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ON NORMAL CR-SUBMANIFOLDS OF S-MANIFOLDS

BY

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0. Introduction. Many authors have studied the geometry of submanifolds of Kaehlerian and Sasakian manifolds. On the other hand, David E. Blair has initiated the study of S-manifolds, which reduce, in particular cases, to Sasakian manifolds ([1, 2]).

I. Mihai ([8]) and L. Ornea ([9]) have investigated CR-submanifolds of S-manifolds. The purpose of the present paper is to study a special kind of such submanifolds, namely the normal CR-submanifolds.

In Sections 1 and 2, we review basic formulas and definitions for submanifolds in Riemannian manifolds and in S-manifolds, respectively, which we shall use later. In Section 3, we introduce normal CR-submanifolds of S-manifolds and we study some properties of their geometry. Finally, in Section 4, we consider those submanifolds in the case of the ambient S-manifold being an S-space form.

1. Preliminaries. Let $\mathcal{N}$ be a Riemannian manifold of dimension n and $\mathcal{M}$ an m-dimensional submanifold of $\mathcal{N}$. Let g be the metric tensor field on $\mathcal{N}$ as well as the induced metric on $\mathcal{M}$. We denote by $\overline{\nabla}$ the covariant differentiation in $\mathcal{N}$ and by ∇ the covariant differentiation in $\mathcal{M}$ determined by the induced metric. Let $T(\mathcal{N})$ (resp. $T(\mathcal{M})$) be the Lie algebra of vector fields in $\mathcal{N}$ (resp. in $\mathcal{M}$) and $T(\mathcal{M})^{\perp}$ the set of vector fields normal to $\mathcal{M}$.

The Gauss–Weingarten formulas are given by

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + \sigma(X,Y)\,, \\ &\overline{\nabla}_X V = -A_V X + D_X V, \qquad X,Y \in T(\mathcal{M}),\ V \in T(\mathcal{M})^{\perp}\,, \end{aligned} \tag{1.1}$$

where D is the connection in the normal bundle, σ is the second fundamental form of $\mathcal{M}$ and A_V the Weingarten endomorphism associated with V. Then

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A_V and σ are related by

$$g(A_V X, Y) = g(\sigma(X,Y), V)\,. \tag{1.2}$$

We denote by $\overline{R}$ and R the curvature tensor fields associated with $\overline{\nabla}$ and ∇, respectively. The Gauss equation is given by

$$\begin{aligned}\overline{R}(X,Y,Z,W) &= R(X,Y,Z,W) + g(\sigma(X,Z),\sigma(Y,W)) \\ &\quad - g(\sigma(X,W),\sigma(Y,Z)), \qquad X,Y,Z,W \in T(\mathcal{M})\,.\end{aligned} \tag{1.3}$$

Moreover, we have the following Codazzi equation:

$$\overline{R}(X,Y,Z,V) = g((\nabla'_X\sigma)(Y,Z),V) - g((\nabla'_Y\sigma)(X,Z),V) \tag{1.4}$$

for any $X,Y,Z \in T(\mathcal{M})$ and $V \in T(\mathcal{M})^{\perp}$, where $\nabla'\sigma$ is the covariant derivative of the second fundamental form given by

$$(\nabla'_X\sigma)(Y,Z) = D_X\sigma(Y,Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) \tag{1.5}$$

for any $X,Y,Z \in T(\mathcal{M})$. Finally, the submanifold $\mathcal{M}$ is said to be *totally geodesic* in $\mathcal{N}$ if its second fundamental form is identically zero, and it is said to be *minimal* if $H \equiv 0$, where H is the mean curvature vector, defined by $H = (1/m)\,\mathrm{trace}\,(\sigma)$.

2. CR-submanifolds of S-manifolds. Let $(\mathcal{N}, g)$ be a Riemannian manifold with $\dim(\mathcal{N}) = 2n+s$. It is said to be an *S-manifold* if there exist on $\mathcal{N}$ an f-structure f ([10]) of rank $2n$ and s global vector fields $\xi_1,\ldots,\xi_s$ (structure vector fields) such that ([1]):

(i) If $\eta_1,\ldots,\eta_s$ are the dual 1-forms of $\xi_1,\ldots,\xi_s$, then

$$\begin{aligned} &f\xi_\alpha = 0, \quad \eta_\alpha \circ f = 0, \quad f^2 = -I + \sum \xi_\alpha \otimes \eta_\alpha\,, \\ &g(X,Y) = g(fX,fY) + \varPhi(X,Y)\,, \end{aligned} \tag{2.1}$$

for any $X, Y \in T(\mathcal{N})$, $\alpha = 1,\ldots,s$, where $\varPhi(X,Y) = \sum \eta_\alpha(X)\eta_\alpha(Y)$.

(ii) The f-structure f is *normal*, that is,

$$[f,f] + 2\sum \xi_\alpha \otimes d\eta_\alpha = 0\,,$$

where $[f,f]$ is the Nijenhuis torsion of f.

