COLLOQUIUM MATHEMATICUM

VOL. LXV 1993 FASC. 2

BOUNDS FOR CHERN CLASSES OF SEMISTABLE VECTOR BUNDLES ON COMPLEX PROJECTIVE SPACES

BY

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This work concerns bounds for Chern classes of holomorphic semistable and stable vector bundles on $\mathbb{P}^n$. Non-negative polynomials in Chern classes are constructed for 4-vector bundles on $\mathbb{P}^4$ and a generalization of the presented method to r-bundles on $\mathbb{P}^n$ is given. At the end of this paper the construction of bundles from complete intersection is introduced to see how rough the estimates we obtain are.

We follow the terminology and notation used in [5].

There are no bounds for the first Chern class $c_1(\mathcal{E})$ of 1-bundles $\mathcal{E}$ on $\mathbb{P}^n$. In the case of 2-bundles the following Bogomolov–Gieseker–Schwarzenberger inequalities (see e.g. [5]) are satisfied:

$$\begin{aligned} &c_1^2 - 4c_2 \le 0 \quad \text{for semistable bundles,}\\ &c_1^2 - 4c_2 < 0 \quad \text{for stable bundles.}\end{aligned}$$

The polynomials above are invariant with respect to tensoring by $\mathcal{O}_{\mathbb{P}^n}(k)$.

Schneider has obtained in [6] the following results for 3-bundles on $\mathbb{P}^n$:

$$\begin{aligned} \text{if } c_1 = 0 \text{ then } &\ |c_3| \le c_2^2 + 5c_2 - 6\,,\\ \text{if } c_1 = -1 \text{ then } &\ |c_3 + 2| \le c_2^2 + 2c_2 - 2\,,\\ \text{if } c_1 = 1 \text{ then } &\ |c_3 - 2| \le c_2^2 + 2c_2 - 2\,, \quad \text{for stable bundles,}\end{aligned}$$

and

$$\begin{aligned} \text{if } c_1 = 0 \text{ then } &\ |c_3| \le c_2^2 + c_2\,,\\ \text{if } c_1 = -1 \text{ then } &\ |c_3| \le c_2^2\,,\\ \text{if } c_1 = -2 \text{ then } &\ |c_3| \le c_2^2 - c_2 - 2\,, \quad \text{for semistable bundles.}\end{aligned}$$

In this paper we obtain the following results for 4-bundles on $\mathbb{P}^4$:

$$\begin{aligned} &\text{if } c_1(\mathcal{E}) = 0 \text{ then } \quad c_4 - \tfrac{15}{2}c_3 + 3c_2^4 + 29c_2^3 + \tfrac{155}{2}c_2^2 + \tfrac{103}{2}c_2 \ge 0\,,\\ &\text{if } c_1(\mathcal{E}) = -1 \text{ then } \quad c_4 - \tfrac{13}{2}c_3 + 3c_2^4 + 35c_2^3 + \tfrac{371}{2}c_2^2 + 359c_2 + 156 \ge 0\,,\end{aligned}$$

if $c_1(\mathcal{E}) = -2$ then $c_4 - \frac{9}{2}c_3 + 3c_2^4 + 35c_2^3 + \frac{347}{2}c_2^2 + \frac{657}{2}c_2 - 6 \geq 0$,
if $c_1(\mathcal{E}) = -3$ then $c_4 - \frac{15}{2}c_3 + 3c_2^4 + 23c_2^3 + \frac{89}{2}c_2^2 + 47c_2 + 6 \geq 0$,
and for stable bundles we get the same polynomials minus 18.

The author wishes to express her thanks to Dr. Michał Szurek for suggesting the problem, introduction to the subject and his help with the construction of examples.

1. The case of stable and semistable 4-bundles on $\mathbb{P}^4$. We can normalize each bundle by twisting it with a suitable line bundle $\mathcal{O}_{\mathbb{P}^n}(k)$. This operation does not affect the stability or semistability and we can express the Chern classes of the twisted bundle by those of the original bundle. This allows us to consider only normalized bundles.

As the Euler–Poincaré characteristic $\chi(\mathcal{E})$ of the bundle $\mathcal{E}$ on $\mathbb{P}^4$ is a polynomial in Chern classes and

$$\chi(\mathcal{E}) \leq h^0(\mathcal{E}) + h^2(\mathcal{E}) + h^4(\mathcal{E})$$

we need to estimate the three components on the right hand side.

By Serre duality and semistability of $\mathcal{E}^*$ we immediately obtain $h^4(\mathcal{E}) = 0$. For $\mathcal{E}$ stable we have $h^0(\mathcal{E}) = 0$ and for $\mathcal{E}$ semistable we obtain $h^0(\mathcal{E}) \leq 3$ (except the case when $\mathcal{E}$ is trivial), according to

LEMMA 1.1 [6, Sect. 2, Hilfssatz]. *If $\mathcal{V}$ is a holomorphic semistable r-vector bundle on $\mathbb{P}^n$ and $c_1(\mathcal{V}) \leq 0$ then either $\mathcal{V} \cong \mathcal{O}^{\oplus r}$ or $h^0(\mathcal{V}) \leq r-1$.*

Now we start to estimate $h^2(\mathcal{E})$. We use

LEMMA 1.2 [6, Sect. 2, Satz 1]. *Let $\mathcal{V}$ be a holomorphic vector bundle on $\mathbb{P}^n$ and $H \subset \mathbb{P}^n$ a hyperplane. Then for $q \leq n-2$,*

$$h^q(\mathcal{V}) \leq \sum_{v \leq 0} h^q(\mathcal{V}_{|H}(v)).$$

From this lemma and Serre duality on $\mathbb{P}^4$ we obtain the estimate

$$h^2(\mathcal{E}) \leq \sum_{j \geq -4} h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j)).$$

