-n}\varphi(x/t)$ where $t > 0$. We say that $f \in h^1(\mathbb{R}^n)$ iff the maximal function $f^*(x) = \sup_{0<t<1} |f * \varphi_t(x)|$ belongs to $L^1(\mathbb{R}^n)$. For properties and other characterizations of $h^1(\mathbb{R}^n)$ we refer to Goldberg [3].

THEOREM 2. *The Riesz transforms R_j and R_j^* are bounded on the local Hardy space $h^1(\mathbb{R}^n)$.*

Both theorems will be proved in Section 4. In the next section we prove certain kernel estimates which are needed in the proof of the theorems.

3. Some estimates on the kernel of the Hermite semigroup. Consider the *Hermite semigroup* e^{-tH}. Since $e^{-tH}f = \sum_{k=0}^{\infty} e^{-(2k+n)t}P_kf$, the kernel of e^{-tH} is given by

$$\sum_{\alpha=0}^{\infty} e^{-(2|\alpha|+n)t}\Phi_\alpha(x)\Phi_\alpha(y).$$

In view of Mehler's formula (see [2]) the above sum equals $(2\pi)^{-n/2}K_t(x,y)$ where

$$K_t(x,y) = (\sinh 2t)^{-n/2} e^{-\varphi(t,x,y)} \tag{3.1}$$

with

$$\varphi(t,x,y) = \tfrac{1}{2}(|x|^2 + |y|^2)\coth 2t - x\cdot y \operatorname{cosech} 2t. \tag{3.2}$$

We can rewrite $\varphi(t,x,y)$ in the following way:

$$\begin{aligned}\varphi(t,x,y) &= \tfrac{1}{2}|x-y|^2\coth 2t + x\cdot y(\coth 2t - \operatorname{cosech} 2t)\\ &= \tfrac{1}{2}|x-y|^2\coth 2t + x\cdot y\tanh t\,.\end{aligned} \tag{3.3}$$

Putting

$$\psi(t,x,y) = \tfrac{1}{4}|x-y|^2\coth 2t + x\cdot y(\coth 2t - \operatorname{cosech} 2t)$$

we write

$$\varphi(t,x,y) = \tfrac{1}{4}|x-y|^2\coth 2t + \psi(t,x,y). \tag{3.4}$$

We claim that $\psi(t,x,y) \ge 0$. This is obvious when $x\cdot y \ge 0$ and when $x\cdot y < 0$ it follows from the inequality $|x-y|^2 \ge -4x\cdot y$.

Thus we have

$$|K_t(x,y)| \le C(\sinh 2t)^{-n/2} e^{-(|x-y|^2/4)\coth 2t}. \tag{3.5}$$

From this we immediately obtain the following estimates for the kernel of e^{-tH}:

$$|K_t(x,y)| \le Ct^{-n/2}e^{-|x-y|^2/(8t)} \qquad \text{if } 0 < t < 1, \tag{3.6}$$

$$|K_t(x,y)| \le Ce^{-nt}e^{-a|x-y|^2} \qquad \text{if } t \ge 1, \tag{3.7}$$

where $a > 0$ is a fixed constant. These estimates follow from the trivial observation that for $0 < t < 1$, $\sinh 2t = O(t)$, $\cosh 2t = O(1)$ and for $t \ge 1$, $\sinh 2t = O(e^{2t}) = \cosh 2t$. To study the conjugate Poisson integrals we also need estimates for the derivatives of $K_t(x,y)$. Such estimates are given in the following two lemmas.

LEMMA 1. *For $0 < t < 1$, there exist positive constants C and a independent of x, y and t such that the following estimates are valid*:

(i) $\left|\dfrac{\partial}{\partial x_j}K_t(x,y)\right| \le Ct^{-(n+1)/2}e^{-a|x-y|^2/t}$,

(ii) $|x_jK_t(x,y)| \le Ct^{-(n+1)/2}e^{-a|x-y|^2/t}$,

(iii) $\left|x_j\dfrac{\partial}{\partial y_i}K_t(x,y)\right| \le Ct^{-n/2-1}e^{-a|x-y|^2/t}$,

(iv) $\left|\dfrac{\partial^2}{\partial x_j\partial y_i}K_t(x,y)\right| \le Ct^{-n/2-1}e^{-a|x-y|^2/t}$.