(iii) $\eta_1 \wedge \ldots \wedge \eta_s \wedge (d\eta_\alpha)^n \neq 0$ and $d\eta_1 = \ldots = d\eta_s = F$, for any α, where F is the fundamental 2-form defined by $F(X,Y) = g(X,fY)$, $X,Y \in T(\mathcal{N})$.

In the case $s = 1$, an S-manifold is a Sasakian manifold. For $s \geq 2$, examples of S-manifolds are given in $[1,2,3,6]$. Thus, the bundle space of a principal toroidal bundle over a Kaehler manifold with certain conditions is an S-manifold. In this way, a generalization of the Hopf fibration $\overline{\pi} : S^{2n+1} \to \mathbb{PC}^n$ is introduced in [1] as a canonical example of an S-manifold playing the role of the complex projective space in Kaehler geometry and the odd-dimensional sphere in Sasakian geometry. This space is given by (see [1, 2] for more details):

$$H^{2n+s} = \{(x_1, \ldots, x_s) \in S^{2n+1} \times \ldots \times S^{2n+1} : \overline{\pi}(x_1) = \ldots = \overline{\pi}(x_s)\}\,.$$

For the Riemannian connection $\overline{\nabla}$ of g on an S-manifold $\mathcal{N}$, the following formulas were also proved in [1]:

$$\overline{\nabla}_X \xi_\alpha = -fX, \quad X \in T(\mathcal{N}),\ \alpha = 1, \ldots, s\,, \tag{2.2}$$

$$(\overline{\nabla}_X f)Y = \sum\{g(fX, fY)\xi_\alpha + \eta_\alpha(Y)f^2X\}, \quad X, Y \in T(\mathcal{N})\,. \tag{2.3}$$

Let $\mathcal{L}$ denote the distribution determined by $-f^2$ and $\mathcal{M}$ the complementary distribution. $\mathcal{M}$ is determined by $f^2 + I$ and spanned by $\xi_1, \ldots, \xi_s$. If $X \in \mathcal{L}$, then $\eta_\alpha(X) = 0$ for any α, and if $X \in \mathcal{M}$, then $fX = 0$.

A plane section π on $\mathcal{N}$ is called an *invariant f-section* if it is determined by a vector $X \in \mathcal{L}(x)$, $x \in \mathcal{N}$, such that $\{X, fX\}$ is an orthonormal pair spanning the section. The sectional curvature of π is called an *f-sectional curvature*. If $\mathcal{N}$ is an S-manifold whose invariant f-sectional curvature is a constant k, then its curvature tensor has the form ([7])

$$\begin{aligned} \overline{R}(X, Y, Z, W) = \sum_{\alpha,\beta}&\{g(fX, fW)\eta_\alpha(Y)\eta_\beta(Z) \\ &- g(fX, fZ)\eta_\alpha(Y)\eta_\beta(W) + g(fY, fZ)\eta_\alpha(X)\eta_\beta(W) \\ &- g(fY, fW)\eta_\alpha(X)\eta_\beta(Z)\} \\ &+ \tfrac{1}{4}(k + 3s)\{g(fX, fW)g(fY, fZ) - g(fX, fZ)g(fY, fW)\} \\ &+ \tfrac{1}{4}(k - s)\{F(X, W)F(Y, Z) - F(X, Z)F(Y, W) \\ &- 2F(X, Y)F(Z, W)\}, \quad X, Y, Z, W \in T(\mathcal{N})\,, \end{aligned} \tag{2.4}$$

and thus, the S-manifold is denoted by $\mathcal{N}(k)$ and it is said to be an *S-space form*. For example, the Euclidean space E^{2n+s} is an S-space form with f-sectional curvature $-3s$ ([6]) and H^{2n+s} is an S-space form with f-sectional curvature $4 - 3s$ ([1]).

Now, let $\mathcal{M}$ be an m-dimensional submanifold immersed in $\mathcal{N}$. $\mathcal{M}$ is said to be an *invariant submanifold* if $\xi_\alpha \in T(\mathcal{M})$ for any α and $fX \in T(\mathcal{M})$ for any $X \in T(\mathcal{M})$. On the other hand, it is said to be an *anti-invariant submanifold* if $fX \in T(\mathcal{M})^\perp$ for any $X \in T(\mathcal{M})$.

Given any vector field $V \in T(\mathcal{M})^\perp$, we write $fV = tV + nV$, where tV (resp. nV) is the tangential component (resp. normal component) of fV. Then t is a tangent-bundle valued 1-form on the normal bundle of $\mathcal{M}$ and n is an endomorphism of the normal bundle of $\mathcal{M}$. Moreover, if n does not vanish, it is an f-structure.