We show that the sum on the right side is finite by finding k_0 which satisfies the condition (1) in the following

LEMMA 1.3 [1, Lemma 3.2]. *Let $Y \subset \mathbb{P}^n$ be a hyperplane and $\mathcal{V}$ a vector bundle on $\mathbb{P}^n$. Let k_0 be an integer such that*

$$h^1(\mathcal{V}_{|Y}(k)) = h^1(\Omega^1_Y \otimes \mathcal{V}_{|Y}(k+1)) = 0 \quad \text{for all } k \geq k_0. \tag{1}$$

Then for every $m \geq k_0 - 1$,

$$h^1(\mathcal{V}(m)) \geq h^1(\mathcal{V}(m+1)),$$

and equality holds if and only if

$$H^1(\mathcal{V}(m)) = 0\,.$$

We begin studying the values of $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k))$. We will discuss in detail the case of $c_1(\mathcal{E}) = 0$ only, because the remaining cases are similar.

THEOREM 1.1 (Spindler) [1, Theorem 2.7]. *Let $\mathcal{V}$ be a semistable vector bundle on $\mathbb{P}^n$ of generic splitting type $a_1 \geq \ldots \geq a_r$. Then it satisfies the Grauert–Mülich condition* (*GM*, *for short*), *i.e.*

$$a_i - a_{i+1} \leq 1 \quad \textit{for } i = 1, \ldots, r-1\,.$$

With the help of this theorem we will be able to determine the generic splitting type of the bundles considered.

LEMMA 1.4. *Let $\mathcal{E}$ be a holomorphic*, *normalized and semistable bundle of rank* 4 *on $\mathbb{P}^4$. Then*

$$h^0(\mathcal{E}_{|\mathbb{P}^2}(-1)) = 0 \quad \textit{and} \quad h^0(\mathcal{E}^*_{|\mathbb{P}^2}(-2)) = 0\,.$$

Proof. We only consider the case of $\mathcal{E}(-1)$; the other case is similar. We use

LEMMA 1.4.1 [2, Lemma 2.3]. *Let $\mathcal{V}$ be a normalized semistable n-vector bundle on $\mathbb{P}^n$. Then its restriction to a hyperplane $H \subset \mathbb{P}^n$ is a semistable bundle except the cases*

$$\mathcal{V} \cong \Omega^1_{\mathbb{P}^n}(-1), \qquad \mathcal{V} \cong T_{\mathbb{P}^n}(-2)\,.$$

If $\mathcal{E}_{|\mathbb{P}^3}$ is semistable and (a_1, a_2, a_3, a_4) is its generic splitting type ($\mathcal{E}_{|\mathbb{P}^3}$ satisfies the GM condition and $\sum_{i=1}^4 a_i = c_1$), then either only one of the a_i is zero or they are all negative. When $a_i = 0$ for some i we use

LEMMA 1.4.2 [6, Sect. 1, Satz 1]. *Let $\mathcal{V}$ be a holomorphic r-bundle on $\mathbb{P}^n$. For a line $L \subset \mathbb{P}^n$ we have*

$$\mathcal{V}_{|L} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \ldots \oplus \mathcal{O}(a_{r-s}) \oplus \mathcal{O}^{\oplus s}\,,$$

where $a_1 \leq a_2 \leq \ldots \leq a_{r-s} < 0$ and $h^0(\mathbb{P}^n, \mathcal{V}) \leq s - 1$. Then $h^0(\mathcal{V}_{|H}) \leq s - 1$, where $H \subset \mathbb{P}^n$ is a general hyperplane.

By taking in this lemma $n = 3$, $H = \mathbb{P}^2$ and from semistability of $\mathcal{E}_{|\mathbb{P}^3}$ we get

$$h^0(\mathcal{E}_{|\mathbb{P}^3}(-1)) \leq s - 1 = 0\,,$$

so $h^0(\mathcal{E}_{|\mathbb{P}^2}(-1)) = 0$.

If all a_i are negative we consider the exact sequence

$$0 \to \mathcal{E}_{|\mathbb{P}^2}(k-1) \to \mathcal{E}_{|\mathbb{P}^2}(k) \to \mathcal{E}_{|L}(k) \to 0$$

where L is a line in $\mathbb{P}^2$ and from the associated cohomology sequence we obtain

$$h^1(\mathcal{E}_{|\mathbb{P}^2}(k-1)) \geq h^1(\mathcal{E}_{|\mathbb{P}^2}(k)) \quad \text{for } k \leq -1$$

because $H^0(\mathcal{E}_{|L}(-1)) = 0$. Now since there exists k_0 such that $H^0(\mathcal{E}_{|\mathbb{P}^2}(k)) = 0$ for $k \leq k_0$ [5, Theorem B], we get $h^0(\mathcal{E}_{|\mathbb{P}^2}(-1)) = 0$.

If $\mathcal{E} \cong \Omega^1_{\mathbb{P}^4}(-1)$ or $\mathcal{E} \cong T_{\mathbb{P}^4}(-2)$ we use the formula

$$T_{\mathbb{P}^n|H} \cong T_H \oplus \mathcal{O}_H(-1)$$

for $H = \mathbb{P}^3$ and then for $H = \mathbb{P}^2$. From Bott's formula (see e.g. [5, Chapter I, §1.1]) and Serre duality we easily calculate $h^0(\mathcal{E}_{|\mathbb{P}^2}(-1)) = 0$ and $h^0(\mathcal{E}^*_{|\mathbb{P}^2}(-2)) = 0$. This completes the proof of Lemma 1.4.

