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ON THE DISJOINT $(0,n)$-CELLS PROPERTY FOR HOMOGENEOUS ANR'S

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A metric space (X,ϱ) satisfies the disjoint $(0,n)$-cells property provided for each point $x\in X$, any map f of the n-cell B^n into X and for each $\varepsilon>0$ there exist a point $y\in X$ and a map $g:B^n\to X$ such that $\varrho(x,y)<\varepsilon$, $\widehat{\varrho}(f,g)<\varepsilon$ and $y\notin g(B^n)$. It is proved that each homogeneous locally compact ANR of dimension >2 has the disjoint $(0,2)$-cells property. If $\dim X=n>0$, X has the disjoint $(0,n-1)$-cells property and X is a locally compact LC^{n-1}-space then local homologies satisfy $H_k(X,X-x)=0$ for $k<n$ and $H_n(X,X-x)\neq 0$.

0. Introduction. All spaces in the paper are assumed to be metric separable and all mappings are continuous. A space X is said to be *homogeneous* if for each couple of points $x,y\in X$ there exists a homeomorphism $h:X\to X$ such that $h(x)=y$. Function spaces are endowed with the compact-open topology. In particular, if Y is locally compact and ϱ is a metric in X, then the space X^Y is metrizable by the metric $\widehat{\varrho}$ defined as follows: represent Y as the union $Y=\bigcup_{m=1}^\infty C_m$, where C_m is compact and $C_m\subset\operatorname{int}C_{m+1}$ for each m; for $f,g\in X^Y$ put $\varrho_m(f,g)=\min\{1/m,\sup\{\varrho(f(y),g(y)):y\in C_m\}\}$ and $\widehat{\varrho}(f,g)=\sup\{\varrho_m(f,g):m=1,2,\ldots\}$. We will say that *maps* $f\in X^Y$ *approximate a given map* $g\in X^Y$ if $\widehat{\varrho}(f,g)$ can be made as small as we wish. Two maps $f,g\in X^Y$ are said to be ε-*close* if $\varrho(f(y),g(y))<\varepsilon$ for each $y\in Y$. As usual, $B^n=\{x\in\mathbb{R}^n:|x|\le 1\}$, $S^{n-1}=\partial B^n=\{x\in\mathbb{R}^n:|x|=1\}$, $I=[0,1]$, B^0 means a one-point space.

The *disjoint* (n,m)-*cells property* of a space X, denoted by $D(n,m)$, is defined as follows: for each $\varepsilon>0$ and any two mappings $f:B^n\to X$ and $g:B^m\to X$ there exist mappings $f':B^n\to X$ and $g':B^m\to X$ such that $\widehat{\varrho}(f,f')<\varepsilon$, $\widehat{\varrho}(g,g')<\varepsilon$ and $f'(B^n)\cap g'(B^m)=\emptyset$. Obviously $D(n,m)\Rightarrow D(n',m')$ for $n'\le n$, $m'\le m$. The properties $D(n,m)$ for $n=m>1$

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are crucial in recognizing manifolds among ANR's of (finite or infinite) dimensions >4 (see [5] as a general reference). The Bing–Borsuk conjecture [1] says that every n-dimensional ($n > 2$) locally compact homogeneous ANR is a manifold or at least a generalized manifold. So far, it is not even known whether homogeneous ANR-spaces of dimension >4 must contain a 2-cell (the property $D(2, 2)$ would imply that).

This paper is concerned with the properties $D(0, n)$ which play a role in recognizing generalized manifolds. The property $D(0, 0)$ of a space X just means that X is dense in itself. A space X satisfies $D(0, 1)$ if and only if X contains no free arcs, i.e. each arc is nowhere dense in X (note that each map $f : I \to X$ can be approximated by a map $g : I \to X$ whose image is a finite union of small arcs in $f(I)$; thus $g(I)$ is nowhere dense in X and $D(0, 1)$ follows). For a homogeneous locally compact ANR X we have $X \in D(0, 1)$ if and only if $\dim X > 1$. Indeed, if $\dim X > 1$, then by the homogeneity arcs are nowhere dense in X; if $\dim X = 1$, then X is a one-manifold [1, Theorem 6.1], hence it contains free arcs. Nontrivial problems start with $n > 1$. Therefore, henceforth, we always assume $n > 1$ when dealing with $D(0, n)$. The main result is that each homogeneous locally compact ANR of dimension >2 has $D(0, 2)$. For such spaces X we also present an easy-to-follow argument that $D(0, n)$ implies $H_k(X, X - \{x\}) = 0$ for $k \le n$ (all homology groups are singular with integer coefficients). The latter result was first established in [10] with a heavy use of algebraic topology. Actually, we show a more natural stronger version for LC^n-spaces and homotopy groups. Moreover, if $\dim X = n$ and $X \in D(0, n - 1)$, then $H_n(X, X - \{x\}) \neq 0$. This generalizes a theorem announced by Łysko [8].