Now, assume that the structure vector fields $\xi_1, \ldots, \xi_s$ are tangent to $\mathcal{M}$ (and so, $\dim(\mathcal{M}) \geq s$). Then $\mathcal{M}$ is called a *CR-submanifold* of $\mathcal{N}$ if there exist two differentiable distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on $\mathcal{M}$ satisfying:

(i) $T(\mathcal{M}) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{M}$, where $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{M}$ are mutually orthogonal to each other.

(ii) The distribution $\mathcal{D}$ is invariant under f, that is, $f\mathcal{D}_x = \mathcal{D}_x$ for any $x \in \mathcal{M}$.

(iii) The distribution $\mathcal{D}^\perp$ is anti-invariant under f, that is, $f\mathcal{D}_x^\perp \subseteq T_x(\mathcal{M})^\perp$ for any $x \in \mathcal{M}$.

We denote by $2p$ and q the real dimensions of $\mathcal{D}_x$ and $\mathcal{D}_x^\perp$ respectively, for any $x \in \mathcal{M}$. Then, if $p = 0$ we have an anti-invariant submanifold tangent to $\xi_1, \dots, \xi_s$, and if $q = 0$ we have an invariant submanifold. The CR-submanifold is called a *generic submanifold* if $q = n - p$, that is, if given $V \in T(\mathcal{M})^\perp$, there exists $Z \in \mathcal{D}^\perp$ such that $V = fZ$.

As an example, it is easy to prove that each hypersurface of $\mathcal{N}$ which is tangent to $\xi_1, \dots, \xi_s$ inherits the structure of CR-submanifold of $\mathcal{N}$.

A CR-submanifold of an S-manifold is said to be $(\mathcal{D}, \mathcal{D}^\perp)$-*geodesic* if $\sigma(X, Z) = 0$ for any $X \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$, and it is said to be $\mathcal{D}^\perp$-*geodesic* if $\sigma(Y, Z) = 0$ for any $Y, Z \in \mathcal{D}^\perp$.

Now, denote by P and Q the projection morphisms of $T(\mathcal{M})$ on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Then, for any $X \in T(\mathcal{M})$, we have $X = PX + QX + \sum \eta_\alpha(X)\xi_\alpha$. Define the tensor field v of type (1, 1) on $\mathcal{M}$ by $vX = fPX$, and the non-null normal-bundle valued 1-form u on $\mathcal{M}$ by $uX = fQX$. Then it is easy to show that:

$$u \circ v = 0\,, \tag{2.5}$$

$$\eta_\alpha \circ u = \eta_\alpha \circ v = 0 \quad \text{for any } \alpha\,, \tag{2.6}$$

$$vX = 0 \quad \text{if and only if} \quad X \in \mathcal{D}^\perp \oplus \mathcal{M}\,, \tag{2.7}$$

$$uX = 0 \quad \text{if and only if} \quad X \in \mathcal{D} \oplus \mathcal{M}\,. \tag{2.8}$$

Moreover, a direct computation gives

$$g(X,Y) = g(vX, vY) + g(uX, uY) + \Phi(X,Y)\,,$$
$$F(X,Y) = g(X, vY), \qquad F(X,Y) = F(vX, vY)\,,$$

for any $X, Y \in T(\mathcal{M})$.

For later use, we recall some lemmas:

Lemma 2.1 ([5]). *Let $\mathcal{M}$ be a CR-submanifold of an S-manifold $\mathcal{N}$. Then*:

$$\nabla_X \xi_\alpha = -vX\,, \tag{2.9}$$

$$\sigma(X, \xi_\alpha) = -uX\,, \tag{2.10}$$

$$A_V \xi_\alpha \in \mathcal{D}^\perp\,, \tag{2.11}$$

for any $X \in T(\mathcal{M})$, $V \in T(\mathcal{M})^\perp$ and $\alpha \in \{1, \dots, s\}$.

LEMMA 2.2 ([5]). *Let $\mathcal{M}$ be a CR-submanifold of an S-manifold N. If $X, Y \in T(\mathcal{M})$, then:*

$$P\nabla_X vY - PA_{uY}X = v\nabla_X Y - \sum \eta_\alpha(Y)PX\,, \tag{2.12}$$

$$Q\nabla_X vY - QA_{uY}X = t\sigma(X,Y) - \sum \eta_\alpha(Y)QX\,, \tag{2.13}$$

$$\sigma(X, vY) + D_X uY = u\nabla_X Y + n\sigma(X,Y)\,, \tag{2.14}$$

$$g(fX, fY) = \eta_\alpha(\nabla_X vY - A_{uY}X)\,. \tag{2.15}$$

From Lemma 2.2 we obtain

$$(\nabla_X v)Y = A_{uY}X + t\sigma(X,Y) - \sum\{\eta_\alpha(Y)f^2X + g(fX,fY)\xi_\alpha\}\,, \tag{2.16}$$

$$(\nabla_X u)Y = n\sigma(X,Y) - \sigma(X, vY)\,, \tag{2.17}$$

for any $X, Y \in T(\mathcal{M})$.

