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WEAK MEROMORPHIC EXTENSION

BY

L. M. HAI, N. V. KHUE AND N. T. NGA (HANOI)

The relation between weak extensibility and extensibility of vector-valued holomorphic functions on open sets and on compact sets has been investigated by many authors, for example Ligocka and Siciak [6] for open sets in a metric vector space, Siciak [9] and Waelbroeck [10] for compact sets in $\mathbb{C}^n$, N. V. Khue and B. D. Tac [8] for compact sets in a nuclear metric vector space. The aim of the present note is to prove some results for Banach-valued meromorphic functions on open sets and on compact sets in $\mathbb{C}^n$.

We recall [7] that a holomorphic function f on a dense open subset G_0 of an open set G in $\mathbb{C}^n$ with values in a sequentially complete locally convex space F is called *meromorphic* on G if for each $z \in G$ there exists a neighbourhood U of z and holomorphic functions g and σ on U with values in F and $\mathbb{C}$ respectively such that $f|G_0 \cap U = g/\sigma|G_0 \cap U$.

Put

$$P(f) = \{z \in G : f \text{ is not holomorphic at } z\}.$$

It is known [7] that $P(f)$ is either empty or a hypersurface in G.

Finally, for each open subset G of $\mathbb{C}^n$, we denote by $\widehat{G}$ the envelope of holomorphy of G.

First we prove the following

THEOREM 1. *Let G be an open set in $\mathbb{C}^n$ and F a Banach space. Assume that f is an F-valued meromorphic function on an open subset X of G such that x^*f can be extended to a meromorphic function $\widehat{x^*f}$ on G for all $x^* \in F^*$, the dual space of F. Then f can be meromorphically extended to G.*

Proof. It suffices to show that f can be meromorphically extended through every point $z \in \partial X$. Fix $z^0 \in \partial X$. Put

$$\begin{aligned} \mathcal{B} = \Big\{ & a^1, \ldots, a^s; b^1, \ldots, b^t \in (\mathbb{Q} + i\mathbb{Q})^n; \\ & r^1, \ldots, r^s; \delta^1, \ldots, \delta^t \in \mathbb{Q}^{+n}; A, B \in \mathbb{Q}^+ : \\ & \text{there exists a neighbourhood } U \text{ of } z^0 \text{ such that} \\ & \qquad \Big[U \setminus \bigcup_j \overline{D}^n(a^j, r^j)\Big]^\wedge = U \Big\} \end{aligned}$$

where for each $z \in \mathbb{C}$ and $r \in \mathbb{R}^{+n}$ we denote by $D^n(z,r)$ the open polydisc centred at z with polyradius r.

For each $\alpha \in \mathcal{B}$, let

$$\begin{aligned} L(\alpha) = \Big\{ & x^* \in F^* : \widehat{x^* f} \text{ is holomorphic on} \\ & U \setminus \Big[\bigcup_j \overline{D}^n(a^j, r^j) \cup \bigcup_j \overline{D}^n(b^j, \delta^j/2) \Big] , \\ & |\widehat{x^* f}(z)| \le A \text{ for } z \notin \bigcup_j \overline{D}^n(a^j, r^j) \cup \bigcup_j \overline{D}^n(b^j, \delta^j/2) , \\ & 1/\widehat{x^* f} \text{ is holomorphic on } \bigcup_j D^n(b^j, \delta^j) , \\ & |1/\widehat{x^* f}(z)| \le B \text{ for } z \in \bigcup_j D^n(b^j, \delta^j) \Big\} \cup \{0\} . \end{aligned}$$

Claim 1. $F^* = \bigcup \{L(\alpha) : \alpha \in \mathcal{B}\}$.