Proof. Since $K_t(x,y)$ is the product of the one-dimensional kernels $K_t(x_j,y_j)$ it is enough to prove the lemma in the one-dimensional case. First consider

$$\frac{\partial}{\partial x}K_t(x,y) = (\sinh 2t)^{-1/2}e^{-\varphi(t,x,y)}(y\,\mathrm{cosech}\,2t - x\coth 2t). \tag{3.8}$$

This can be written as the sum of

$$A_t(x,y) = (\sinh 2t)^{-3/2}e^{-\varphi(t,x,y)}(y-x)$$

and

$$B_t(x,y) = -2(\sinh t)^2(\sinh 2t)^{-3/2}e^{-\varphi(t,x,y)}x.$$

Since $0<t<1$, we immediately get

$$|A_t(x,y)| \le Ct^{-3/2}|x-y|e^{-|x-y|^2/(4t)} \le Ct^{-1}e^{-|x-y|^2/(8t)}. \tag{3.9}$$

Since $|B_t(x,y)| \le C|xK_t(x,y)|$ it is enough to prove the estimate (ii) of the lemma.

First assume that $|x| \le 4|y|$. Then

$$|xK_t(x,y)| \le Ct^{-1/2}|2xy|^{1/2}e^{-\varphi(t,x,y)}.$$

When $xy \ge 0$ we have

$$|xK_t(x,y)| \le Ct^{-1/2}|2xy|^{1/2}e^{-xy\tanh t}e^{-|x-y|^2/(2t)} \tag{3.10}$$

and this gives the estimate

$$|xK_t(x,y)| \le Ct^{-1}e^{-|x-y|^2/(2t)}$$

as $\tanh t$ behaves like t for $0<t<1$ so that $(txy)^{1/2}e^{-xy\tanh t}$ is bounded. When $xy<0$, $|2xy| = -2xy \le (x-y)^2$ and so

$$|xK_t(x,y)| \le Ct^{-1/2}|x-y|e^{-|x-y|^2/(4t)} \le Ct^{-1}e^{-|x-y|^2/(8t)}.$$

This settles the case when $|x| \le 4|y|$.

Next assume that $|x| > 4|y|$. In this case when $xy \ge 0$,

$$\begin{aligned}\tfrac12\varphi(t,x,y) &= \tfrac14(\coth 2t)(x^2+y^2) - \tfrac12 xy\,\mathrm{cosech}\,2t\\ &\ge \tfrac14(\coth 2t)(x^2-2|xy|) \ge \tfrac18(\coth 2t)x^2.\end{aligned}$$

This proves that

$$|xK_t(x,y)| \le Ct^{-1/2}|x|e^{-(\coth 2t)x^2/8}e^{-|x-y|^2/(4t)}$$

and this is certainly bounded by a constant times $t^{-1}\exp(-|x-y|^2/(4t))$. When $xy<0$,

$$\tfrac12\varphi(t,x,y) = \tfrac14(\coth 2t)(x^2+y^2) - \tfrac12 xy\,\mathrm{cosech}\,2t \ge \tfrac14(\coth 2t)x^2$$

and we get the same estimate as before.

Now consider the second derivative of $K_t(x,y)$:

$$\frac{\partial^2}{\partial x\partial y}K_t(x,y) = (\sinh 2t)^{-3/2}e^{-\varphi} - 2(\sinh t)^2(\sinh 2t)^{-5/2}x(x-y\cosh 2t)e^{-\varphi}. \tag{3.11}$$

It follows that $\partial^2 K_t/\partial x\partial y$ is a sum of the terms $(\sinh 2t)^{-3/2}e^{-\varphi}$, $(\sinh t)^2 \times (\sinh 2t)^{-5/2}x(x-y)e^{-\varphi}$ and $(\sinh t)^4(\sinh 2t)^{-5/2}xye^{-\varphi}$. All the terms can be estimated as before to get (iii). The estimation of $x(\partial/\partial y)K_t(x,y)$ is similarly done.