3. Normal CR-submanifolds of an S-manifold. In this section, let $\mathcal{M}$ be a CR-submanifold of an S-manifold $\mathcal{N}$. We say that $\mathcal{M}$ is a *normal CR-submanifold* of $\mathcal{N}$ if

$$N_v(X,Y) = 2tdu(X,Y) - 2\sum F(X,Y)\xi_\alpha \tag{3.1}$$

for any $X, Y \in T(\mathcal{M})$, where N_v denotes the Nijenhuis torsion of v. Notice that (3.1) is equivalent to

$$\begin{aligned}(\nabla_{vX} v)Y - (\nabla_{vY} v)X + v((\nabla_Y v)X - (\nabla_X v)Y)&\\ = t((\nabla_X u)Y - (\nabla_Y u)X) - 2\sum F(X,Y)\xi_\alpha\,.&\end{aligned} \tag{3.2}$$

THEOREM 3.1. *A CR-submanifold $\mathcal{M}$ of an S-manifold $\mathcal{N}$ is normal if and only if*

$$A_{uY}vX = vA_{uY}X \tag{3.3}$$

for any $X \in \mathcal{D}$ and any $Y \in \mathcal{D}^\perp$.

Proof. If we define the tensor field

$$\begin{aligned}S(X,Y) = (\nabla_{vX} v)Y - (\nabla_{vY} v)X + v((\nabla_Y v)X - (\nabla_X v)Y)&\\ - t((\nabla_X u)Y - (\nabla_Y u)X) + 2\sum F(X,Y)\xi_\alpha, \qquad X, Y \in T(\mathcal{M})\,,&\end{aligned}$$

then $\mathcal{M}$ is normal if and only if S is identically zero. A direct expansion, by using (2.16) and (2.17), gives

$$S(X,Y) = A_{uY}vX - vA_{uY}X - A_{uX}vY + vA_{uX}Y, \qquad X, Y \in T(\mathcal{M})\,. \tag{3.4}$$

Now, if $\mathcal{M}$ is a normal CR-submanifold of $\mathcal{N}$, (3.3) follows from (3.4) since $uX = 0$ for any $X \in \mathcal{D}$.

Conversely, if (3.3) holds, we shall prove that S vanishes by using the decomposition $T(\mathcal{M}) = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{M}$. First, since $uX = 0$ for any $X \in \mathcal{D}$ and

$v\xi_\alpha = 0 = u\xi_\alpha$ for any α, we observe from (3.3) and (3.4) that $S(X,Y) = 0$ for any $X \in \mathcal{D}$ and any $Y \in T(\mathcal{M})$.

Moreover, if $Y \in \mathcal{D}^\perp$, from (2.11) we have $A_{uY}\xi_\alpha \in \mathcal{D}^\perp$, and so $vA_{uY}\xi_\alpha = 0$ for any α. Consequently, $S(X,\xi_\alpha) = 0$ for any α and any $X \in T(\mathcal{M})$.

Finally, if $Y, Z \in \mathcal{D}^\perp$, (3.4) becomes

$$S(X,Y) = v(A_{fX}Y - A_{fY}X)\,,$$

since $vX = vY = 0$ and $uX = fX$, $uY = fY$. But, from (1.1), (1.2) and (2.3), we easily show that $A_{fX}Y = A_{fY}X$. ∎

COROLLARY 3.2. *A CR-submanifold $\mathcal{M}$ of an S-manifold $\mathcal{N}$ is normal if and only if*

$$g(\sigma(X,vY) + \sigma(Y,vX), fZ) = 0\,, \tag{3.5}$$

$$g(\sigma(X,Z), fW) = 0\,, \tag{3.6}$$

for any $X, Y \in \mathcal{D}$ and any $Z, W \in \mathcal{D}^\perp$.

Proof. Since v is skew-symmetric, from (3.3) we see that $\mathcal{M}$ is normal if and only if

$$g(\sigma(X,vY), uZ) = -g(\sigma(Y,vX), uZ) \tag{3.7}$$

for any $X \in T(\mathcal{M})$, $Y \in \mathcal{D}$, $Z \in \mathcal{D}^\perp$.

Now, if $\mathcal{M}$ is normal, from (3.7) we get (3.5) taking $X \in \mathcal{D}$ and (3.6) taking $X \in \mathcal{D}^\perp$. Conversely, if (3.5) and (3.6) are satisfied, we observe that (3.7) is satisfied if $X \in \mathcal{D}$ and if $X \in \mathcal{D}^\perp$. Finally, if $X \in \mathcal{M}$, we have $vX = 0$ and, by using (2.5) and (2.10), $\sigma(X,vY) = 0$ for any $Y \in \mathcal{D}$. So, (3.7) holds for any $X \in T(\mathcal{M})$. ∎

COROLLARY 3.3. *Each normal generic submanifold of an S-manifold is $(\mathcal{D},\mathcal{D}^\perp)$-geodesic.*

LEMMA 3.4. *Let $\mathcal{M}$ be a normal CR-submanifold of an S-manifold $\mathcal{N}$. Then the following assertions are satisfied*:

$$\sigma(fX,Z) = f\sigma(X,Z)\,, \tag{3.8}$$

$$t\sigma(fX,fX) = t\sigma(X,X)\,, \tag{3.9}$$

$$A_{fZ}X \in \mathcal{D}\,, \tag{3.10}$$

for any $X \in \mathcal{D}$ and any $Z \in \mathcal{D}^\perp$.