By Lemma 1.4 we have $h^0(\mathcal{E}^*_{|\mathbb{P}^2}(-1)) = 0$ and $h^0(\mathcal{E}_{|\mathbb{P}^2}(-1)) = 0$ so by Serre duality also $h^2(\mathcal{E}^*_{|\mathbb{P}^2}(-1)) = 0$ and we conclude

$$-\chi(\mathcal{E}^*_{|\mathbb{P}^2}) = h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-1)).$$

Using $c_1(\mathcal{V}(k)) = c_1(\mathcal{V}) + 4k$, $c_2(\mathcal{V}(k)) = 6k^2 + 3kc_1(\mathcal{V}) + c_2(\mathcal{V})$ [5, §1.2] and the Riemann–Roch formula on $\mathbb{P}^2$, i.e.

$$\chi(\mathcal{V}) = \tfrac{1}{2}c_1^2(\mathcal{V}) - c_2(\mathcal{V}) + \tfrac{3}{2}c_1(\mathcal{V}) + r, \qquad r = \operatorname{rank}\mathcal{V},$$

we can easily calculate

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-1)) = c_2(\mathcal{E}^*_{|\mathbb{P}^2}).$$

Similarly we obtain $h^1(\mathcal{E}_{|\mathbb{P}^2}(-2)) = c_2(\mathcal{E}_{|\mathbb{P}^2})$ and from Serre duality

$$h^1(\mathcal{E}_{|\mathbb{P}^2}(-2)) = h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-1))$$

so $c_2(\mathcal{E}^*_{|\mathbb{P}^2}) = c_2(\mathcal{E}_{|\mathbb{P}^2})$ (for short, we will write $c_2(\mathcal{E}_{|\mathbb{P}^2}) = c_2$).

From the exact sequence

$$0 \to \mathcal{E}^*_{|\mathbb{P}^2}(-1) \to \mathcal{E}^*_{|\mathbb{P}^2} \to \mathcal{E}^*_{|L} \to 0$$

where L is a line in $\mathbb{P}^2$ and from the cohomology sequence we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-1)) \geq h^1(\mathcal{E}^*_{|\mathbb{P}^2})$$

because $H^1(\mathcal{E}^*_{|L}) = 0$ (since $\mathcal{E}^*_{|L} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{E}^*_{|L} = \mathcal{O}(-1) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$). We now use

Lemma 1.5 (Le Potier) [1, Lemma 2.17]. *Let $\mathcal{V}$ be a vector bundle on $\mathbb{P}^2$ and $a_1 \geq \ldots \geq a_r$ its generic splitting type. Then*

$$h^1(\mathcal{V}(m)) \geq h^1(\mathcal{V}(m+1))$$

for $m \geq -a_r - 1$, and we have equality if and only if $H^1(\mathbb{P}^2, \mathcal{V}(m)) = 0$.

With the help of the lemma above we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k+1)) \leq h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k)) \quad \text{for } k \geq 0.$$

The same results can be obtained for $\mathcal{E}$ $(c_1(\mathcal{E}_{|\mathbb{P}^2}) = c_1(\mathcal{E}^*_{|\mathbb{P}^2}))$ so we also have

$$c_2 = h^1(\mathcal{E}_{|\mathbb{P}^2}(-1)) \geq h^1(\mathcal{E}_{|\mathbb{P}^2}),$$
$$h^1(\mathcal{E}_{|\mathbb{P}^2}(k)) \geq h^1(\mathcal{E}_{|\mathbb{P}^2}(k+1)) \quad \text{for } k \geq 0.$$

By Serre duality we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-3)) \leq h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-2)) = c_2,$$
$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k-1)) \leq h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k)) \quad \text{for } k \leq -3.$$

Finally, we obtain

$$(*) \qquad h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k)) = \begin{cases} 0 & \text{if } k \leq -c_2-3, \\ k+c_2+3 & \text{if } -c_2-3 \leq k \leq -3, \\ c_2 & \text{if } -3 \leq k \leq 0, \\ -k+c_2 & \text{if } 0 \leq k \leq c_2, \\ 0 & \text{if } k \geq c_2. \end{cases}$$

Hence we can estimate

$$\sum_{j=-\infty}^{\infty} h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j)) \leq \tfrac{1}{2}(2c_2+6)c_2 = c_2^2 + 3c_2.$$

To apply Lemma 1.3 we start to seek j_0 such that for $j \geq j_0$,

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(j)) = 0.$$

LEMMA 1.6 [4, Corollary 3.1.1]. *Let $\mathcal{V}$ be a semistable bundle on $\mathbb{P}^n$ with* $\operatorname{rank}\mathcal{V} \leq 2n-2$ *and* $c_1(\mathcal{V}) = d\cdot\operatorname{rank}\mathcal{V}$, $d \in \mathbb{Z}$. *Then for a general hyperplane* $H \subset \mathbb{P}^n$, $\mathcal{V}_{|H}$ *is a semistable bundle.*

Putting in this lemma $n = 4$, $H = \mathbb{P}^3$ and then $n = 3$, $H = \mathbb{P}^2$ we conclude that $\mathcal{E}^*_{|\mathbb{P}^2}$ is a semistable bundle. The tensor product of semistable bundles is semistable so $\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}$ is semistable.

We show that $h^0(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)) = 0$.

Suppose that $0 \neq s \in H^0(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1))$. Then we have the imbedding

$$\mathcal{O}_{\mathbb{P}^2} \hookrightarrow \mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1).$$

But $\mu(\mathcal{O}_{\mathbb{P}^2}) = 0$ and $\mu(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)) = -4/8 = -1/2$ (recall that $\mu(\mathcal{V}) = c_1(\mathcal{V})/\operatorname{rank}\mathcal{V}$) because $c_1(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)) = -4$, which we calculate e.g. from the generic splitting type of $\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)$ $(\Omega^1_{\mathbb{P}^2}{}_{|L} = \mathcal{O}(-1) \oplus \mathcal{O}(-2))$. We thus get a contradiction with semistability of $\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)$.