Let $x^* \in F^*$, $x^* \neq 0$. Since $\operatorname{codim}(P(\widehat{x^* f}) \cap P(1/\widehat{x^* f})) \ge 2$ we can find [4] holomorphic functions $h_1, \ldots, h_n$ on $\overline{D}^n(z^0, \varepsilon) \Subset G$ such that

$$P(\widehat{x^* f}) \cap P(1/\widehat{x^* f}) \cap \overline{D}^n(z^0, \varepsilon) = \{z \in \overline{D}^n(z^0, \varepsilon) \,:\, h_{n-1}(z) = h_n(z) = 0\}$$

and the map $h : \overline{D}^n(z^0, \varepsilon) \to \mathbb{C}^n$ defined by $h_1, \ldots, h_n$ has discrete fibres. Hence $h : U \to D^n(0, \delta)$ is proper for some neighbourhood U of z^0 and $\delta \in \mathbb{Q}^{+n}$. Put

$$W = D^{n-2}(0, \delta_1, \ldots, \delta_{n-2}) \times D^2(0, \delta_{n-1}, \delta_n) .$$

Then $h^{-1}(\overline{W})$ is a neighbourhood of $P(\widehat{x^* f}) \cap P(1/\widehat{x^* f}) \cap \overline{U}$ in $\overline{U}$. Since $[D^n(0,\delta) \setminus \overline{W}]^\wedge = D^n(0, \delta)$ and $h : U \to D^n(0, \sigma)$ is a branched covering map [3] we have $[U \setminus h^{-1}(\overline{W})]^\wedge = U$. Cover now $h^{-1}(\overline{W})$ by $D^n(a^1, r^1), \ldots$ $\ldots, D^n(a^s, r^s), a^1, \ldots, a^s \in (\mathbb{Q} + i\mathbb{Q})^n, r^1, \ldots, r^s \in \mathbb{Q}^{+n}$, such that

$$\Big[U \setminus \bigcup_j \overline{D}^n(a^j, r^j) \Big]^\wedge = U .$$

Since

$$\Big[\overline{U} \cap P(x^* f) \setminus \bigcup_j D^n(a^j, r^j) \Big] \cap P(1/\widehat{x^* f}) = \emptyset$$

we can find $b^1, \ldots, b^t \in (\mathbb{Q} + i\mathbb{Q})^n$ and $\delta^1, \ldots, \delta^t \in \mathbb{Q}^{+n}$ such that

$$\Big[U \setminus \bigcup_j D^n(a^j, r^j) \Big] \cap P(\widehat{x^* f}) \subseteq \bigcup_j D^n(b^j, \delta^j/2)$$

and

$$\overline{D}^n(b^j, \delta^j) \cap P(1/\widehat{x^* f}) = \emptyset \quad \text{ for } j = 1, \ldots, t .$$

Then $x^* \in L(\alpha)$ with $\alpha = \{a^1, \ldots, a^s; r^1, \ldots, r^s; \delta^1, \ldots, \delta^t, A, B\}$ where $A, B \in \mathbb{Q}^+$ and

$$A \geq \sup\Big\{|\widehat{x^* f}(z)| : z \in U \setminus \Big[\bigcup_j \overline{D}^n(a^j, r^j) \cup \bigcup_j \overline{D}^n(b^j, \delta^j/2)\Big]\Big\},$$

$$B \geq \sup\Big\{|1/\widehat{x^* f}(z)| : z \in \bigcup_j D^n(b^j, \delta^j)\Big\}.$$

Claim 2. *$L(\alpha)$ is closed in F^* for every $\alpha \in \mathcal{B}$.*

Let $\{x_a^*\} \subset L(\alpha)$ converge to x^* in F^*. Since $\{\widehat{x_a^* f}\}$ and $\{1/\widehat{x_a^* f}\}$ are bounded in $H(U \setminus [\bigcup_j \overline{D}^n(a^j, r^j) \cup \bigcup_j \overline{D}^n(b^j, \delta^j/2)])$ and $H(\bigcup_j D^n(b^j, \delta^j))$ respectively, by the Montel Theorem without loss of generality we can assume that $\{\widehat{x_a^*}\}$ and $\{1/\widehat{x_a^* f}\}$ converge to h and g respectively. Hence by uniqueness we have $x^* \in L(\alpha)$.