Proof. (3.8) follows easily from (1.1), (2.3) and (3.6). Now, from (3.5) we get (3.9). Finally, from (3.6) we have $g(A_{fZ}X,Y) = 0$ for any $Y \in \mathcal{D}^\perp$, and from (2.10) we have $\eta_\alpha(A_{fZ}X) = 0$ for any α. Consequently, (3.10) holds. ∎

In [5], CR-products of S-manifolds are defined as CR-submanifolds such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable and locally they are Riemannian products $\mathcal{M}_1 \times \mathcal{M}_2$, where $\mathcal{M}_1$ (resp. $\mathcal{M}_2$) is a leaf of $\mathcal{D} \oplus \mathcal{M}$ (resp. $\mathcal{D}^\perp$). Moreover, from Theorem 3.1 and Proposition 3.2 in [5], we deduce that a CR-submanifold $\mathcal{M}$ of an S-manifold $\mathcal{N}$ is a CR-product if and only if one of the following assertions is satisfied:

$$A_{f\mathcal{D}^\perp} f\mathcal{D} = 0\,, \tag{3.11}$$

$$g(\sigma(X,Y), fZ) = 0, \quad X \in \mathcal{D},\ Y \in T(\mathcal{M}),\ Z \in \mathcal{D}^\perp\,, \tag{3.12}$$

$$\nabla_Y X \in \mathcal{D} \oplus \mathcal{M}, \quad X \in \mathcal{D},\ Y \in T(\mathcal{M})\,. \tag{3.13}$$

Then, from (3.6), we can prove the following:

PROPOSITION 3.5. *A CR-product in an S-manifold is a normal CR-submanifold.*

THEOREM 3.6. *Let $\mathcal{M}$ be a normal CR-submanifold of an S-manifold $\mathcal{N}$. Then $\mathcal{M}$ is a CR-product if and only if $\mathcal{D} \oplus \mathcal{M}$ is integrable.*

Proof. We recall that $\mathcal{D} \oplus \mathcal{M}$ is integrable if and only if

$$\sigma(X, fY) = \sigma(fX, Y) \tag{3.14}$$

for any $X, Y \in \mathcal{D}$ ([8]).

Now, the necessary condition is obvious, by definition. Conversely, we prove (3.12). Let $X \in \mathcal{D}$. If $Y \in \mathcal{D}^\perp$, then (3.12) is (3.6). On the other hand, if $Y \in \mathcal{M}$, from (2.8) and (2.10) we get $\sigma(X,Y) = 0$. Finally, if $Y \in \mathcal{D}$, from (3.5) and (3.14), (3.12) holds. ∎

To finish this section, we recall that a submanifold $\mathcal{M}$ of an S-manifold $\mathcal{N}$ is said to be *totally f-umbilical* ([9]) if there exists a normal vector field V such that

$$\sigma(X,Y) = g(fX, fY)V + \sum\{\eta_\alpha(Y)\sigma(X,\xi_\alpha) + \eta_\alpha(X)\sigma(Y,\xi_\alpha)\} \tag{3.15}$$

for any $X, Y \in T(\mathcal{M})$. These submanifolds have been studied in [4]. We can prove the following:

PROPOSITION 3.7. *A totally f-umbilical CR-submanifold of an S-manifold is a normal CR-submanifold.*

Proof. From (3.15) we easily get (3.5) and (3.6). ∎

4. Normal CR-submanifolds of S-space forms. Let $\mathcal{N}(k)$ be an S-space form and let $\mathcal{M}$ be a CR-submanifold of $\mathcal{N}(k)$. Then, by using (2.4), the Codazzi equation (1.4) gives

$$\begin{aligned}(\nabla'_X\sigma)(Y,Z) - (\nabla'_Y\sigma)(X,Z) = ((k-s)/4)\{&g(X,vZ)uY \\ &- g(Y,vZ)uX + 2g(X,vY)uZ\}\,,\end{aligned} \tag{4.1}$$

for any $X, Y, Z \in T(\mathcal{M})$. Now, we have:

Proposition 4.1. *If $\mathcal{M}$ is a normal CR-submanifold of $\mathcal{N}(k)$, then*

$$\begin{aligned}\overline{R}(X, fX, Z, fZ) = 2s - 2\|A_{fZ}X\|^2 - 2\|\sigma(X, Z)\|^2 \\ + 2g(t\sigma(Z, Z), t\sigma(X, X))\end{aligned} \tag{4.2}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof. By using (1.4) and (1.5), we have

$$\begin{aligned}\overline{R}(X, fX, Z, fZ) = g(D_X\sigma(fX, Z) - D_{fX}\sigma(X, Z), fZ) \\ -g(\sigma([X, fX], Z), fZ) + g(\sigma(X, \nabla_{fX}Z) - \sigma(fX, \nabla_X Z), fZ)\,.\end{aligned} \tag{4.3}$$

Now, from (1.1), (2.3), (3.6) and (3.8), a direct expansion gives

$$g(D_X\sigma(fX, Z) - D_{fX}\sigma(X, Z), fZ) = -2\|\sigma(X, Z)\|^2\,. \tag{4.4}$$

On the other hand, by using (3.6) again,

$$\begin{aligned}g(\sigma([X, fX], Z), fZ) = g(\sigma(Q[X, fX], Z), fZ) \\ + \sum g(\sigma(\eta_\alpha([X, fX])\xi_\alpha, Z), fZ)\,.\end{aligned} \tag{4.5}$$

But, from (2.2) and since X and Z are unit vector fields, we see that $\eta_\alpha([X, fX]) = 2$ for any α. Moreover, from (2.13), we obtain $Q[X, fX] = t\sigma(X, X) + t\sigma(fX, fX)$. Then, taking into account (2.10) and (3.9), (4.5) becomes

$$g(\sigma([X, fX], Z), fZ) = 2g(\sigma(t\sigma(X, X), Z), fZ) - 2s\,. \tag{4.6}$$

However, since $Z \in \mathcal{D}^\perp$ and by using (1.2) and (2.13), it is easy to show that $g(\sigma(t\sigma(X, X), Z), fZ) = -g(t\sigma(X, X), t\sigma(Z, Z))$. Substituting this in (4.6), we have

$$g(\sigma([X, fX], Z), fZ) = -2s - 2g(t\sigma(X, X), t\sigma(Z, Z))\,. \tag{4.7}$$

Finally, since $\eta_\alpha(\nabla_{fX}Z) = \eta_\alpha(\nabla_X Z) = 0$ for any α, from (2.12), (3.5) and (3.6) we get

$$\begin{aligned}&g(\sigma(X, \nabla_{fX}Z) - \sigma(fX, \nabla_X Z), fZ) \\ &\qquad = g(\sigma(X, P\nabla_{fX}Z + fP\nabla_X Z), fZ) \\ &\qquad = g(A_{fZ}X, P\nabla_{fX}Z - PA_{fZ}X)\,.\end{aligned} \tag{4.8}$$

But, by using (2.12) and (4.3), it easy to check that $P\nabla_{fX}Z = -PA_{fZ}X$. Consequently and taking into account (3.10), (4.8) gives

$$\begin{aligned}g(\sigma(X, \nabla_{fX}Z) - \sigma(fX, \nabla_X Z), fZ) &= -2g(A_{fZ}X, PA_{fZ}X) \\ &= -2\|A_{fZ}X\|^2\,.\end{aligned} \tag{4.9}$$

Then, substituting (4.4), (4.7) and (4.9) in (4.3), we complete the proof. ■

Proposition 4.2. *Let $\mathcal{M}$ be a normal CR-submanifold of an S-space form $\mathcal{N}(k)$. Then*

$$\|\sigma(X,Z)\|^2 + \|A_{fZ}X\|^2 - g(t\sigma(X,X), t\sigma(Z,Z)) = (k+3s)/4 \tag{4.10}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof. From (2.4), we have $\overline{R}(X, fX, Z, fZ) = -(k-s)/2$. Then, from (4.2), the proof is complete. ∎

Corollary 4.3. *If $\mathcal{M}$ is a normal $\mathcal{D}^\perp$-geodesic CR-submanifold of an S-space form $\mathcal{N}(k)$, then $k \geq -3s$.*

Proposition 4.4. *If $\mathcal{M}$ is a normal CR-submanifold of an S-space form $\mathcal{N}(k)$ such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable, then $k \geq -3s$ and $\mathcal{M}$ is a CR-product.*

Proof. From Theorem 3.6, $\mathcal{M}$ is a CR-product. Now, from (3.12), we have $g(\sigma(X,Y), fZ) = 0$ for any $X, Y \in \mathcal{D}$. Then, if $X \in \mathcal{D}$ is a unit vector field, $t\sigma(X,X) = 0$ and, by using (4.10), $k \geq -3s$. ∎

For the $(2n+s)$-dimensional euclidean S-space form $E^{2n+s}(-3s)$ (see [6]), we can prove:

Theorem 4.5. *If $\mathcal{M}$ is a normal $(\mathcal{D}, \mathcal{D}^\perp)$-geodesic and $\mathcal{D}^\perp$-geodesic CR-submanifold of $E^{2n+s}(-3s)$, then $\mathcal{M}$ is a CR-product.*