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1. Auxiliary results. The book [9] is a good reference for basic theory of ANR's. For convenience we recall here three facts about ANR's and their counterparts for LC^n-spaces.

(1.1) *Open subsets of an ANR* (LC^n*-space*) *are again ANR's* (LC^n*-spaces*).

(1.2) *If X is an ANR* (LC^n*-space*), $\varepsilon > 0$ *and f is a map from a compact space Y* (*with* $\dim Y \le n$, *resp.*) *into X, then there is a $\delta > 0$ such that if another map $g : Y \to X$ is δ-close to f, then f and g are ε-homotopic.*

(1.3) *The small homotopy extension property, which means the homotopy extension property where all homotopies involved are limited by an arbitrarily fixed number $\varepsilon > 0$* (*and come from spaces of dimension at most n in case of LC^n-spaces*).

We are going to use the following Effros theorem.

PROPOSITION 1.4. *If* X *is a homogeneous locally compact space with metric* ϱ, $a \in X$ *and* $\varepsilon > 0$, *then there exists* $\delta > 0$ *such that if* $\varrho(x,a) < \delta$, *then there is a homeomorphism* $h : X \to X$ *satisfying* $h(a) = x$ *and* $\widehat{\varrho}(h, \mathrm{id}_X) < \varepsilon$ (the number δ is called an *Effros* δ for ε and a).

Remark. Effros' theorem has usually been formulated for compact spaces. However, its proof, as that in [3, p. 584], runs unchanged for locally compact spaces X due to the fact that the group of all selfhomeomorphisms of X is a Borel subset of X^X [6].

PROPOSITION 1.5. *Suppose* X *is a homogeneous locally compact ANR with metric* ϱ, $U \subset X$ *is an open neighborhood of a point* x *with compact closure*, $0 < \varepsilon < \varrho(x, X-U)$ *and* $K = X - N_\varepsilon(X-U)$ *where* $N_\varepsilon(X-U)$ *denotes the open* ε*-ball around* $X-U$. *Then there exists a* $\delta > 0$ *such that if* $\varrho(x,y) < \delta$, $y \in U$, *then there is a mapping* $g : U \to U$ *which is* ε*-close to the identity* id_U *on* U, $g|K$ *is a homeomorphism and* $g(y) = x$.

Proof. Represent X as the union of compact subsets C_m such that $C_m \subset \mathrm{int}\ C_{m+1}$ for $m=1,2,\ldots$ There exists n such that $U \subset C_n$. Since U is an ANR, there is a positive number $\eta < \min\{\varepsilon, 1/n\}$ such that if a mapping $f : K \to U$ is η-close to id_K, then f is ε-homotopic to id_K in U. Consider an Effros' δ for η and x. Take $y \in U$ with $\varrho(x,y) < \delta$. By Proposition 1.4 there exists a homeomorphism $h : X \to X$ such that $h(y) = x$ and $\widehat{\varrho}(h, \mathrm{id}_X) < \eta$. By the definition of $\widehat{\varrho}$, $h|K$ is η-close to id_K. It follows from (1.3) that $h|K$ extends to a mapping $g : U \to U$ which is ε-close to id_U. ■

Let us recall the notion of a Cantor manifold. A locally compact n-dimensional space is called a *Cantor manifold* if no subset of dimension less than $n-1$ separates it. If the space is infinite-dimensional, then it is called a *Cantor manifold* if no finite-dimensional subset separates it. A locally compact locally connected space is a *local Cantor manifold* if each connected open subset is a Cantor manifold of the same dimension.

The following theorem was stated in [7] (see also [6]).

PROPOSITION 1.6. *Any locally compact locally connected homogeneous space is a local Cantor manifold.*

On the other hand, one can recognize local Cantor manifolds by means of local homology groups.

PROPOSITION 1.7. *Let* X *be a locally compact locally connected space and* $n > 1$. *If* $H_k(X, X-\{x\}) = 0$ *for every* $x \in X$ *and* $k < n$, *then* $\dim X \geq n$. *In the case where* $\dim X = n$ *at each point*, X *is a local Cantor manifold.*

Proof. Let U be an open connected subset of X. From [4, Lemma 2.1] and the excision we have $H_1(U, U-A) = 0$ for each closed subset A of X whose dimension is less than $n-1$. This means that $U-A$ is connected.

