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SOME MODELS OF GEOMETRIES AND A FUNCTIONAL EQUATION

BY

R. C. POWERS, T. RIEDEL AND P. K. SAHOO (LOUISVILLE, KENTUCKY)

1. Introduction. Let $\mathbb{R}$ denote the set of real numbers. Let $\mathcal{M}_p$ be the family of all straight lines in $\mathbb{R}^2$ which are parallel to the y axis and of all curves of the form $y = p(x+\alpha)+\beta$, where p is a fixed function and α, β run over $\mathbb{R}$. In [3] and [4], Faber, Grünbaum, Kuczma and Mycielski proved that if there exists a continuous bijection of $\mathbb{R}^2$ onto $\mathbb{R}^2$ which induces a map of the family of all straight lines onto $\mathcal{M}_p$, then p must be a polynomial of degree 2. Let $\mathcal{N}_p$ consist of all planes parallel to the z-axis and all surfaces of the form

$$z = p(x+\alpha,\, y+\beta)+\gamma\,,$$

where $p : \mathbb{R}^2 \to \mathbb{R}$, and α, β, and γ are real constants. In [3], the following problem was raised: Characterize those functions p for which $\mathcal{N}_p$ is continuously isomorphic to the family of all planes in $\mathbb{R}^3$. In this paper, we solve the problem posed in [3] through a functional equation and show that p is a polynomial of degree 2.

The paper is organized as follows. In Section 2, we determine the general solution of a functional equation which is instrumental in proving the main result. In Section 3, we provide the answer to the question of Faber, Kuczma and Mycielski.

2. A functional equation. For $x, y \in \mathbb{R}^2$, let

$$\langle x, y\rangle = \sum_{i=1}^{2} x_i y_i \tag{2.1}$$

denote the inner product between x and y, where x_i and y_i are the ith components of x and y, respectively. Let x^T denote the transpose of x in $\mathbb{R}^2$. Let $e_1 := (1,0)$ and $e_2 := (0,1)$ be the basis elements of $\mathbb{R}^2$. For $f : \mathbb{R}^2 \to \mathbb{R}$, let

$$f[x,y] := f(x)+f(y)-f(x+y)-f(0) \tag{2.2}$$

be the Cauchy difference of f. Furthermore, let

$$f\{y\} := (f[e_1, y], f[e_2, y])\,. \tag{2.3}$$

Then, for $y \in \mathbb{R}^2$, clearly $f\{y\} \in \mathbb{R}^2$. Now we determine the general solution of a functional equation which is instrumental in establishing our main result.

THEOREM 1. *The continuous map $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the functional equation*

$$f[x, y] = \langle x, f\{y\}\rangle \qquad (x,\, y \in \mathbb{R}^2) \tag{FE}$$

if and only if

$$f(x) = Ax^T + xBx^T + \alpha \tag{SO}$$

where $A = (a \;\; b)$, $B = \left(\begin{smallmatrix} c & e \\ e & d\end{smallmatrix}\right)$, and a, b, c, d, e, α are arbitrary real constants.

Proof. Using (2.1)–(2.3), (FE) can be written as

$$\begin{aligned} f(x_1, x_2) + f(y_1, y_2) - f(x_1 + y_1, x_2 + y_2) - f(0, 0) \\ = x_1[f(1, 0) - f(0, 0) + f(y_1, y_2) - f(1 + y_1, y_2)] \\ + x_2[f(0, 1) - f(0, 0) + f(y_1, y_2) - f(y_1, 1 + y_2)] \end{aligned} \tag{2.4}$$

for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Defining $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x_1, x_2) := f(x_1, x_2) - f(0, 0) \tag{2.5}$$

we get from (2.4),

$$\begin{aligned} g(x_1, x_2) + g(y_1, y_2) - g(x_1 + y_1, x_2 + y_2) \\ = x_1[g(y_1, y_2) - g(1 + y_1, y_2) + g(1, 0)] \\ + x_2[g(y_1, y_2) - g(y_1, 1 + y_2) + g(0, 1)]\,. \end{aligned} \tag{2.6}$$