Proof. From (4.10), we have $A_{fZ}X = 0$ for any $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$. From (3.11), $\mathcal{M}$ is a CR-product. ∎

Corollary 4.6. *Every normal $\mathcal{D}^\perp$-geodesic generic submanifold of $E^{2n+s}(-3s)$ is a CR-product.*

Finally, consider the $(2n+s)$-dimensional S-space form $H^{2n+s}(4-3s)$ (see [1]). Let $\mathcal{M}$ be a CR-submanifold of $H^{2n+s}(4-3s)$. Denote by ν the complementary distribution of $f\mathcal{D}^\perp$ in $T(\mathcal{M})^\perp$. Then $f\nu \subseteq \nu$. Let $\{E_1, \dots, E_{2p}\}$, $\{F_1, \dots, F_q\}$, $\{N_1, \dots, N_r, fN_1, \dots, fN_r\}$ be local fields of orthonormal frames on $\mathcal{D}$, $\mathcal{D}^\perp$ and ν, respectively, where $2r$ is the real dimension of ν. For later use, we shall prove:

Lemma 4.7. *If $\mathcal{M}$ is a CR-product in $H^{2n+s}(4-3s)$, then*

$$\|\sigma(X,Z)\| = 1 \tag{4.11}$$

for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$.

Proof. We observe that $\mathcal{M}$ is a normal CR-submanifold due to Proposition 3.5, and so (4.10) holds with $(k+3s)/4 = 1$. Moreover, from (3.11), we have $A_{fZ}X = 0$ and, from (3.12), $t\sigma(X,X) = 0$. ∎

LEMMA 4.8. *Let $\mathcal{M}$ be a CR-product in $H^{2n+s}(4-3s)$. Then the vector fields $\sigma(E_i, F_a)$, $i = 1, \ldots, 2p$, $a = 1, \ldots, q$, are $2pq$ orthonormal vector fields on ν.*

Proof. From (4.11) and by linearity, we get

$$g(\sigma(E_i, Z), \sigma(E_j, Z)) = 0$$

for any $i, j = 1, \ldots, 2p$, $i \neq j$ and any unit vector field $Z \in \mathcal{D}^{\perp}$. Now, from (3.6), if $q = 1$, the proof is complete. On the other hand, if $q \geq 2$, by linearity again, we have

$$g(\sigma(E_i, F_a), \sigma(E_j, F_b)) + g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0$$

for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$. Next, by using (3.13) and the Bianchi identity, we obtain $R(X, Y, Z, W) = 0$ for any $X, Y \in \mathcal{D}$, $Z, W \in \mathcal{D}^{\perp}$. But, if $i \neq j$ and $a \neq b$, (2.4) gives $\overline{R}(E_i, E_j, F_a, F_b) = 0$. Then, from the Gauss equation (1.3), we get

$$g(\sigma(E_i, F_a), \sigma(E_j, F_b)) - g(\sigma(E_i, F_b), \sigma(E_j, F_a)) = 0$$

for any $i, j = 1, \ldots, 2p$, $i \neq j$, $a, b = 1, \ldots, q$, $a \neq b$, and this completes the proof. ■

Now, we shall study the normal CR-submanifolds of $H^{2n+s}(4-3s)$:

THEOREM 4.9. *Let $\mathcal{M}$ be a normal CR-submanifold of $H^{2n+s}(4-3s)$ such that the distribution $\mathcal{D} \oplus \mathcal{M}$ is integrable. Then*:

(a) *$\mathcal{M}$ is a CR-product $\mathcal{M}_1 \times \mathcal{M}_2$.*

(b) *$n \geq pq + p + q$.*

(c) *If $n = pq + p + q$, then $\mathcal{M}_1$ is an invariant totally geodesic submanifold immersed in $H^{2n+s}(4-3s)$.*

(d) *$\|\sigma\|^2 \geq 2q(2p + s)$.*

(e) *If $\|\sigma\|^2 = 2q(2p + s)$, then $\mathcal{M}_1$ is an S-space form of constant f-sectional curvature $4 - 3s$ and $\mathcal{M}_2$ has constant curvature 1.*

(f) *If $\mathcal{M}$ is a minimal submanifold, then*

$$\varrho \leq 4p(p+1) + 2p(q+s) + q(q-1),$$

where ϱ denotes the scalar curvature and equality holds if and only if $\|\sigma\|^2 = 2q(2p+s)$.

Proof. (a) follows directly from Proposition 4.4. Now, from Lemma 4.8, $\dim(\nu) = 2(n-p) - 2q \geq 2pq$. So, (b) holds.