By Serre duality and semistability of $\mathcal{E}_{|\mathbb{P}^2} \otimes T_{\mathbb{P}^2}$ we also have $h^2(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)) = 0$ so

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)) = -\chi(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(1)).$$

If A is a bundle of rank 2 and B of rank 4 then one has

$$c_1(A\otimes B)=4c_1(A)+2c_1(B)\,,$$
$$c_2(A\otimes B)=6c_1^2(A)+4c_2(A)+c_1^2(B)+2c_2(B)+7c_1(A)c_1(B)\,,$$

so

$$c_1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(1))=-4\quad\text{and}\quad c_2(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(1))=2c_2+10\,,$$

and finally we get $-\chi(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(1))=2c_2=h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(1))$. Tensoring the exact sequence

$$0\to\mathcal{E}^*_{|\mathbb{P}^2}(-1)\to\mathcal{E}^*_{|\mathbb{P}^2}\to\mathcal{E}^*_{|L}\to 0$$

by $\Omega^1_{\mathbb{P}^2}(2)$ we deduce

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(1))\ge h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(2))$$

from the associated cohomology sequence, because $H^1(\mathcal{E}^*_{|L}\otimes\Omega^1_{\mathbb{P}^2}(2)_{|L})=0$. We also have either

$$\mathcal{E}^*_{|L}\otimes\Omega^1_{\mathbb{P}^2|L}\cong\mathcal{O}(-2)^{\oplus 4}\oplus\mathcal{O}(-1)^{\oplus 4}$$

or

$$\mathcal{E}^*_{|L}\otimes\Omega^1_{\mathbb{P}^2|L}\cong\mathcal{O}(-3)\oplus\mathcal{O}(-2)^{\oplus 3}\oplus\mathcal{O}(-1)^{\oplus 3}\oplus\mathcal{O}\,,$$

so by Le Potier's Lemma 1.5 we obtain

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(j))\ge h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(j+1))$$

for $j\ge 2$, and equality occurs if and only if $H^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(j))=0$. Finally, we conclude that for $j\ge j_0=2c_2+2$,

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}(j))=0\,.$$

From the formula $(*)$ we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k))=0\qquad\text{for } k\ge k_0=c_2\,.$$

Applying Lemma 1.3 to these results we obtain

$$h^1(\mathcal{E}^*_{|\mathbb{P}^3}(l))\ge h^1(\mathcal{E}^*_{|\mathbb{P}^3}(l+1))\qquad\text{for } l\ge 2c_2\,,$$

and equality holds if and only if $H^1(\mathcal{E}^*_{|\mathbb{P}^3}(l))=0$. Using Lemma 1.2 we can estimate

$$h^1(\mathcal{E}^*_{|\mathbb{P}^3}(2c_2))\le\sum_{j\le 0}h^1(\mathcal{E}^*_{|\mathbb{P}^2}(2c_2+j))=\sum_{j=-\infty}^{\infty}h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))\le c_2^2+3c_2$$

and with the aid of the inequality above we conclude

$$(**)\qquad h^1(\mathcal{E}^*_{|\mathbb{P}^3}(l))=0\qquad\text{for } l\ge l_0=c_2^2+5c_2\,,$$

$$\binom{*}{*}\qquad \sum_{j=2c_2}^{c_2^2+5c_2-1}h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j))\le\tfrac{1}{2}(c_2^2+3c_2)(c_2^2+3c_2+1)\,.$$

By Lemma 1.2 we have an estimate

$$h^2(\mathcal{E}) \leq \sum_{j\geq -4} h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j))$$

and applying $(**)$ we obtain

$$h^2(\mathcal{E}) \leq \sum_{j=-4}^{c_2^2+5c_2-1} h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j))\,.$$

By the same Lemma 1.2 and from $(*)$ we get

$$\begin{array}{ll}
j=-4: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}(-4)) \leq \sum_{j\leq 0} h^1(\mathcal{E}^*_{|\mathbb{P}^2}(-4+j)) \leq \frac{1}{2}c_2(c_2-1)\,,\\
j=-3: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}(-3)) \leq \frac{1}{2}c_2(c_2+1)\,,\\
j=-2: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}(-2)) \leq \frac{1}{2}c_2(c_2+1)+c_2 = \frac{1}{2}c_2(c_2+3)\,,\\
j=-1: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}(-1)) \leq \frac{1}{2}c_2(c_2+1)+2c_2 = \frac{1}{2}c_2(c_2+5)\,,\\
j=0: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}) \leq \frac{1}{2}c_2(c_2+5)+c_2\,,\\
\vdots & \vdots\\
j=c_2-1: & h^1(\mathcal{E}^*_{|\mathbb{P}^3}(c_2-1)) \leq \frac{1}{2}c_2(c_2+5)+c_2+(c_2-1)+\ldots+1\,.
\end{array}$$

If $j \in \{c_2, c_2+1, \ldots, 2c_2-1\}$ then

$$h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j)) \leq \sum_{k=-\infty}^{\infty} h^1(\mathcal{E}^*_{|\mathbb{P}^2}(k)) \leq c_2^2+3c_2$$

so

$$\sum_{j=c_2}^{2c_2-1} h^1(\mathcal{E}^*_{|\mathbb{P}^3}(j)) \leq (c_2^2+3c_2)c_2.$$