Letting $x_2 = y_2 = 0$ in (2.6), we get

$$g(x_1, 0) + g(y_1, 0) - g(x_1 + y_1, 0) = x_1[g(y_1, 0) - g(1 + y_1, 0) + g(1, 0)]\,. \tag{2.7}$$

Defining

$$p(x_1) := g(x_1, 0) \qquad (x_1 \in \mathbb{R})\,, \tag{2.8}$$

from (2.7), we obtain

$$p(x_1) + p(y_1) - p(x_1 + y_1) = x_1[p(y_1) + p(1) - p(y_1 + 1)]\,. \tag{2.9}$$

Interchanging x_1 and y_1 in (2.9) and using the resulting expression with (2.9), we obtain

$$x_1[p(y_1) + p(1) - p(y_1 + 1)] = y_1[p(x_1) + p(1) - p(x_1 + 1)] \tag{2.10}$$

for all $x_1, y_1 \in \mathbb{R}$. Hence

$$p(y_1) + p(1) - p(y_1 + 1) = c_0 y_1, \tag{2.11}$$

where c_0 is a real constant. Inserting (2.11) into (2.9), we get

$$p(x_1) + p(y_1) - p(x_1 + y_1) = c_0 x_1 y_1 \,. \tag{2.12}$$

Define

$$\phi(x_1) := p(x_1) + \frac{1}{2} c_0\, x_1^2 \,. \tag{2.13}$$

Then by (2.13), (2.12) reduces to

$$\phi(x_1) + \phi(y_1) = \phi(x_1 + y_1) \tag{2.14}$$

for all $x_1, y_1 \in \mathbb{R}$.

Since f is continuous, p is also continuous. Therefore, with (2.14), (2.13) yields (see [1, p. 13])

$$p(x_1) = a x_1 - \frac{1}{2} c_0 x_1^2 \,, \tag{2.15}$$

where a is an arbitrary constant.

Similarly, we let $x_1 = y_1 = 0$ in (2.6) to get

$$g(0,x_2) + g(0,y_2) - g(0,x_2+y_2) = x_2[g(0,y_2) + g(0,1) - g(0,y_2+1)] \,. \tag{2.16}$$

Defining

$$q(x_2) := g(0,x_2) \qquad (x_2 \in \mathbb{R}) \tag{2.17}$$

we obtain from (2.16)

$$q(x_2) + q(y_2) - q(x_2 + y_2) = x_2[q(y_2) + q(1) - q(y_2 + 1)] \tag{2.18}$$

for all $x_2, y_2 \in \mathbb{R}$. This equation is similar to (2.9) and hence, we obtain

$$q(x_2) = b x_2 - \frac{1}{2}\, d_0\, x_2^2, \tag{2.19}$$

where b and d_0 are constants in $\mathbb{R}$.

Next, letting $x_1 = y_2 = 0$ in (2.6), we obtain

$$g(0,x_2) + g(y_1,0) - g(y_1,x_2) = x_2[g(y_1,0) - g(y_1,1) + g(0,1)] \,, \tag{2.20}$$

which is

$$q(x_2) + p(y_1) - g(y_1,x_2) = x_2\, [p(y_1) - g(y_1,1) + g(0,1)] \,. \tag{2.21}$$

Similarly, letting $x_2 = y_1 = 0$ in (2.6), we get

$$g(x_1,0) + g(0,y_2) - g(x_1,y_2) = x_1[g(0,y_2) - g(1,y_2) + g(1,0)] \,, \tag{2.22}$$

which is

$$p(x_1) + q(y_2) - g(x_1,y_2) = x_1[q(y_2) - g(1,y_2) + g(1,0)] \,. \tag{2.23}$$