Next, suppose that $n = pq + p + q$. If $X, Y, Z \in \mathcal{D}$ and $W \in \mathcal{D}^{\perp}$, from (2.4), $\overline{R}(X, Y, Z, W) = 0$ and, by using a similar proof to that of Lemma 4.8, $\overline{R}(X, Y, Z, W) = 0$. So, the Gauss equation gives

$$g(\sigma(X, W), \sigma(Y, Z)) - g(\sigma(X, Z), \sigma(Y, W)) = 0 \tag{4.12}$$

for any $X, Y, Z \in \mathcal{D}$ and any $W \in \mathcal{D}^{\perp}$. Since from Proposition 3.2 of [5], $\sigma(fX, Z) = f\sigma(X, Z)$, if we put $Y = fX$, we have, by using (3.8), $g(\sigma(fX, W), \sigma(X, Z)) = 0$. Now, if we put $Z = fY$, then $g(\sigma(X, Y), \sigma(X, W)) = 0$ for any $X, Y \in \mathcal{D}$ and $W \in \mathcal{D}^{\perp}$. Thus, by linearity, we get $g(\sigma(X, W), \sigma(Y, Z)) + g(\sigma(X, Z), \sigma(Y, W)) = 0$ for any $X, Y, Z \in \mathcal{D}$ and any $W \in \mathcal{D}^{\perp}$ and so, from (4.12),

$$g(\sigma(X, W), \sigma(Y, Z)) = 0, \qquad X, Y, Z \in \mathcal{D},\ W \in \mathcal{D}^{\perp}. \tag{4.13}$$

Since now $\dim(\nu) = 2pq$, (4.13) implies that $\sigma(X, Y) = 0$ for any $X, Y \in \mathcal{D}$. Consequently, (c) holds from Theorem 2.4(ii) of [5].

Assertions (d) and (e) follow from Theorem 4.2 of [5]. Finally, if $\mathcal{M}$ is a minimal normal CR-submanifold of $H^{2n+s}(4-3s)$, then a straightforward computation gives

$$\varrho = 4p(p+1) + 2s(p+q) + q(q-1) + 6pq - \|\sigma\|^2 .$$

Then, by using (d), the proof is complete. ∎

THEOREM 4.10. *Let $\mathcal{M}$ be a normal, $(\mathcal{D}, \mathcal{D}^{\perp})$-geodesic and $\mathcal{D}^{\perp}$-geodesic CR-submanifold of $H^{2n+s}(4-3s)$. Then*:

(a) $\|A_{fZ}X\| = 1$ *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^{\perp}$*,

(b) $\|\sigma\|^2 \geq 2q(p+s)$ *and equality holds if and only if $\sigma(\mathcal{D}, \mathcal{D}) \in f\mathcal{D}^{\perp}$.*

Proof. (a) follows inmediately from (4.10). Now, consider the above local fields of orthonormal frames for $\mathcal{D}$, $\mathcal{D}^{\perp}$ and ν. Since $\sigma(\mathcal{D}, \mathcal{D}^{\perp}) = \sigma(\mathcal{D}^{\perp}, \mathcal{D}^{\perp}) = 0$, a direct computation gives

$$\|\sigma\|^2 = 2qs + \sum_{i,j=1}^{2p} \|\sigma(E_i, E_j)\|^2 .$$

But

$$\|\sigma(E_i, E_j)\|^2 = \sum_{a=1}^{q} g(A_{fF_a}E_i, E_j)^2 + \sum_{l=1}^{r} \{g(A_{N_l}E_i, E_j)^2 + g(A_{fN_l}E_i, E_j)^2\} . \tag{4.14}$$

On the other hand, since $\sigma(\mathcal{D}, \mathcal{D}^{\perp}) = 0$, we see that $A_{fF_a}E_i$, $A_{N_l}E_i$, $A_{fN_l}E_i \in \mathcal{D}$ for any $i = 1, \ldots, 2p$, $a = 1, \ldots, q$ and $l = 1, \ldots, r$. So, from (a), we get

$$\sum_{i,j=1}^{2p} \Big[\sum_{a=1}^{q} g(A_{fF_a}E_i, E_j)^2 + \sum_{l=1}^{r} \{g(A_{N_l}E_i, E_j)^2 + g(A_{fN_l}E_i, E_j)^2\} \Big]$$

$$= \sum_{i=1}^{2p} \Big[\sum_{a=1}^{q} \|A_{fF_a} E_i\|^2 + \sum_{l=1}^{r} \{ \|A_{N_l} E_i\|^2 + \|A_{fN_l} E_i\|^2 \} \Big] \geq 2pq\,.$$

Consequently, $\|\sigma\|^2 \geq 2q(p+s)$ and, from (4.14), equality holds if and only if $\sigma(\mathcal{D},\mathcal{D}) \in f\mathcal{D}^\perp$. ■

Finally, from (3.6), (4.10) and (4.14), we can prove:

COROLLARY 4.11. *Let $\mathcal{M}$ be a normal, generic and $\mathcal{D}^\perp$-geodesic CR-submanifold of $H^{2n+s}(4-3s)$. Then*:

(a) $\|A_{fZ}X\| = 1$ *for any unit vector fields $X \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp$*,
(b) $\|\sigma\|^2 = 2q(p+s)$.

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