LEMMA 2. *For $t \ge 1$ the following estimates are valid with two positive constants C and b independent of x, y and t*:

(i) $|x_j K_t(x,y)| \le Ce^{-nt}e^{-b|x-y|^2}$,

(ii) $\left|\dfrac{\partial}{\partial x_j}K_t(x,y)\right| \le Ce^{-nt}e^{-b|x-y|^2}$.

The proof is very similar to that of the previous lemma. We have to use the fact that when $t \ge 1$ both $\cosh 2t$ and $\sinh 2t$ behave like e^{2t}. The details are omitted.

To establish the boundedness of the Riesz transforms on the local Hardy space we need certain estimates for the derivatives of the function $a(x,\xi)$ defined by the integral

$$a(x,\xi) = \int_0^\infty t^{-1/2}(\cosh 2t)^{-n/2}e^{-b(t,x,\xi)}\,dt \tag{3.12}$$

where

$$b(t,x,\xi) = \tfrac{1}{2}\tanh 2t(|x|^2+|\xi|^2) + 2ix\cdot\xi\sinh^2 t\cdot \operatorname{sech} 2t. \tag{3.13}$$

For this function $a(x,\xi)$ the following is valid.

LEMMA 3. *For all multiindices α and β there exist constants $C_{\alpha,\beta}$ independent of x and ξ such that*

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{-1-|\alpha|-|\beta|}.$$

Proof. The proof is straightforward. The integral taken from one to infinity has exponential decay as a function of $|x|^2+|\xi|^2$. So it is enough to estimate

$$a_0(x,\xi) = \int_0^1 t^{-1/2}(\cosh 2t)^{-n/2}e^{-b(t,x,\xi)}\,dt.$$

For $0 < t < 1$, $\tanh 2t$ behaves like t and $\cosh 2t = O(1)$ so that

$$|a_0(x,\xi)| \le C\int_0^1 t^{-1/2}e^{-\varepsilon t(|x|^2+|\xi|^2)}\,dt$$

and this gives the estimate

$$|a_0(x,\xi)| \le C(1+|x|+|\xi|)^{-1}.$$

In the case of $(\partial/\partial\xi_j)a_0(x,\xi)$ we have two terms. The first term,

$$\xi_j \int_0^1 t^{-1/2}(\cosh 2t)^{-n/2}(\tanh t)e^{-b(t,x,\xi)}\,dt$$

is bounded by

$$|\xi|(1+|x|^2+|\xi|^2)^{-3/2} \le C(1+|x|+|\xi|)^{-2}.$$

The other term has a better bound since the derivative falling on $x\cdot\xi(\sinh^2 t)$ $\times(\operatorname{sech} 2t)$ brings down $x_j(\sinh^2 t)$. Derivatives with respect to x and higher order derivatives are similarly dealt with. This completes the proof of the lemma.

4. Conjugate Poisson integrals and Riesz transforms for the Hermite expansions. We first prove the following L^2 result for the maximal conjugate Poisson integrals.

PROPOSITION 4.1. *Assume that $n \ge 2$. Then*

$$\|U_j f\|_2 \le C\|f\|_2, \qquad \|U_j^* f\|_2 \le C\|f\|_2.$$

P r o o f. Recall that for f in $L^2(\mathbb{R}^n)$, $R_j f$ has the Hermite expansion

$$R_j f = \sum_{\alpha=0}^{\infty}(2\alpha_j+2)^{1/2}(2|\alpha|+n)^{-1/2}(f,\Phi_\alpha)\Phi_{\alpha+e_j}$$

where the series converges in the L^2 norm. From this it follows that

$$e^{-t(H-2)^{1/2}}(R_j f) = R_j(e^{tH^{1/2}}f) = u_j(x,t). \tag{4.1}$$

Thus

$$U_j f(x) = \sup_{0<t<1}|u_j(x,t)| = \sup_{0<t<1}|e^{-t(H-2)^{1/2}}R_j f(x)|$$

and since R_j is bounded on $L^2(\mathbb{R}^n)$ it is enough to show that

$$\sup_{0<t<1}|e^{-t(H-2)^{1/2}}f(x)| \le CMf(x) \tag{4.2}$$

where M is the Hardy–Littlewood maximal function.