Finally, using the results above and $\binom{*}{*}$ we obtain

$$\begin{aligned}
h^2(\mathcal{E}) &\leq \sum_{j\geq -4} h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))\\
&\leq \tfrac{1}{2}c_2(c_2-1)+\tfrac{1}{2}c_2(c_2+1)+\tfrac{1}{2}c_2(c_2+3)\\
&\quad +\tfrac{1}{2}c_2(c_2+5)(c_2+1)+c_2^2+(c_2-1)^2+\ldots+1+c_2(c_2^2+3c_2)\\
&\quad +\tfrac{1}{2}(c_2^2+3c_2)(c_2^2+3c_2+1)\\
&= \tfrac{1}{2}c_2^4+\tfrac{29}{6}c_2^3+13c_2^2+\tfrac{17}{3}c_2\,.
\end{aligned}$$

We have

$$\chi(\mathcal{E}) = \tfrac{1}{12}c_2^2 - \tfrac{1}{6}c_4 + \tfrac{5}{4}c_3 - \tfrac{35}{12}c_2 + 4$$

for a 4-vector bundle $\mathcal{E}$ on $\mathbb{P}^4$ with $c_1 = 0$, because

$$\begin{aligned}\chi(\mathcal{V}) = {} & \tfrac{1}{24}(c_1^4 + 4c_1c_3 - 4c_1^2c_2 + 2c_2^2 - 4c_4) \\ & + \tfrac{5}{12}(c_1^3 - 3c_1c_2 + 3c_3) + \tfrac{35}{24}(c_1^2 - 2c_2) + \tfrac{25}{12}c_1 + r\end{aligned}$$

for every r-vector bundle $\mathcal{V}$ on $\mathbb{P}^4$.

At the beginning of this chapter we got the two inequalities:

$$\begin{aligned}&\chi(\mathcal{E}) \le h^2(\mathcal{E}) + 3 \quad && \text{if } \mathcal{E} \text{ is semistable and non-trivial,}\\ &\chi(\mathcal{E}) \le h^2(\mathcal{E}) && \text{if } \mathcal{E} \text{ is stable.}\end{aligned}$$

Using our last result we obtain the following non-negative polynomials:

$$c_4 - \tfrac{15}{2}c_3 + 3c_2^4 + 29c_2^3 + \tfrac{155}{2}c_2^2 + \tfrac{103}{2}c_2 \ge 0$$

for semistable non-trivial bundles

and

$$c_4 - \tfrac{15}{2}c_3 + 3c_2^4 + 29c_2^3 + \tfrac{155}{2}c_2^2 + \tfrac{103}{2}c_2 - 18 \ge 0$$

for stable bundles.

In the cases of $c_1(\mathcal{E}) = -1, -2, -3$ we apply almost the same procedure.

There is a difference when we want to estimate $h^0(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(k))$ and $h^2(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(k))$ because we cannot use Lemma 1.6. We just take Lemma 1.2 and get an estimate

$$h^0(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(k)) \le \sum_{j \le 0} h^0(\mathcal{E}^*_{|\mathbb{P}^1} \otimes \Omega^1_{\mathbb{P}^1}(k+j)).$$

There exists j_0 such that $h^0(\mathcal{E}^*_{|\mathbb{P}^1} \otimes \Omega^1_{\mathbb{P}^1}(k+j)) = 0$ for $j \le j_0$ so the sum on the right side is finite and the non-trivial values are easily calculated by taking the generic splitting type of $\mathcal{E}^*_{|\mathbb{P}^1} \otimes \Omega^1_{\mathbb{P}^1}(k+j)$ and from Bott's formula [5, Chapter I, §1.1]. Using Serre duality in a similar way we can estimate $h^2(\mathcal{E}^*_{|\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2}(k))$. Finally, we obtain the following non-negative polynomials for semistable, non-trivial 4-bundles on $\mathbb{P}^4$:

if $c_1(\mathcal{E}) = -1$ then

$$c_4 - \tfrac{13}{2}c_3 + 3c_2^4 + 35c_2^3 + \tfrac{371}{2}c_2^2 + 359c_2 + 156 \ge 0,$$

if $c_1(\mathcal{E}) = -2$ then

$$c_4 - \tfrac{9}{2}c_3 + 3c_2^4 + 35c_2^3 + \tfrac{347}{2}c_2^2 + \tfrac{657}{2}c_2 - 6 \ge 0,$$

if $c_1(\mathcal{E}) = -3$ then

$$c_4 - \tfrac{15}{2}c_3 + 3c_2^4 + 23c_2^3 + \tfrac{89}{2}c_2^2 + 47c_2 + 6 \ge 0,$$

and for stable bundles we get the same polynomials minus $3 \times 6 = 18$.

2. Generalization to semistable r-bundles on $\mathbb{P}^n$. In this chapter we will need a more general version of Lemma 1.2:

THEOREM 2.1 [1, Theorem 1.6a]. *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^n$ and $Y \subset \mathbb{P}^n$ be a complete intersection. Then for $q \le \dim Y$,*

$$h^q(\mathcal{E}) \le \sum_{v\ge 0} h^q(\mathcal{E}_{|Y} \otimes S^v N_Y^*)$$

where $S^v N_Y^*$ is the v-th symmetric power of the conormal bundle $N^*_{Y/\mathbb{P}^n}$.

Since for every bundle $\mathcal{E}$ on $\mathbb{P}^n$,

$$\chi(\mathcal{E}) \le h^0(\mathcal{E}) + h^2(\mathcal{E}) + \ldots + h^{2k}(\mathcal{E}), \qquad k = \left[\frac{n}{2}\right],$$

and the Euler–Poincaré characteristic $\chi(\mathcal{E})$ is a polynomial in Chern classes, we have to estimate the components on the right side.