But if $\dim X < n$, then X contains a basis of open sets with boundaries of dimension less than $n-1$. Therefore $\dim X \geq n$. The second part of the proposition now easily follows. ∎

Recall that a subset A of X is called *locally k-coconnected* (k-LCC) if for each $a \in A$ any neighborhood U of a contains a neighborhood V of a such that each map of S^k into $V - A$ can be extended to a map of B^{k+1} into $U - A$. The condition LCC^n means k-LCC for all $k = 0, 1, \ldots, n$.

PROPOSITION 1.8 [7]. *If X is a homogeneous locally compact space, then X has the property $D(0,n)$ if and only if the following condition $D^*(0,n)$ holds: for each point $x \in X$ any mapping $f : B^n \to X$ can be approximated by mappings with images omitting x. If X is an LC^n-space of dimension greater than* 1, *then $D^*(0,n)$ is equivalent to a singleton $\{x\}$ being LCC^{n-1} for each $x \in X$.*

Remark. Condition $D^*(0,n)$ follows from $D(0,n)$ by Proposition 1.4. The second part of the above proposition was formulated in [7] under superfluous assumptions that X be a compact ANR and $\dim X > 2$, but the proof runs for an LC^n-space X; the assumption $\dim X > 2$ was used there to derive $D(0,1)$ from $D(1,1)$ but, as we have seen in the previous section, $D(0,1)$ is a consequence of $\dim X > 1$.

We add the following nice description of the property $D(0,n)$.

PROPOSITION 1.9. *If X is a homogeneous locally compact space, then X has $D(0,n)$ if and only if the set of all mappings of B^n into X with nowhere dense images is dense in the mapping space X^{B^n}.*

Proof. Suppose X satisfies $D(0,n)$. Let $\{d_1, d_2, \ldots\}$ be a countable dense subset of X and $D_m = \{d_1, \ldots, d_m\}$. It easily follows from Proposition 1.8 that, given any finite subset A of X, each mapping of B^n into X can be approximated by mappings omitting A. Hence the set $\mathcal{F}_m$ of all mappings of B^n into $X - D_m$ is open and dense in X^{B^n}. Now, the set $\bigcap_{m=1}^{\infty} \mathcal{F}_m$ consists of mappings with nowhere dense images and is dense in X^{B^n} by the Baire Category Theorem. The proof of the converse implication is left to the reader. ∎

2. Main results

PROPOSITION 2.1. *Assume a locally compact LC^n-space X satisfies $D^*(0,n)$, $n > 1$. If U is an open nonempty subset of X, $x \in X$ and $z \in U - \{x\}$, then the inclusion-induced homomorphism i_* between the k-th homotopy groups $\pi_k(U - \{x\}, z)$ and $\pi_k(U, z)$ is an isomorphism for $0 < k < n$ and it is an epimorphism for $k = n$.*

P r o o f. Recall that $D(0,n) \Rightarrow D(0,k)$ for $k \leq n$. To show that i_* is one-to-one for $0 < k < n$ take two maps f and g of the cube I^k into $U - \{x\}$ which are joined by a homotopy $H : I^k \times I \to U$ such that $H(\partial I^k \times I) = \{z\}$. By $D^*(0,n)$, H is approximated, arbitrarily closely, by a map $H' : I^k \times I \to U - \{x\}$. If H' is close enough to H, then by (1.1)–(1.3) the map $H|\partial I^{k+1}$ has an extension $\overline{H} : I^k \times I \to U - \{x\}$. Hence f and g represent the same element of $\pi_k(U - \{x\}, z)$. To prove that i_* is onto for $0 < k \leq n$ let $f : I^k \to U$ be a map such that $f(\partial I^k) = \{z\}$. Then f is approximated by a map $f' : I^k \to U - \{x\}$ (property $D^*(0,n)$). Set $K = I^k \times \{0\} \cup \partial I^k \times I$ and consider the map $H : K \to U$ defined by $H(p,0) = f(p)$ for $p \in I^k$ and $H(\partial I^k \times I) = \{z\}$. It follows from (1.1)–(1.3) that if f' is close enough to f, then there is a small homotopy $G : I^k \times I \to U$, where $G(p,0) = f(p)$, $G(p,1) = f'(p)$, such that $G|K$ is homotopic to H in U. Thus H extends to a homotopy $\overline{H} : I^k \times I \to U$ which approximates G. Then the map g defined by $g(p) = \overline{H}(p,1)$ approximates f', so we can assume that g maps I^k into $U - \{x\}$. Moreover, the homotopy $\overline{H}$ joins f and g and $\overline{H}(\partial I^k \times I) = \{z\}$. ■

R e m a r k. That the fundamental groups $\pi_1(U - \{x\}, z)$ and $\pi_1(U, z)$ are isomorphic follows also from [5, Proposition 3, p. 144].