Comparing (2.21) and (2.23), we see that

$$\begin{aligned} g(x_1,x_2) &= (1-x_2)\, p(x_1) + q(x_2) + x_2\, [g(x_1,1) - g(0,1)] \\ \text{also} &= p(x_1) + (1-x_1) q(x_2) + x_1[g(1,x_2) - g(1,0)] \,. \end{aligned} \tag{2.24}$$

Hence, we get

$$x_2[g(x_1,1)-g(0,1)-p(x_1)] = x_1[g(1,x_2)-g(1,0)-q(x_2)]\,.$$

Therefore

$$g(x_1,1)-g(0,1) = p(x_1)+\delta x_1\,,$$

where δ is a constant, and by (2.24) and the above, we get

$$\begin{aligned} g(x_1,x_2) &= (1-x_2)\,p(x_1)+q(x_2)+x_2\,[p(x_1)+\delta x_1] \\ &= p(x_1)+q(x_2)+\delta x_1x_2\,. \end{aligned} \tag{2.25}$$

By (2.5), (2.15) and (2.19), (2.25) yields

$$f(x_1,x_2) = ax_1-\frac{1}{2}c_0x_1^2+bx_2-\frac{1}{2}d_0x_2^2+\delta x_1x_2+\alpha\,. \tag{2.26}$$

Now renaming the constants $-\frac{1}{2}c_0$, $-\frac{1}{2}d_0$, and δ as c, d and $2e$, respectively, we get the asserted solution (SO).

The converse is easy to verify. This completes the proof of the theorem.

R e m a r k 1. Although, in Theorem 1, we have assumed f to be continuous, the general solution of the functional equation (FE) can be obtained without this assumption. In this case, the general solution of (FE) would be $f(x) = A(x)+xBx^T+\alpha$, where A is a biadditive function. Furthermore, (FE) can easily be generalized to the case of n variables. We leave the details to the reader.

3. The main result. Let $\mathcal{P}$ denote the family of all planes in $\mathbb{R}^3$. Here, the following subsets of $\mathcal{P}$ and $\mathcal{N}_p$ are of particular interest. Let

$$\mathcal{V}_x = \bigcup_{a\in\mathbb{R}}\{\{(a,y,z)\in\mathbb{R}^3 \mid a \text{ is some fixed constant}\}\}$$

be the set of all planes parallel to yz-plane and similarly, let

$$\mathcal{V}_y = \bigcup_{b\in\mathbb{R}}\{\{(x,b,z)\in\mathbb{R}^3 \mid b \text{ is some fixed constant}\}\}$$

denote the set of all planes parallel to xz-plane.

An *isomorphism* from $\mathcal{P}$ onto $\mathcal{N}_p$ is a bijection of $\mathbb{R}^3$ which induces a bijection from $\mathcal{P}$ onto $\mathcal{N}_p$. Similarly, an *automorphism* of $\mathcal{P}$ is a bijection of $\mathbb{R}^3$ which induces a bijection from $\mathcal{P}$ onto $\mathcal{P}$. The next result characterizes the automorphisms of $\mathcal{P}$ and is known as the “Fundamental Theorem of Projective Geometry” [2, p. 156].

LEMMA. *Every bijection ϕ of $\mathbb{R}^3$ onto itself which induces a bijection of $\mathcal{P}$ onto itself is affine linear, that is,*

$$\phi(x,y,z) = (a_1x+b_1y+c_1z+u,\ a_2x+b_2y+c_2z+v,\ a_3x+b_3y+d_3z+w)$$

where

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0\,,$$

and a_i, b_i, c_i $(i = 1, 2, 3)$ *are arbitrary constants.*

Now adopting a technique similar to the proof of the Theorem in [4], we proceed to prove our main result.

