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NONCOMMUTATIVE ANALOGS OF SYMMETRIC POLYNOMIALS

BY

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1. Introduction. Our aim is to introduce and investigate several analogs of the (commutative) symmetric polynomials (compare [2], Section I.2) in the case of the semigroup algebra of the free noncommutative semigroup with a finite number of generators—this is the algebra of noncommutative polynomials—and in the case of the group algebra of the free noncommutative group with a finite number of generators (for the free group see [3], Section 1.2, for the (semi)group algebra see [1], Definition 5.73).

The general idea is to consider expressions of the form

$$\sum_{i_1,\ldots,i_q} x_{i_1}^{h_1}\ldots x_{i_q}^{h_q},$$

where h_j's are nonzero integers and any two consecutive i_j, i_{j+1} are different.

Remarkably, the vector spaces spanned by these functions are algebras. Moreover, many properties of ordinary symmetric functions hold in this new situation.

The algebras m (of Section 4) and λ (of Section 7) are basic while C and M are variations on the same principle.

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2. Notation and terminology. We write $\mathbb{N}=\{0,1,2,\ldots\}$, $\mathbb{N}_+=\mathbb{N}\backslash\{0\}$. We fix a commutative ring K with unit, an integer $k\ge 2$ and free generators $x_1,\ldots,x_k$ of the free noncommutative group $\mathbb{F}_k$. Let $\mathbb{P}_k$ mean the (free noncommutative) semigroup $\mathbb{P}_k\subseteq\mathbb{F}_k$ with unit generated by $x_1,\ldots,x_k$. The symbols $K(\mathbb{F}_k)$ and $K(\mathbb{P}_k)$ denote the group algebra of $\mathbb{F}_k$ and the semigroup algebra of $\mathbb{P}_k$ respectively.

We say that a subset I of noncommutative algebra A is *algebraically independent* if for every $a_1,\ldots,a_i\in I$ and a polynomial f of i noncommutating variables the equality $f(a_1,\ldots,a_i)=0$ implies $f=0$.

We write $B<A$ if B is a subalgebra over K of A and $K\subset B$. Notice $K(\mathbb{P}_k)<K(\mathbb{F}_k)$.

The algebra over K generated by the sum of its subset T and K is denoted by $\mathrm{Alg}(T)$. We call a set T an *algebraic basis* of an algebra $A < K(\mathbb{F}_k)$ if T is algebraically independent and $\mathrm{Alg}(T) = A$.

If T is a set then T^i, T^∞ denote respectively the i-fold and countable products of T, $T^i = \{\emptyset\}$ for $i < 1$, $T_\infty = \bigcup_{i=0}^{\infty} T^i$.

Moreover, if a_t's belong to an algebra with unit $\mathbf{1}$ and with zero $\mathbf{0}$ then we put $\prod_{t=1}^{i} a_t = a_1 a_2 \ldots a_i$ if $i \in \mathbb{N}_+$ (notice the ordering of a_j's), $\prod_{t\in\emptyset} a_t = \mathbf{1}$, $\sum_{t\in\emptyset} a_t = \mathbf{0}$.

A sequence $(i_t)_{t=p}^{q}$, where p, q are integers, is usually denoted by $i_{p,q}$, $i_{p,q} = \emptyset$ for $p > q$.

The symbol $[\ldots]$ denotes the operation of "writing in" elements of a finite sequence into a sequence, that is,

$$(\ldots, a, [i_{p,q}], b, \ldots) = (\ldots, a, i_p, i_{p+1}, \ldots, i_q, b, \ldots),$$

for instance $(\ldots, 1, [(2,3)], 4, \ldots) = (\ldots, 1, 2, 3, 4, \ldots)$.

For $i_{p,p+q} \in \{1, \ldots, k\}_\infty, h_{r,r+q} \in \mathbb{Z}_\infty$ and $\alpha \in \{-1, 1\}$ we set

$$x_{i_{p,p+q}}^{h_{r,r+q}} = \prod_{t=0}^{q} x_{i_{p+t}}^{h_{r+t}}, \qquad \alpha h_{r,r+q} = (\alpha h_{r+t})_{t=0}^{q}$$

and we let $1_{p,p+q} \in \{1\}^{q+1}$ be the sequence consisting of 1's.

We say that the condition $W(i_{p,p+q})$ holds iff $p, q \in \mathbb{Z}$, $-1 \le q$, $i_{p,p+q} \in \{1, \ldots, k\}^{q+1}$ and any two consecutive i_j, i_{j+1} are different.