But in view of the subordinate identity

$$e^{-\alpha} = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-1/u}u^{-3/2}e^{-\alpha^2 u/4}\,du \tag{4.3}$$

we have the formula

$$e^{-t(H-2)^{1/2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-1/u} u^{-3/2} e^{-(u/4)t^2(H-2)}\, du. \tag{4.4}$$

As we are assuming $n \geq 2$, (4.2) will follow once we show that

$$\sup_{0<u<\infty} |e^{-u(H-n)} f(x)| \leq CMf(x). \tag{4.5}$$

This is an immediate consequence of the estimates (3.6) and (3.7) for the kernel K_u. This completes the proof of the proposition.

We now come to the proof of the main theorem. In view of Proposition 4.1 and the Marcinkiewicz interpolation theorem it is enough to prove the following weak type inequality.

THEOREM 4.1. *Assume $n \geq 2$. The maximal conjugate Poisson integrals are of weak type* $(1,1)$.

Proof. We imitate the standard proof for the weak type inequality of Calder/on–Zygmund singular integrals. Given f in $L^1(\mathbb{R}^n)$ we take the Calder/on–Zygmund decomposition $f = g + b$ (see [8]). Suppose we are given a singular integral operator T defined by a kernel $K(x,y)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\, dy. \tag{4.6}$$

If we assume that T is bounded on $L^2(\mathbb{R}^n)$ then the term Tg is taken care of, i.e. we obtain

$$|\{x : |Tg(x)| > \lambda\}| < C\|f\|_1/\lambda.$$

To obtain the same inequality for Tb, what we really need is the estimate

$$\left|\frac{\partial}{\partial y_j} K(x,y)\right| \leq C|x-y|^{-n-1} \tag{4.7}$$

for the derivatives of the kernel K. Then establishing the weak type inequality for Tb is well known.

Suppose now we have a kernel $k_t(x,y)$ depending on a parameter t and suppose we are interested in the weak type inequality for the maximal operator

$$\sup_{0<t<1} |T_t f(x)| = \sup_{0<t<1} \left| \int k_t(x,y) f(y)\, dy\right|.$$

If we know that $\sup_{0<t<1} |T_t f|$ is bounded on $L^2(\mathbb{R}^n)$ then as before the term $\sup_{0<t<1} |T_t g|$ is taken care of. The weak type inequality for $\sup_{0<t<1} |T_t b|$ can be established if we know that

$$\sup_{0<t<1} \left|\frac{\partial}{\partial y_j} k_t(x,y)\right| \leq C|x-y|^{-n-1}, \qquad j = 1, \ldots, n, \tag{4.8}$$

with C independent of t. The proof is merely an imitation of the proof of the t-independent case.

Now the conjugate Poisson integral $A_j H^{-1/2} e^{-tH^{1/2}}$ is given by a kernel $K_t^j(x,y)$. This kernel can be calculated in the following way. From the subordinate identity (4.3) we obtain the formula

$$\alpha^{-1} e^{-\alpha} = \frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-1/u} u^{-1/2} e^{-\alpha^2 u/4}\, du \tag{4.9}$$

from which we get

$$H^{-1/2} e^{-tH^{1/2}} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t^2/(4u)} u^{-1/2} e^{-uH}\, du. \tag{4.10}$$

Thus the kernel K_t^j is given by

$$K_t^j(x,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-t^2/(4u)} u^{-1/2} \Big(-\frac{\partial}{\partial x_j} + x_j \Big) K_u(x,y)\, du. \tag{4.11}$$

In view of Proposition 4.1 we need only consider the term b in the Calder/on–Zygmund decomposition. Write

$$\int K_t^j(x,y) b(y)\, dy = \int L_t^j(x,y) b(y)\, dy + \int M_t^j(x,y) b(y)\, dy$$

where

$$L_t^j(x,y) = \frac{1}{\sqrt{\pi}} \int_0^1 e^{-t^2/(4u)} u^{-1/2} \Big(-\frac{\partial}{\partial x_j} + x_j \Big) K_u(x,y)\, du, \tag{4.12}$$

$$M_t^j(x,y) = \frac{1}{\sqrt{\pi}} \int_1^\infty e^{-t^2/(4u)} u^{-1/2} \Big(-\frac{\partial}{\partial x_j} + x_j \Big) K_u(x,y)\, du. \tag{4.13}$$