Applying the Baire Theorem to $F^* = \bigcup\{L(\alpha) : \alpha \in \mathcal{B}\}$ we can find $\alpha \in \mathcal{B}$ such that $\operatorname{Int} L(\alpha) \neq \emptyset$. Let $x_0^* \in \operatorname{Int} L(\alpha)$. For each $x^* \in F^*$ take $\delta > 0$ such that $x_0^* + \delta x^* \in \operatorname{Int} L(\alpha)$. Then from the relation

$$U \cap P(\widehat{x^* f}) \subseteq (P(\widehat{x_0^* f}) \cup P(\widehat{x_0^* f} + \delta \widehat{x^* f})) \cap U$$

it follows that

$$U \cap P(\widehat{x^* f}) \subseteq \Big[\bigcup_j \overline{D}^n(a^j, r^j) \cup \bigcup_j \overline{D}^n(b^j, \delta^j)\Big] \cap U$$

where $\alpha = \{a^j, b^j; r^j, \delta^j; A, B\}$ and U is defined by α.

Now for each $x^* \in F^*$ we denote by $C(\widehat{x^* f})$ the number of irreducible components of $P(\widehat{x^* f}) \cap W$, where $W = U \setminus \bigcup_j \overline{D}^n(a^j, r^j)$. Observe that $P(\widehat{x^* f}) \cap W = Z(1/\widehat{x^* f}) \cap W$, the zero set of $1/\widehat{x^* f}|W$. For each $k \geq 0$ put

$$A_k = \{x^* \in F^* : C(\widehat{x^* f}) \leq k\}.$$

We shall prove that A_k is closed in F^* for every $k \geq 0$. Let $\{x_j^*\} \subset A_k$ converge to $x^* \in F^*$. For each $z \in Z(1/\widehat{x^* f}) \cap W$ take a complex line L containing z such that $1/\widehat{x^* f}$ is non-constant on $L \cap W$. Then by the Hurwitz Theorem for every $j > j_0$ there exists $z_j \in Z(1/\widehat{x_j^* f}) \cap W$ such that $z_j \to z$. This yields $C(\widehat{x^* f}) \leq k$. Hence $x^* \in A_k$.

From the Baire Theorem we have $\operatorname{Int} A_k \neq \emptyset$ for some $k \geq 0$. Thus

$$m = \sup\{C(\widehat{x^* f}) : x^* \in F^*\} \leq 2k.$$

Claim 3. *There exists a finite set A in F^* such that*

$$U \cap \bigcup\{P(\widehat{x^* f}) : x^* \in F^*\} = U \cap \bigcup\{P(\widehat{x^* f}) : x^* \in A\}.$$

Indeed, otherwise we can find a finite set B in F^* such that

$$C\Big(U \cap \bigcup\{P(\widehat{x^*f}) : x^* \in B\}\Big) \geq m^2 .$$

Let $y^* \in E^*$ be such that

$$P(\widehat{y^*f}) \cap W \not\subseteq \bigcup\{P(\widehat{x^*f}) : x^* \in B\} \cap W .$$

Then

$$C\Big(\widehat{y^*f} + \sum\{\widehat{x^*f} : x^* \in B\}\Big) \geq m^2 - m > m .$$

This is impossible.

CLAIM 4. *f is meromorphic on U.*

By Claim 3, f is holomorphic on $W \setminus V$, where $V = \bigcup\{P(\widehat{x^*f}) : x^* \in A\}$ and A is some finite set in F^*. Let z^1 be a regular point of V. Then there are local coordinates $(u_1, \ldots, u_n)$ in a neighbourhood Z of z^1 in U such that $V \cap Z = Z(u_1)$. In $Z \setminus Z(u_1)$ we have

$$f(u_1, v) = \sum_{k=-\infty}^{\infty} c_k(v) u_1^k$$

where $v = (u_2, \ldots, u_n)$. By the Baire Theorem and since $\widehat{x^*f}$ is meromorphic on Z for $x^* \in F^*$ it follows that $c_k = 0$ for every $k < p$. Thus f is meromorphic on $Z \setminus S(V)$, where $S(V)$ denotes the singular locus of V. Since $\operatorname{codim} S(V) \geq 2$ we have

$$\Big[U \setminus \Big[\bigcup_j \overline{D}^n(a^j, r^j) \cup S(V)\Big]\Big]^{\wedge} = U .$$

From [1] we conclude that f is meromorphic on U.