Let $l(y)$ be the length of a reduced word z, where $z = y \in \mathbb{F}_k$ (compare [3], Chapters 1.4 and 1.1). Every function $f \in K(\mathbb{F}_k)$ can be written in a unique way as

$$f = \sum_{y\in\mathbb{F}_k} a_y y,$$

and we set $d(f) = \max\{l(y) : a_y \ne 0\}$, $d(\mathbf{0}) = \infty$.

The characteristic function of $\{0\} \subseteq \mathbb{Z}$ is denoted by δ.

In the following to denote the value of a function f at an element which is a sequence $(i_p, \ldots, i_q)$ we often write $f(i_p, \ldots, i_q)$ instead of $f((i_p, \ldots, i_q))$ and this should not be misleading.

3. Auxiliary definitions. For calculating coefficients in the products of our symmetric functions we need a useful function L. In the proofs that some sets are algebraic bases we apply the orderings $<_1, \ldots, <_4$ defined below, and the functions I_1 and I_2 are used in proving algebraic independence and in defining $<_3$ and $<_4$.

Notice that L below depends only on its first argument and on whether other arguments are 0 or not.

3.1. DEFINITION. The function $L : \mathbb{N} \times \mathbb{Z}^4 \to K$ is given by

$$L(a,b,c,d,e) = \begin{cases} 1 & \text{if } a = 0, \\ 0 & \text{if } ab \neq 0, \\ (k-1)^{a-1} & \text{if } b = 0 \text{ and } ac \neq 0, \\ (k-2)(k-1)^{a-1} & \text{if } b = c = 0 \text{ and } a \neq 0 \neq de, \\ (k-1)^a & \text{if } b = c = de = 0 \text{ and } a(d^2+e^2) \neq 0, \\ k(k-1)^{a-1} & \text{if } b = c = d = e = 0 \text{ and } a \neq 0. \end{cases}$$

Notice that I_1 and I_2 below are injections.

3.2. DEFINITION. (a) Let $I_1 : (\mathbb{Z}\backslash\{0\})_\infty \to (\mathbb{Z}\backslash\{0\})_\infty$ be given by induction as follows: if $q, r \in \mathbb{N}_+$, $z \in \mathbb{Z}\backslash\{0\}$, $z_{1,q} \in (\mathbb{Z}\backslash\{0\})^q$ and $I_1(z_{1,q}) = \varepsilon_{1,r}$ then $I_1(\emptyset) = \emptyset$, $I_1(z) = \operatorname{sgn}(z)1_{1,|z|}$, and

$$I_1([z_{1,q}], z) = \begin{cases} ([\varepsilon_{1,r-1}], \varepsilon_r + \operatorname{sgn}(z), [\operatorname{sgn}(z)1_{1,|z|-1}]) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } \operatorname{sgn}(\varepsilon_r) = \operatorname{sgn}(z), \\ ([\varepsilon_{1,r}], [\operatorname{sgn}(z)1_{1,|z|}]) \qquad\quad \text{if } \operatorname{sgn}(\varepsilon_r) = -\operatorname{sgn}(z). \end{cases}$$

(b) Let $I_2 : (\{-1,1\}_\infty)_\infty \to (\mathbb{Z}\backslash\{0\})_\infty$ be given by the following induction: if $r \in \mathbb{N}_+$, $h \in \{-1,1\}_\infty$, $j \in (\{-1,1\}_\infty)_\infty\backslash\{\emptyset\}$ and $I_2(j) = \varepsilon_{1,r}$ then $I_2(\emptyset) = \emptyset$, $I_2(h) = (1, [h])$, and

$$I_2([j], h) = ([\varepsilon_{1,r-1}], \varepsilon_r + \operatorname{sgn}(\varepsilon_r), [\operatorname{sgn}(\varepsilon_r)h])\,.$$

3.3. DEFINITION. We define orderings $<_1, <_2$ and $<_3$ on $(\mathbb{Z}\backslash\{0\})_\infty$. Let $h_{1,q} \neq l_{1,s} \in (\mathbb{Z}\backslash\{0\})_\infty$. In the following for nonempty $h_{1,q}$, $l_{1,s}$ we write

$$\nu = \min\{t \in \{1,2,\ldots,\min\{q,s\}\} : h_t \neq l_t\}\,.$$

We define:

(a) $h_{1,q} <_1 l_{1,s}$ iff $\sum_{t=1}^q |h_t| > \sum_{u=1}^s |l_u|$ or $(\sum_{t=1}^q |h_t| = \sum_{u=1}^s |l_u|$ and $(|h_\nu| < |l_\nu|$ or $(|h_\nu| = |l_\nu|$ and ω_1 holds$)))$. The condition ω_1 is chosen to make $<_1$ a linear ordering; for instance, ω_1 holds iff $h_\nu = -l_\nu > 0$.