PROPOSITION 2.2. *If* $i_* : \pi_k(U - \{x\}) \to \pi_k(U)$ *is a monomorphism for each* $x \in X$ *and each* U *from a basis* $\mathcal{U}$ *of open connected subsets of an* LC^n*-space* X $(0 < k < n)$, *then* $\{x\}$ *is* k*-LCC.*

P r o o f. Write $\mathcal{U}_x = \{U \in \mathcal{U} : x \in U\}$. Suppose W is an open neighborhood of x. Choose $U_2 \subset U_1 \subset U_0 \subset W$ such that $U_i \in \mathcal{U}_x$ and any map from an at most n-dimensional space into U_{i+1} is homotopic in U_i to a constant map, $i = 0, 1$. Fix a point s of the sphere S^k and consider a map $f : (S^k, s) \to (U_2 - \{x\}, f(s))$.

This map is homotopic in U_1 to a constant map g. Since U_1 is arcwise connected, we can assume that $g(S^k) = f(s)$. Suppose $H : S^k \times I \to U_1$ is a homotopy such that $H(p,0) = f(p)$, $H(p,1) = f(s)$. Put $K = S^k \times \{0,1\} \cup \{s\} \times I$ and define $G : K \to U_2$ by $G(p,0) = f(p)$ for $p \in S^k$ and $G(z) = f(s)$ elsewhere. Then G and $H|K$ are homotopic in U_0. So G extends to a mapping $\overline{G} : S^k \times I \to U_0$. This means that f represents the identity element in the group $\pi_k(U_0, f(s))$, hence in $\pi_k(U_0 - \{x\}, f(s))$ as well. It follows that f admits an extension $\bar{f} : B^{k+1} \to U_0 - \{x\}$. ■

The next theorem is a consequence of Propositions 2.1, 2.2 and 1.8.

THEOREM 2.3. *If* X *is a homogeneous locally compact* LC^n*-space of dimension greater than* 1, *then* X *satisfies* $D(0,n)$, $n > 1$, *if and only if for each basis* (*equivalently*, *there exists a basis*) $\mathcal{U}$ *of open connected subsets of* X *and for any* $x \in X$ *and* $U \in \mathcal{U}$ *the inclusion* $i : U - \{x\} \subset U$ *is an* n*-equivalence* (*in the sense of* [11]).

From the Whitehead theorem [11], excision and exactness properties and from Proposition 1.7 we get the following corollary.

COROLLARY 2.4. *Suppose X is a homogeneous locally compact LC^n-space satisfying $D(0,n)$, $n > 1$. Then $H_k(X, X - \{x\}) = 0$ for each $x \in X$ and $k \le n$. Moreover*, $\dim X > n$.

Theorem 2.3, Corollary 2.4 and Proposition 1.7 imply

THEOREM 2.5. *Let X be an n-dimensional homogeneous locally compact LC^{n-1}-space satisfying $D(0, n-1)$, $n > 2$. Then*

(a) *$\pi_k(U, U - \{x\}) = 0$ for $k < n$ and for each open connected nonempty $U \subset X$, but, in case that $X \in ANR$, for all sufficiently small open connected neighborhoods V of x we have $\pi_n(V, V - \{x\}) \neq 0$;*

(b) *$H_k(X, X - \{x\}) = 0$ for $k < n$ and $H_n(X, X - \{x\}) \neq 0$.*

THEOREM 2.6. *If X is a homogeneous locally compact ANR of dimension > 2, then X satisfies $D(0, 2)$.*

Proof. We will prove that $\{p\} \in \mathrm{LCC}^1$ for arbitrary $p \in X$ (Proposition 1.8). To this end let U be an open subset of X containing p and V be an open neighborhood of p which is contractible in U. We can assume that U is connected and its closure is compact.