THEOREM 2. *Let* $p : \mathbb{R}^2 \to \mathbb{R}$. *There exists a continuous isomorphism* θ *from* $\mathcal{P}$ *onto* $\mathcal{N}_p$ *if and only if* $p(x,y) = d_1x^2 + d_2y^2 + 2d_3xy + d_4x + d_5y + d_6$ *with* $d_1d_2 - d_3^2 \neq 0$. *Here,* d_i $(i = 1, \ldots, 6)$ *are real constants.*

Proof. Let $\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be a continuous bijection which induces an isomorphism from $\mathcal{P}$ onto $\mathcal{N}_p$. Then, for any affine linear automorphism α of $\mathcal{P}$, which by the previous lemma has the form

$$\alpha(x,y,z) = (a_1x + b_1y + c_1z + u,\ a_2x + b_2y + c_2z + v,\ a_3x + b_3y + c_3z + w)$$

with

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0\,,$$

the composition $\kappa = \theta \circ \alpha$ is still a continuous isomorphism from $\mathcal{P}$ onto $\mathcal{N}_p$.

Note that $\mathcal{V}_x$ and $\mathcal{V}_y$ are maximal sets of parallel planes belonging to both $\mathcal{P}$ and $\mathcal{N}_p$. Since κ is an isomorphism, the images of $\mathcal{V}_x$ and $\mathcal{V}_y$ are maximal sets of parallel planes/surfaces in $\mathcal{N}_p$. Thus, we can choose α such that κ maps $\mathcal{V}_x$ onto $\mathcal{V}_x$ and $\mathcal{V}_y$ onto $\mathcal{V}_y$. Hence, κ is of the form

$$\kappa(x,y,z) = (f(x),\ g(y),\ h(x,y,z)), \tag{3.1}$$

and by choosing α appropriately, we may assume that $f(0) = 0$ and $g(0) = 0$.

Now consider the two-parameter group of automorphisms of $\mathcal{P}$ and $\mathcal{N}_p$ whose elements are defined by

$$\tau_{s,t}(x,y,z) = (x + s,\ y + t,\ z)\,.$$

Since κ maps $\mathcal{P}$ to $\mathcal{N}_p$, we see by the previous lemma that

$$\tau'_{s,t} = \kappa^{-1} \circ \tau_{s,t} \circ \kappa \tag{3.2}$$

is affine linear, and hence, by (3.1), of the form

$$\tau'_{s,t}(x,y,z) = (\widetilde{a}(s)x + u(s),\ \widetilde{b}(t)y + v(t),\ a(s,t)x + b(s,t)y + c(s,t)z + w(s,t))\,.$$

For $s \neq 0$, $\tau'_{s,t}$ does not fix any of the planes in $\mathcal{V}_x$, hence the equation $\widetilde{a}(s)x + u(s) = x$ cannot have a solution. This yields $\widetilde{a}(s) \equiv 1$. Similarly for $t \neq 0$, $\widetilde{b}(t)y + v(t) = y$ cannot have a solution and thus $\widetilde{b}(t) \equiv 1$. Thus, $\tau'_{s,t}$

has the form

$$\begin{aligned}(3.3)\qquad &\tau'_{s,t}(x,y,z)\\ &\quad = (x+u(s),\, y+v(t),\, a(s,t)x+b(s,t)y+c(s,t)z+w(s,t))\,.\end{aligned}$$

Using $\kappa\circ\tau'_{s,t}=\tau_{s,t}\circ\kappa$, we obtain

$$f(x+u(s)) = f(x)+s \quad\text{and}\quad g(y+v(t)) = g(y)+t\,. \tag{3.4}$$

Letting $x=0$ and $y=0$ in (3.4), we see that

$$u(s) = f^{-1}(s) \quad\text{and}\quad v(t) = g^{-1}(t) \tag{3.5}$$

since $f(0)=g(0)=0$. Thus u and v are continuous. Furthermore, since the $\tau'_{s,t}$ form a group, we have