(b) $h_{1,q} <_2 l_{1,s}$ iff $\sum_{t=1}^q |h_t| > \sum_{u=1}^s |l_u|$ or $(\sum_{t=1}^q |h_t| = \sum_{u=1}^s |l_u|$ and $(|h_\nu| > |l_\nu|$ or $(|h_\nu| = |l_\nu|$ and ω_2 holds$)))$, where ω_2 is a condition making $<_2$ linear; for instance, ω_2 is equivalent to ω_1.

(c) $h_{1,q} <_3 l_{1,s}$ iff $I_1(h_{1,q}) <_2 I_1(l_{1,s})$.

3.4. DEFINITION. A linear ordering $<_4$ on $(\{-1,1\}_\infty)_\infty$ is given by the following formula:

$$h <_4 l \quad \text{iff} \quad I_2(h) <_2 I_2(l)\,,$$

where $h, l \in (\{-1,1\}_\infty)_\infty$.

4. The algebra m. We now introduce our first version of symmetric functions. These are functions $S(h) \in K(\mathbb{F}_k)$ (Definition 4.1) which are analogous to complete symmetric functions. The crucial Lemma 4.2, expressing the product of two $S(h)$'s as a linear combination of $S(h)$'s, shows that the linear subspace m of $K(\mathbb{F}_k)$ spanned by the $S(h)$'s is in fact a subalgebra.

Then we introduce two subsets n, e, which are analogs of the polynomials $\sum_i x_i^l$ and of elementary symmetric polynomials respectively. It turns out that both these sets are algebraic bases of m. Moreover, the Euler formula holds.

At the end we remark that m consists of functions invariant under a length preserving action of a product G of permutation groups.

4.1. Definition. If $h = \emptyset$ or h is a finite sequence of zeros then $S(h) = \mathbf{1}$, and

$$S(h) = \sum_{W(j_{1,s})} x_{j_{1,s}}^{l_{1,s}}$$

for other $h \in \mathbb{Z}_\infty$, where the sequence $l_{1,s}$ arises from h by omission of zeros.

The vector space spanned by the $S(h)$'s is denoted by m.

Every element $f \in m$ can be written in a unique way as

$$f = \sum_{h \in (\mathbb{Z}\setminus\{0\})_\infty} a_h S(h)\,,$$

where $a_h \in K$. We call a_h the coefficient of $S(h)$ in f.

Practical use of the following Lemma 4.2 is made easier by the fact that if $h_{q+1-u} + l_u \neq 0$ for an index u then

$$\sum_{w=1}^{t-1} |h_{q+1-w} + l_w| \neq 0 \qquad \text{for } t > u\,,$$

$$L\Big(t, \sum_{w=1}^{t-1} |h_{q+1-w} + l_w|, h_{q+1-t} + l_t, q-t, s-t\Big) = 0$$

and we actually sum over t until $h_{t+1} + l_t \neq 0$.

4.2. Lemma. *Let* $q, s \in \mathbb{N}$, $h_{1,q}$, $l_{1,s} \in (\mathbb{Z}\setminus\{0\})_\infty$ *and* $l_0 = h_{q+1} = 0$. *Then*

$$S(h_{1,q})S(l_{1,s}) = \sum_{t=0}^{\min(q,s)} L\Big(t, \sum_{u=1}^{t-1} |h_{q+1-u} + l_u|, h_{q+1-t} + l_t, q-t, s-t\Big) \cdot S([h_{1,q-t}], h_{q+1-t} + l_t, [l_{t+1,s}])\,.$$

Proof. If $q = 0$ then $S(\emptyset)S(l_{1,s}) = S(l_{1,s})$, and similarly for $s = 0$. Let $q, s \geq 1$. Set $v = \max\{t \in \{0, 1, \ldots, \min(q,s)\} : h_{q+1} + l_0 = h_q + l_1 = \ldots = h_{q+1-t} + l_t = 0\}$. If $v = 0$ then

$$S(h_{1,q})S(l_{1,s}) = S([h_{1,q}], [l_{1,s}]) + S([h_{1,q-1}], h_q + l_1, [l_{2,s}])\,.$$