The estimates of Lemma 2 show that

$$|M_t^j(x,y)| \le C \int_1^\infty e^{-t^2/(4u)} u^{-1/2} e^{-nu} e^{-b|x-y|^2}\, du \tag{4.14}$$

from which it follows that

$$\sup_{0<t<1} \Big| \int M_t^j(x,y) b(y)\, dy \Big| \le C \int e^{-b|x-y|^2} |b(y)|\, dy$$

and so it is immediate that this term satisfies the weak type $(1,1)$ inequality. From the estimates of Lemma 1 it follows that

$$\begin{aligned} \Big| \frac{\partial}{\partial y_j} L_t^j(x,y) \Big| &\le C \int_0^1 u^{-1/2} u^{-n/2-1} e^{-a|x-y|^2/u}\, du \\ &\le C|x-y|^{-n-1} \end{aligned} \tag{4.15}$$

with C independent of t. Therefore, by the previous remarks it is clear that the term corresponding to L_t^j is also of weak type $(1,1)$. This completes the proof of Theorem 4.1.

To end this section we give a proof of Theorem 2 on the boundedness of R_j and R_j^* on the local Hardy space. For that purpose let us recall the definition of the symbol class $S_{1,0}^m$. By a symbol of class $S_{1,0}^m$ we mean a function σ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ which satisfies the estimates

$$|D_x^\alpha D_\xi^\beta \sigma(x,\xi)| \le C(1+|\xi|)^{m-|\beta|} \tag{4.16}$$

for all multiindices α and β where the constant C is independent of x and ξ. (In the usual definition it is assumed that C is a function of x but we are interested in symbols which satisfy estimates uniformly in x.) Such a symbol σ defines a pseudodifferential operator $\sigma(x,D)$ by

$$\sigma(x,D)f(x) = \int e^{ix\cdot\xi}\hat{f}(\xi)\sigma(x,\xi)\,d\xi. \tag{4.17}$$

In [3] Goldberg has shown that if $\sigma \in S_{1,0}^0$ then $\sigma(x,D)$ maps h^1 boundedly into itself. So, to prove Theorem 2 we need only prove the following proposition.

PROPOSITION 4.2. *R_j and R_j^* are pseudodifferential operators whose symbols belong to $S_{1,0}^0$.*

Proof. Since $R_j f = A_j H^{-1/2} f$ it is enough to show that $H^{-1/2}$ is a pseudodifferential operator whose symbol belongs to $S_{1,0}^{-1}$. But

$$H^{-1/2} = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}e^{-tH}\,dt$$

and so the symbol $a(x,\xi)$ of $H^{-1/2}$ is given by

$$a(x,\xi) = \frac{1}{\sqrt{\pi}}\int_0^\infty t^{-1/2}\sigma_t(x,\xi)\,dt \tag{4.18}$$

where $\sigma_t(x,\xi)$ is the symbol of e^{-tH}. Now for f in the Schwartz class

$$e^{-tH}f = \sum e^{-(2|\alpha|+n)t}(f,\Phi_\alpha)\Phi_\alpha \tag{4.19}$$

and the relations $(f,\Phi_\alpha) = (\widehat{f},\widehat{\Phi}_\alpha)$ and $\widehat{\Phi}_\alpha = (-i)^{|\alpha|}\Phi_\alpha$ show that

$$\sigma_t(x,\xi) = e^{-ix\cdot\xi}\sum e^{-(2|\alpha|+n)t}i^{|\alpha|}\Phi_\alpha(x)\Phi_\alpha(\xi). \tag{4.20}$$

In view of Mehler's formula one obtains

$$\sigma_t(x,\xi) = (2\pi)^{-n/2}(\cosh 2t)^{-n/2}e^{-b(t,x,\xi)} \tag{4.21}$$

where

$$b(t,x,\xi) = \tfrac{1}{2}(\tanh 2t)(|x|^2+|\xi|^2) + 2ix\cdot\xi(\sinh^2 t)\,\mathrm{sech}\,2t.$$

This shows that, with some constant C_0,

$$a(x,\xi) = C_0 \int_0^\infty t^{-1/2}(\cosh 2t)^{-n/2} e^{-b(t,x,\xi)}\, dt. \tag{4.22}$$

Now the proposition follows from the estimates of Lemma 3.

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