Suppose first that $f : S^1 \to V - \{p\}$ has one-dimensional image and let $F_0 : B^2 \to U$ be an extension of f. Take a point $q \in U - F_0(B^2)$ and an arc A in $U - f(S^1)$ joining p and q. Such an arc exists because X is a local Cantor manifold (Proposition 1.6). Define $M = \{x \in A$: there exists a mapping $F : B^2 \to U - \{x\}$ such that $F|S^1 = f\}$. We are going to show that M is closed. Suppose $x \in \mathrm{cl}\, M$. Let $0 < \varepsilon < \frac{1}{2}\varrho(x, X - U)$ and ε satisfy the condition that if a map $f' : S^1 \to U - \{x\}$ is ε-close to f, then f' is homotopic to f in the ANR $U - \{x\}$. Take a point $y \in M$ such that $\varrho(x, y) < \delta$ where δ is a number as in Proposition 1.5. Let $F : B^2 \to U - \{y\}$ be an extension of f. If g is a map guaranteed by Proposition 1.5, then gF maps B^2 into $U - \{x\}$ and $gF|S^1 = gf$ is homotopic to f in $U - \{x\}$. It follows from the homotopy extension property for $U - \{x\}$ that f has an extension $F_1 : B^2 \to U - \{x\}$. That means that $x \in M$. The set M is evidently nonempty and open in A, hence $M = A$. We have shown that $p \in M$ which means that the condition LCC^1 is satisfied by mappings with one-dimensional images.

In the general case any mapping $f : S^1 \to V - \{p\}$ can be approximated by mappings $f' : S^1 \to V - \{p\}$ with one-dimensional images ($f'(S^1)$ can be viewed as a finite union of small arcs in $f(S^1)$; details of this standard procedure are left to the reader). If f' is sufficiently close to f, then the two mappings are homotopic in $U - \{p\}$. Since f' extends to a mapping $F : B^2 \to U - \{p\}$, so does f by the homotopy extension property for $U - \{p\}$. ∎

The three-dimensional case calls special attention.

COROLLARY 2.7. *Let X be a homogeneous locally compact ANR. If* $\dim X > 2$, *then* $H_k(X, X - \{x\}) = 0$ *for any* $x \in X$ *and* $k < 3$. *If* $\dim X = 3$, *then* $H_3(X, X - \{x\}) \neq 0$.

The author does not know whether a homogeneous locally compact ANR of dimension greater than $n > 2$ must satisfy $D(0, n)$.

3. Final remarks. Let us recall property Δ of Borsuk [2]: a space X has *property* $\Delta(n)$ if for every point $x \in X$ every neighborhood U of x contains a neighborhood V of x such that each compact nonempty set $A \subset V$ of dimension at most $n - 1$ is contractible in a subset of U of dimension at most $\dim A+1$; *property* Δ means $\Delta(n)$ for every n. If X is a locally compact ANR satisfying $\Delta(n)$ and K is a compact space of dimension at most n, then the set of mappings $f : K \to X$ with $\dim f(K) \leq \dim K$ is dense in X^K (see the proof of [2, (2.1), p. 164]). It follows that $\Delta(n)$ implies $D(0, n)$ for locally compact ANR's of dimension greater than n at each point. Thus Theorem 2.5 generalizes the following result announced in [8] (unfortunately, its proof has never been published): if X is an n-dimensional compact homogeneous ANR which satisfies condition Δ, then $H_k(X, X - \{x\}) = 0$ for $k < n$ and $H_n(X, X - \{x\}) \neq 0$.

Each local Cantor manifold X of dimension at least three has $D(1, 1)$ (see the proof of [4, Proposition 2.2]). If X is, additionally, an LC^1-space, then $X \times \mathbb{R}$ has $D(1, 2)$ and $X \times \mathbb{R}^2$ has $D(2, 2)$ [4]. When $X \times \mathbb{R}$ has $D(2, 2)$ is, however, a deeper question. One of central problems on generalized manifolds is to learn whether their products with the real line $\mathbb{R}$ are genuine manifolds. It is thus important to be able to detect $D(2, 2)$ for such products of dimension at least five. It follows from a characterization of $D(1, 2)$ in [4] that each ANR X of dimension at least four which is a local Cantor manifold satisfying $\Delta(2)$ has $D(1, 2)$, hence the product $X \times \mathbb{R}$ has $D(2, 2)$. Propositions 1.6 and 1.7 show possible applications of this remark.

OBSERVATION 3.1. *Let X be a locally compact ANR of dimension at least four satisfying* $\Delta(2)$. *If X is either homogeneous or a generalized manifold, then X has* $D(1, 2)$ *and* $X \times \mathbb{R}$ *has* $D(2, 2)$.

The above observation improves [10, Corollary 5.5] and restates (a correct part of) [10, Theorem 4.6].

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