Let $v > 0$. Then

$$
\begin{aligned}
&S(h_{1,q})S(l_{1,s}) \\
&= \Big(\sum_{W(i_{1,q})} x_{i_{1,q}}^{h_{1,q}}\Big)\Big(\sum_{W(j_{1,s})} x_{j_{1,s}}^{l_{1,s}}\Big) = \sum_{W(i_{1,q+s})} x_{i_{1,q+s}}^{([h_{1,q}],[l_{1,s}])} \\
&\quad + (k-2) \sum_{W(i_{1,q+s-2})} x_{i_{1,q+s-2}}^{([h_{1,q-1}],[l_{2,s}])} \\
&\quad + (k-2)(k-1) \sum_{W(i_{1,q+s-4})} x_{i_{1,q+s-4}}^{([h_{1,q-2}],[l_{3,s}])} + \ldots \\
&\quad + (k-2)(k-1)^t \sum_{W(i_{1,q+s-2(t+1)})} x_{i_{1,q+s-2(t+1)}}^{([h_{1,q-1-t}],[l_{t+2,s}])} + \ldots \\
&\quad + \begin{cases} (k-2)(k-1)^{v-1} \sum_{W(i_{1,q+s-2v})} x_{i_{1,q+s-2v}}^{([h_{1,q-v}],[l_{v+1,s}])} & \\ \quad + (k-1)^v \sum_{W(i_{1,q+s-1-2v})} x_{i_{1,q+s-1-2v}}^{([h_{1,q-v-1}],h_{q-v}+l_{v+1},[l_{v+2,s}])} & \\ & \text{if } v < \min\{q,s\}, \\ (k-1)^v \sum_{W(i_{1,q+s-2v})} x_{i_{1,q+s-2v}}^{([h_{1,q-v}],[l_{v+1,s}])} & \\ & \text{if } v = \min\{q,s\} < \max\{q,s\}, \\ k(k-1)^{v-1} & \text{if } v = q = s. \ \blacksquare \end{cases}
\end{aligned}
$$

4.3. COROLLARY. $m < K(\mathbb{F}_k)$. ■

4.4. DEFINITION. We put $n = \{\sum_{i=1}^k x_i^l : l \in \mathbb{Z}\backslash\{0\}\}$. These are analogs of the polynomials $\sum_i x_i^l$.

4.5. THEOREM. (a) $\mathrm{Alg}(\{f \in n : d(f) \le i\}) = \mathrm{Alg}(\{g \in m : d(g) \le i\})$ *for every* $i \in \mathbb{N}$.

(b) $\mathrm{Alg}(n) = m$.

Proof. To show that if $q \in \mathbb{N}$, $h_{1,q} \in (\mathbb{Z}\backslash\{0\})^q$ and $S(h_{1,q}) \in \{g \in m : d(g) \le i\}$ then $S(h_{1,q}) \in \mathrm{Alg}(\{f \in n : d(f) \le i\})$ we apply induction on q. We have $S(h_{1,1}) \in n$. If $q > 1$ then, by Lemma 4.2,

$$
\begin{aligned}
S(h_{1,q}) &= S(h_1)S(h_{2,q}) - L(1,0,h_1+h_2,0,q-2)S(h_1+h_2,[h_{3,q}]) \\
&\in \mathrm{Alg}(\{f \in n : d(f) \le i\}) \quad \text{ by the inductive assumption. } \blacksquare
\end{aligned}
$$

4.6. THEOREM. *The set n is algebraically independent. Thus it forms an algebraic basis of m.*

Proof. Every polynomial f over K with elements of n as noncommutative variables is of the form

$$
f = \sum_{q\in\mathbb{N},h_{1,q}\in(\mathbb{Z}\backslash\{0\})_\infty} a_{h_{1,q}} P(h_{1,q}),
$$

where $a_{h_{1,q}} \in K$, $P(h_{1,q}) = \prod_{t=1}^{q} S(h_t)$ and all but finitely many $a_{h_{1,q}}$ are equal to 0.