By substituting $Y = \mathbb{P}^{2l+1}$ in the theorem above we get

$$h^{2l}(\mathcal{E}) \le \sum_{v\ge 0} h^{2l}(\mathcal{E}_{|\mathbb{P}^{2l+1}} \otimes S^v N^*_{\mathbb{P}^{2l+1}}),$$

but

$$N^*_{\mathbb{P}^{2l+1}/\mathbb{P}^n} = (n-2l-1)\mathcal{O}_{\mathbb{P}^{2l+1}}(-1)$$

so

$$S^v N^*_{\mathbb{P}^{2l+1}} = \sum_{v\ge 0} \binom{n-2l-2+v}{v} \mathcal{O}_{\mathbb{P}^{2l+1}}(-v)$$

and we obtain

$$h^{2l}(\mathcal{E}) \le \sum_{v\ge 0} \binom{n-2l-2+v}{v} h^{2l}(\mathcal{E}_{|\mathbb{P}^{2l+1}}(-v)).$$

Then immediately by Serre duality we get

$$(*) \qquad h^{2l}(\mathcal{E}) \le \sum_{v\ge 0} \binom{n-2l-2+v}{v} h^{1}(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(-2l-2+v)).$$

Now we need to show that the sum $(*)$ above is finite and estimate the values of $h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(v))$ by polynomials in the second Chern class of $\mathcal{E}$.

We first study $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))$. As we have

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j)) = -\chi(\mathcal{E}^*_{|\mathbb{P}^2}(j)) + h^0(\mathcal{E}^*_{|\mathbb{P}^2}(j)) + h^2(\mathcal{E}^*_{|\mathbb{P}^2}(j))$$

and $\chi(\mathcal{E}^*_{|\mathbb{P}^2}(j))$ is a polynomial in Chern classes, we shall estimate $h^0(\mathcal{E}^*_{|\mathbb{P}^2}(j))$ and $h^2(\mathcal{E}^*_{|\mathbb{P}^2}(j))$.

From Lemma 1.2 we get

$$h^0(\mathcal{E}^*_{|\mathbb{P}^2}(j)) \le \sum_{v\le 0} h^0(\mathcal{E}^*_{|\mathbb{P}^1}(j+v))$$

where $h^0(\mathcal{E}^*_{|\mathbb{P}^1}(j+v))=0$ for v small enough. We have finitely many possibilities for the generic splitting type $a_1^*\le\ldots\le a_r^*$ of $\mathcal{E}^*(k)$. Therefore using $h^0(\mathcal{E}^*_{|\mathbb{P}^1}(k))=\sum_{i=1}^r h^0(\mathcal{O}(a_i^*))$ where

$$h^0(\mathcal{O}(a_i^*))=\begin{cases}-a_{i+1}^* & \text{if } a_i^*\ge 0,\\ 0 & \text{if } a_i^*<0,\end{cases}$$

we can calculate the values of $h^0(\mathcal{E}^*_{|\mathbb{P}^1}(k))$.

Finally, taking the maximum of those values we are able to estimate $h^0(\mathcal{E}^*_{|\mathbb{P}^2}(j))$ and, by Serre duality, $h^2(\mathcal{E}^*_{|\mathbb{P}^2}(j))$, so we obtain an estimate for $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))$.

Now from Le Potier's Lemma we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(m))\ge h^1(\mathcal{E}^*_{|\mathbb{P}^2}(m+1))\,, \tag{1}$$

$$h^1(\mathcal{E}_{|\mathbb{P}^2}(n))\ge h^1(\mathcal{E}_{|\mathbb{P}^2}(n+1)) \tag{2}$$

for $m\ge -a_r^*-1$ and $n\ge -a_r-1$, and equalities hold if and only if $H^1(\mathcal{E}^*_{|\mathbb{P}^2}(m))=0$ and $H^1(\mathcal{E}_{|\mathbb{P}^2}(n))=0$.

We can easily calculate the minimal a_r for normalized r-bundles with $c_1(\mathcal{E})$ fixed: it is

$$a_r=\left[\frac{c_1(\mathcal{E})}{r}-\frac{r-1}{2}\right].$$

By Serre duality $h^1(\mathcal{E}_{\mathbb{P}^2}(-a_r-1))=h^1(\mathcal{E}_{\mathbb{P}^2}(a_r^*-2))$ so (2) implies

$$h^1(\mathcal{E}^*_{|\mathbb{P}^2}(n))\le h^1(\mathcal{E}^*_{|\mathbb{P}^2}(n+1)) \tag{2'}$$

for $n\le a_r^*-2$.

Finally, from (1) and (2′) we conclude that there are a finite number of j such that $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))\ne 0$ so we can get an estimate for $\sum_{j=-\infty}^{\infty}h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))$ because we have one for $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))$.

Using this result we estimate $h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(v))$. Taking, in Theorem 2.1, $n=2l+1$, $Y=\mathbb{P}^2$, $\mathcal{E}=\mathcal{E}^*_{|\mathbb{P}^{2l+1}}$, $q=1$ we obtain

$$h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(v))\le\sum_{j\ge 0}h^1(\mathcal{E}^*_{|\mathbb{P}^2}\otimes S^jN^*_{\mathbb{P}^2}(v))\,;$$

but

$$N^*_{\mathbb{P}^2/\mathbb{P}^n}=(2l+1-2)\mathcal{O}_{\mathbb{P}^2}(-1)=(2l-1)\mathcal{O}_{\mathbb{P}^2}(-1)$$

so

$$h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(v))\le\sum_{j\ge 0}\binom{2l-2+j}{j}h^1(\mathcal{E}^*_{|\mathbb{P}^2}(v-j))\,.$$

Now we will show that the sum $(*)$ is finite.