The theorem is proved.

Remarks. 1) Theorem 1 is also true when F is replaced by a sequentially complete locally convex space E for which E^* is a Baire space.

2) Since every Fréchet space which does not have a continuous norm contains a subspace isomorphic to $\mathbb{C}^\infty$ [2] and since the function $z \mapsto (1/z, 1/z^2, \ldots)$ is not meromorphic at $0 \in \mathbb{C}$, it follows that if Theorem 1 holds for F, then F has a continuous norm.

THEOREM 2. *Let G be a non-empty subset of a compact set K of $\mathbb{C}^n$ and let f be a function on G with values in a Banach space F such that x^*f can be extended to a meromorphic function $\widehat{x^*f}$ on a neighbourhood of K for all $x^* \in F^*$. Then f is meromorphic on a neighbourhood of K.*

Proof. For each $z \in \mathbb{C}^n$ consider $\mathcal{B}(z)$ constructed as $\mathcal{B}$ in Theorem 1 with z^0 replaced by z. From the proof of Theorem 1 (Claim 1), for every $x^* \in F^*$ and every $z \in K$ we can find $\alpha \in \mathcal{B}(u)$, $u \in (\mathbb{Q} + i\mathbb{Q})^n$, such that $x^* \in L(\alpha)$ and $z \in U(\alpha)$, where $U(\alpha)$ is defined by α. Thus by the compactness of K we have

$$F^* = \bigcup\{L(\alpha) : \alpha \in \widetilde{\mathcal{B}}\}, \quad L(\alpha) = \bigcap_k L(\alpha^k),$$

where

$$\widetilde{\mathcal{B}} = \{\alpha = (\alpha^1, \ldots, \alpha^m) \in \mathcal{B}(z^1) \times \ldots \times \mathcal{B}(z^m), \\ z^1, \ldots, z^m \in (\mathbb{Q} + i\mathbb{Q})^n : K \subset \bigcup_j U(\alpha^j)\}.$$

Moreover, as in Theorem 1 (Claim 2), $L(\alpha)$ is closed for every $\alpha \in \widetilde{\mathcal{B}}$. Using the Baire Theorem we can find $\alpha \in \widetilde{\mathcal{B}}$ for which $\operatorname{Int} L(\alpha) \neq \emptyset$. Then similarly to Theorem 1 (Claim 2) for $U = \bigcup_j U(\alpha^j)$ we find that, for all $x^* \in F^*$, $\widehat{x^* f}$ is meromorphic on U,

$$P(\widehat{x^* f}) \subset \bigcup_j \overline{D}^n(a^{j,k}, r^{j,k}) \cup \bigcup_j \overline{D}^n(b^{j,k}, \delta^{j,k}/2)$$

and

$$\Big[\bigcup_k \Big[U(\alpha^k) \setminus \Big[\bigcup_j \overline{D}^n(a^{j,k}, r^{j,k}) \cup \bigcup_j \overline{D}^n(b^{j,k}, \delta^{j,k}/2)\Big]\Big]\Big]^{\wedge} \supseteq \bigcup_k U(\alpha^k)$$

where $\alpha = (\alpha^1, \ldots, \alpha^m), \alpha^k = (a^{j,k}; b^{j,k}; r^{j,k}; \delta^{j,k}; A^k, B^k), k = 1, \ldots, m$. Hence as in Theorem 1 (Claims 3–4) we obtain a meromorphic extension of f to a neighbourhood of K.

The theorem is proved.

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DEPARTMENT OF MATHEMATICS
PEDAGOGICAL INSTITUTE OF HANOI I
HANOI, VIETNAM

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