$$\tau'_{q,r}\circ\tau'_{s,t} = \tau'_{q+s,r+t}\,. \tag{3.6}$$

Thus, from (3.6), we have

$$u(q+s) = u(q)+u(s) \quad\text{and}\quad v(r+t) = v(r)+v(t)\,. \tag{3.7}$$

Since u and v are continuous, the Cauchy equations in (3.7) yield

$$u(s) = \xi s \quad\text{and}\quad v(t) = \eta t\,, \tag{3.8}$$

where ξ and η are nonzero arbitrary constants. Again, we can refine our choice of α (the affine linear transformation) to obtain $\xi=\eta=1$. Thus, from (3.5), we get $f(x)=x$ and $g(y)=y$; and from (3.1),

$$\kappa(x,y,z) = (x,y,h(x,y,z))\,. \tag{3.9}$$

Now we note that, for $m\in\mathbb{R}$, $\sigma_m(x,y,z)=(x,y,z+m)$ is the only one-parameter group of automorphisms of $\mathcal{P}$ and $\mathcal{N}_p$ which fixes $\mathcal{V}_x$ and $\mathcal{V}_y$, respectively and has no fixed points for $m\neq 0$. Thus, the mapping

$$\sigma_m \to \sigma'_m = \kappa^{-1}\circ\sigma_m\circ\kappa \tag{3.10}$$

is an automorphism of this group. This implies that

$$\sigma'_m(x,y,z) = (x,\, y,\, z+l(m)) \tag{3.11}$$

for some continuous function $l:\mathbb{R}\to\mathbb{R}$. Since $\sigma'_{m+n}=\sigma'_m\circ\sigma'_n$, it follows that

$$l(m) = \zeta m \qquad \text{for some } \zeta\neq 0\,. \tag{3.12}$$

Since $\sigma'_m=\kappa^{-1}\circ\sigma_m\circ\kappa$, we have

$$\kappa\circ\sigma'_m = \sigma_m\circ\kappa \tag{3.13}$$

and hence

$$h(x,y,z+l(m)) = h(x,y,z)+m\,. \tag{3.14}$$

Letting $z = 0$ and $m = z/\zeta$, we obtain from (3.14) and (3.12),

$$h(x, y, z) = h(x, y, 0) + \frac{z}{\zeta}\,, \tag{3.15}$$

and by again refining our choice of the affine linear transformation α we may assume that $\zeta = 1$.

By (3.9), κ maps the xy-plane to the set $\{(x, y,\, h(x, y, 0)) \,|\, x, y \in \mathbb{R}\}$ and this belongs to $\mathcal{N}_p$. It follows that there exist constants c_1, c_2 and c_3 such that

$$h(x, y, 0) = p(x + c_0, y + c_1) + c_2 =: \psi(x, y)\,. \tag{3.16}$$

Thus (3.15) and (3.16) with $\zeta = 1$ yield

$$h(x, y, z) = \psi(x, y) + z\,, \tag{3.17}$$

and hence from (3.9), we obtain

$$\kappa(x, y, z) = (x, y, z + \psi(x, y))\,. \tag{3.18}$$

Substituting (3.18) into (3.2), we obtain, on the one hand,

$$\tau'_{s,t}(x, y, z) = (x + s, y + t, z + \psi(x, y) - \psi(x + s, y + t))\,, \tag{3.19}$$

and from (3.3),

$$\tau'_{s,t}(x, y, z) = (x + s,\, y + t,\, a(s, t)x + b(s, t)y + c(s, t)z + w(s, t))\,. \tag{3.20}$$

Setting these two equal yields for the z-component:

$$z + \psi(x, y) - \psi(x + s, y + t) = a(s, t)x + b(s, t)y + c(s, t)z + w(s, t) \tag{3.21}$$

and it immediately follows that $c(s, t) \equiv 1$. Thus

$$\psi(x, y) - \psi(x + s, y + t) = a(s, t)x + b(s, t)y + w(s, t)\,. \tag{3.22}$$

Now we let $x = y = 0$:

$$\psi(0, 0) - \psi(s, t) = w(s, t)\,. \tag{3.23}$$

Also, letting $x = 1, y = 0$ in (3.22), we see that

$$\psi(1, 0) - \psi(1 + s, t) = a(s, t) + \psi(0, 0) - \psi(s, t)\,. \tag{3.24}$$