Applying Lemma 1.3 we get

$$h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(m))\ge h^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(m+1))$$

for $m \geq j_0 - 1$ and equality holds if and only if $H^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(m)) = 0$ where for $j \geq j_0$,

$$h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}}(j)) = h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}} \otimes \Omega^1_{\mathbb{P}^{2l}}(j+1)) = 0\,.$$

To find j_0 we will be looking for l_0 and l'_0 which satisfy

$$\begin{aligned} &h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}}(l)) = 0 && \text{if } \ l \geq l_0\,,\\ &h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}} \otimes \Omega^1_{\mathbb{P}^{2l}}(l')) = 0 && \text{if } l' \geq l'_0\,. \end{aligned}$$

Then j_0 will be equal to $\max(l_0, l'_0 - 1)$. Now once again from Lemma 1.3 we obtain

$$(**) \qquad h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}}(s)) \geq h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}}(s+1))$$

for $s \geq s_0$, and equality holds if and only if $H^1(\mathcal{E}^*_{|\mathbb{P}^{2l+1}}(s)) = 0$; moreover, s_0 satisfies $h^1(\mathcal{E}^*_{|\mathbb{P}^{2l-1}}(t)) = h^1(\mathcal{E}^*_{|\mathbb{P}^{2l-1}} \otimes \Omega^1_{\mathbb{P}^{2l-1}}(t+1)) = 0$ for $t \geq s_0 - 1$.

By Theorem 2.1 we get the estimate

$$h^1(\mathcal{E}^*_{|\mathbb{P}^{2l}}(s_0 - 1)) \leq \sum_{v \geq 0} \binom{2l-3+v}{v} h^1(\mathcal{E}^*_{|\mathbb{P}^2}(s_0 - 1 - v))$$

and from our previous consideration of $h^1(\mathcal{E}^*_{|\mathbb{P}^2}(j))$ we can represent this estimate as a polynomial $w(c_2)$ in the second Chern class. Using $(**)$ we conclude that $l_0 = w(c_2) + s_0 - 1$. Similarly we can find l'_0. In this way the task of finding j_0 is replaced by the problem of finding s_0, s'_0 which satisfy the conditions of Lemma 1.3. By analogy we can replace the search for s_0, s'_0 by looking for four other numbers which we determine by substituting, in Lemma 1.3, $n = 2l-1$, $Y = \mathbb{P}^{2l-1}$ and taking for $\mathcal{V}$ a suitable restriction.

Further we proceed by iteration until $Y = \mathbb{P}^1$ in Lemma 1.3 and finally we obtain explicitly 2^{2l} numbers which enable us to calculate j_0.

3. Construction of examples. In this last section, in order to see how rough the estimates we obtained are, we present some theorems which are helpful in constructing semistable vector bundles on $\mathbb{P}^n$ from complete intersection. Finally, we construct an example of a semistable 4-bundle on $\mathbb{P}^4$ and calculate the value of its non-negative polynomial found in Section 1.

THEOREM 3.1 [5, Chapter I, §5]. *Let Y be a locally complete intersection of codimension 2 in $\mathbb{P}^n$ $(n \geq 3)$ with sheaf of ideals $\mathcal{I}_Y \subset \mathcal{O}_{\mathbb{P}^n}$ and with $[\det N_{Y/\mathbb{P}^n}](-k)$ (the determinant of the normal bundle) generated by $n-1$ global sections. Then there exists an exact sequence*

$$0 \to \mathcal{O}^{r-1}_{\mathbb{P}^n} \to \mathcal{E} \to \mathcal{I}_Y(k) \to 0$$

where $\mathcal{E}$ is a bundle of rank r.

THEOREM 3.2 [3]. *Let $\mathcal{E}$ be a bundle of rank r on $\mathbb{P}^n$.*

(a) *$\mathcal{E}$ is semistable if and only if $H^0(\Lambda^q\mathcal{E}\otimes\mathcal{O}(-i))=0$ for each $q<r$ and $i>\mu q$ (where $\mu(\mathcal{E})=c_1(\mathcal{E})/\operatorname{rank}\mathcal{E}$).*

(b) *If $H^0(\Lambda^q\mathcal{E}\otimes\mathcal{O}(-i))=0$ for $q<r$ and $i\geq\mu q$ then $\mathcal{E}$ is stable.*

THEOREM 3.3. *Let Y be a complete intersection of two hyperplanes in $\mathbb{P}^n$ ($n\geq 4$) given by two equations of degree d_1, d_2 respectively and $d_1,d_2>0$. Then for $k\leq d_1+d_2$ the bundle $[\det N_{Y/\mathbb{P}^n}](-k)$ is generated by $n-1$ global sections.*

Proof. We have an isomorphism

$$[\det N_{Y/\mathbb{P}^n}](-k)\cong\mathcal{O}_Y(d_1+d_2-k)$$

so

$$\begin{aligned}h^0(\det N_{Y/\mathbb{P}^n}(-k))&=h^0(\mathcal{O}_Y(d_1+d_2-k))\\&=\binom{d_1+d_2+n-2}{n-2}\geq n-1=\dim Y+1\,.\end{aligned}$$

We conclude that the sections of $H^0(N_{Y/\mathbb{P}^n}(-k))$ are forms of degree d_1+d_2-k. For each $y\in Y$ we can find a form which is non-trivial at this point.

THEOREM 3.4. *Let Y be the intersection of two hyperplanes in $\mathbb{P}^n$ given by equations of degree $d_1,d_2>0$ respectively and $k\leq d_1+d_2-1$. Then the Chern classes of the bundle $\mathcal{E}$ in the exact sequence*

$$0\to\mathcal{O}^3_{\mathbb{P}^n}\to\mathcal{E}\to\mathcal{T}_Y\otimes\mathcal{O}_Y(k)\to 0$$

are

$$\begin{aligned}c_1(\mathcal{E})&=k\,, & c_3(\mathcal{E})&=d_1d_2(d_1+d_2-k)\,,\\ c_2(\mathcal{E})&=d_1d_2\,, & c_4(\mathcal{E})&=d_1d_2(d_1+d_2-k)^2\,.\end{aligned}$$

Proof. $c(\mathcal{E})=c(\mathcal{T}_Y(k))\cdot c(\mathcal{O}^3_{\mathbb{P}^n})=c(\mathcal{T}_Y(k))$ because $c(\mathcal{O}_{\mathbb{P}^n})=1$.

Tensoring the Koszul complex by $\mathcal{O}_{\mathbb{P}^n}(k)$ we obtain

$$0\to\mathcal{O}_{\mathbb{P}^n}(-d_1-d_2+k)\to\mathcal{O}_{\mathbb{P}^n}(-d_1+k)\oplus\mathcal{O}_{\mathbb{P}^n}(-d_2+k)\to\mathcal{T}_Y(k)\to 0\,.$$

From this sequence we get

$$c(\mathcal{T}_Y(k))=\frac{c(\mathcal{O}_{\mathbb{P}^n}(-d_1+k)\oplus\mathcal{O}_{\mathbb{P}^n}(-d_2+k))}{c(\mathcal{O}_{\mathbb{P}^n}(-d_1-d_2+k))}\,;$$

but $c(\mathcal{O}_{\mathbb{P}^n}(j))=1+jh$, $h\in H^2(\mathbb{P}^n,\mathbb{Z})$, so

$$c(\mathcal{T}_Y(k))=\frac{(1-(d_1-k)h)(1-(d_2-k)h)}{1-(d_1+d_2-k)h}$$

and by quick calculation we obtain the assertion of the theorem.

Now we construct an example of a semistable 4-vector bundle $\mathcal{E}$ on $\mathbb{P}^n$ with $c_1(\mathcal{E})=0$.

Let Y be the intersection of two hyperplanes in $\mathbb{P}^4$ given by two equations of degree d_1, d_2 respectively and $d_1, d_2 > 0$. By Theorem 3.3, the bundle $\det N_{Y/\mathbb{P}^4}$ is generated by three global sections.

We notice that Y satisfies the assumption of Theorem 3.1 (since Y is a global complete intersection it is also a local one) so we get the exact sequence

$$0 \to \mathcal{O}^3_{\mathbb{P}^4} \to \mathcal{E} \to \mathcal{T}_Y \to 0$$

where $\mathcal{E}$ is a bundle of rank 4.

To show that $\mathcal{E}$ is semistable we consider the diagram

$$\begin{array}{ccccccccc} & & & & & 0 & & & \\ & & & & & \downarrow & & & \\ 0 & \to & \mathcal{O}^3_{\mathbb{P}^4} & \to & \mathcal{E} & \to & \mathcal{T}_Y & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & \mathcal{O}_{\mathbb{P}^4} & \to & \mathcal{O}_Y & \to & 0 \end{array}$$

Let us restrict it to a line L which does not meet Y. Then $\mathcal{O}_{Y|L} = 0$ and we get the exact sequence

$$0 \to \mathcal{O}^3_{|L} \to \mathcal{E}_{|L} \to \mathcal{O}_{|L} \to 0\,.$$

Since $\operatorname{Ext}(\mathcal{O}_L, \mathcal{O}^3_L) = H^1(\mathcal{O}^3_L) = 0$, we get $\mathcal{E}_{|L} = \mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L \oplus \mathcal{O}_L$. By Theorem 3.4 we have $c_1(\mathcal{E}) = 0$ so $\mu(\mathcal{E}) = 0$ and by Theorem 3.2 we conclude that $\mathcal{E}$ is semistable if and only if $H^0(\Lambda^q \otimes \mathcal{O}(-2)) = 0$ for each $q < r$ and $i > \mu q = 0$, so it suffices to show that the bundle $[\Lambda^q\mathcal{E}](-1)$ has only trivial sections for $q = 1, 2, 3$.

Suppose that one of the bundles above has a non-trivial section. Then its restriction to an arbitrary line L is a section of the bundle

$$[\Lambda^q\mathcal{E}]_{|L} = \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) \oplus \ldots \oplus \mathcal{O}_L(-1)\,.$$

We can choose the line L on which there exist some points where the section S is non-trivial. Then $S_{|L} \neq 0$ and $S_{|L} \in H^0([\Lambda^q\mathcal{E}]_{|L}(-1))$, but

$$H^0([\Lambda^q\mathcal{E}]_{|L}(-1)) = H^0(\mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) \oplus \ldots \oplus \mathcal{O}_L(-1)) = 0\,,$$

so we obtain a contradiction.

Finally, we calculate the value of the non-negative polynomial from Section 1 for the bundle we have just constructed.

By Theorem 3.4 we get

$$c_2(\mathcal{E}) = d_1d_2, \quad c_3(\mathcal{E}) = d_1d_2(d_1+d_2)\,,$$
$$c_4(\mathcal{E}) = d_1d_2(d_1+d_2)^2\,,$$

and substituting $x = d_1 + d_2$ and $y = d_1d_2$ we obtain the value

$$xy\left(x - \frac{15}{2}\right) + \left(3y^4 + 29y^3 + \frac{155}{2}y^2 + \frac{103}{2}y\right).$$

When $d_1, d_2 \in \mathbb{N}$ it is easy to see that the polynomial above has a minimal value for $d_1 = 1$, $d_2 = 1$ and then the value is 150.

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Reçu par la Rédaction le 28.5.1992;
en version modifiée le 16.2.1993