Thus (3.24) yields

$$a(s, t) = \psi(1, 0) - \psi(1 + s, t) - \psi(0, 0) + \psi(s, t)\,. \tag{3.25}$$

Similarly, letting $x = 0$ and $y = 1$ in (3.22), we get

$$b(s, t) = \psi(0, 1) - \psi(s, 1 + t) - \psi(0, 0) + \psi(s, t)\,. \tag{3.26}$$

Hence, from (3.22), (3.25) and (3.26), we obtain

$$\begin{aligned} \psi(x, y) - \psi(x + s, y + t) - \psi(0, 0) + \psi(s, t) \\ = x[\psi(1, 0) - \psi(1 + s, t) - \psi(0, 0) + \psi(s, t)] \\ + y[\psi(0, 1) - \psi(s, 1 + t) - \psi(0, 0) + \psi(s, t)] \end{aligned} \tag{3.27}$$

for all $x, y, s, t \in \mathbb{R}$. By Theorem 1, the solution to (3.27) is

$$\psi(x,y) = d_1x^2 + d_2y^2 + 2d_3xy + \delta_4 x + \delta_5 y + \delta_6\,, \tag{3.28}$$

where $d_1, d_2, d_3, \delta_4, \delta_5, \delta_6$ are real constants. From (3.22), we know that

$$\psi(x+s, y+t) = \psi(x,y) - ax - by - w\,. \tag{3.29}$$

That is, if $\psi(x,y)$ is given by (3.28), then there exist $a = a(s,t)$, $b = b(s,t)$ and $w = w(s,t)$ such that (3.29) is true. Thus, any element in $\mathcal{N}_p$ is the image under κ of a plane belonging to $\mathcal{P}$. Conversely, $z = -ax - by - w$ is the equation of an arbitrary nonvertical plane in $\mathcal{P}$ and its image under κ belongs to $\mathcal{N}_p$ if there exist $s = s(a,b)$ and $t = t(a,b)$ such that (3.29) is true. Substituting (3.28) into (3.29), we see that ψ satisfies (3.29) if and only if

$$\begin{pmatrix} 2d_1 & 2d_3 \\ 2d_3 & 2d_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix} \tag{3.30}$$

and

$$(s \;\; t) \begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} + (\delta_4 \; \delta_5) \begin{pmatrix} s \\ t \end{pmatrix} = w\,. \tag{3.31}$$

In order to have a unique solution for s and t (uniqueness is needed so that κ is one-to-one from $\mathcal{P}$ onto $\mathcal{N}_p$), it follows that

$$\det \begin{pmatrix} d_1 & d_3 \\ d_3 & d_2 \end{pmatrix} \neq 0\,,$$

which is $d_1d_2 - d_3^2 \neq 0$. Hence by (3.16), there exists a continuous isomorphism θ from $\mathcal{P}$ onto $\mathcal{N}_p$ if and only if $p(x,y) = d_1x^2 + d_2y^2 + 2d_3xy + d_4x + d_5y + d_6$ with $d_1d_2 - d_3^2 \neq 0$. Here, d_i $(i = 1, \ldots, 6)$ are real constants.

Remark 2. Our Theorem 2 has a natural generalization to $\mathbb{R}^n$. In this case, the function $p : \mathbb{R}^{n-1} \to \mathbb{R}$ is of the form $p(x) = ax^T + xBx^T + \alpha$, where a is an arbitrary constant in $\mathbb{R}^{n-1}$, B is an $n-1$ by $n-1$ real symmetric matrix with nonzero determinant, and α is a real constant. Again we leave the details to the reader.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LOUISVILLE
LOUISVILLE, KENTUCKY 40292,
U.S.A.
E-mail: RCPOWE01@ULKYVX.LOUISVILLE.EDU
T0RIED01@ULKYVX.LOUISVILLE.EDU
PKSAHO01@ULKYVX.LOUISVILLE.EDU

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