All the elements $S(l_{1,s}) \in K(\mathbb{F}_k)$, where $l_{1,s} \in (\mathbb{Z}\backslash\{0\})_\infty$, appearing with nonzero coefficients in $P(h_{1,q}) \in K(\mathbb{F}_k)$ for an $h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty$, satisfy $\sum_{t=1}^{q} |h_t| \geq \sum_{u=1}^{s} |l_u|$. If equality holds then every l_u is a sum of some h_t's which are of the same sign and

$$\begin{aligned}&(|l_1| > |h_1| \text{ or } (l_1 = h_1 \text{ and } |l_2| > |h_2|) \text{ or}\\ &(l_1 = h_1 \text{ and } l_2 = h_2 \text{ and } |l_3| > |h_3|) \text{ or } \ldots \text{ or}\\ &(l_1 = h_1 \text{ and } l_2 = h_2 \text{ and } \ldots \text{ and } l_q = h_q))\,,\end{aligned}$$

which means that $h_{1,q} <_1 l_{1,s}$.

Therefore, $S(l_{1,s})$ appears with coefficient 0 in $P(h_{1,q}) \in K(\mathbb{F}_k)$ if $h_{1,q} >_1 l_{1,s}$ and $l_{1,s} \neq h_{1,q}$.

Now, by induction in $(\mathbb{Z}\backslash\{0\})_\infty$ with respect to $<_1$, one can show that $a_{h_{1,q}}$ is the coefficient of $S(h_{1,q})$ in f and therefore $a_{h_{1,q}} = 0$. ∎

4.7. Definition. Let $e = \{S(\operatorname{sgn}(i)1_{1,|i|}) : i \in \mathbb{Z}\backslash\{0\}\}$; these are analogs of elementary symmetric polynomials.

4.8. Proposition (Euler formula). *If* $i \in \mathbb{N}_+$ *and* $\varepsilon \in \{-1, 1\}$ *then*

$$\sum_{t=0}^{i} (-1)^t S(\varepsilon 1_{1,t}) S(\varepsilon(i-t)) = \sum_{t=0}^{i} (-1)^t S(\varepsilon(i-t)) S(\varepsilon 1_{1,t}) = 0\,.$$

Proof. It suffices to apply Lemma 4.2 and to consider the differences between the products for t and $t+1$. ∎

4.9. Proposition. (a) $\operatorname{Alg}(\{f \in e : d(f) \leq i\}) = \operatorname{Alg}(\{g \in m : d(g) \leq i\})$ *for every* $i \in \mathbb{N}$.

(b) $\operatorname{Alg}(e) = m$.

Proof. (a) is a consequence of Theorem 4.5 and Proposition 4.8. ∎

4.10. Theorem. *The set* e *is algebraically independent. Thus it forms an algebraic basis of* m.

Proof. Let

$$Q(h_{1,q}) = \prod_{t=1}^{q} S(I_1(h_t)) \qquad \text{for } h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty$$

(I_1 is defined in 3.2) and let

$$f = \sum_{q \in \mathbb{N}, h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty} a_{h_{1,q}} Q(h_{1,q}) = 0\,,$$

where $a_{h_{1,q}} \in K$ and all but finitely many $a_{h_{1,q}}$ are 0.

All the elements $S(l_{1,s}) \in K(\mathbb{F}_k)$, where $l_{1,s} \in (\mathbb{Z}\backslash\{0\})_\infty$, which have nonzero coefficients in a fixed $Q(h_{1,q})$, satisfy

$$\sum_{t=1}^{q} |h_t| \geq \sum_{u=1}^{s} |l_u| \,.$$

If equality holds then every l_u is a sum

$$l_u = \operatorname{sgn}(h_t) + \operatorname{sgn}(h_{t+1}) + \ldots + \operatorname{sgn}(h_p)\,,$$

with all signs equal to 1 or all signs equal to -1. Therefore, $I_1(h_{1,q}) <_2 l_{1,s}$. Moreover, the coefficient of $S(I_1(h_{1,q}))$ in $Q(h_{1,q})$ is 1.

Finally, one can apply induction in $(\mathbb{Z}\backslash\{0\})_\infty$ with respect to $<_3$ and show that each $a_{h_{1,q}}$ is the coefficient of $S(I_1(h_{1,q}))$ in f and therefore $a_{h_{1,q}} = 0$. ∎

4.11. Remark. The algebra m consists of functions invariant under a length preserving action of the group $G = S_k \times (S_{k-1})^\infty$ on $K(\mathbb{F}_k)$, where S_l denotes the permutation group of $\{1, \ldots, l\}$. The action does not preserve multiplication in $\mathbb{F}_k$ for $k > 2$. It is defined as follows.

Let $i\langle j\rangle = i - 1 + \operatorname{sgn}(j - i)$ for $i, j \in \mathbb{N}$ and let

$$\phi : \{i_{1,q} : q \in \mathbb{N}_+ \text{ and } W(i_{1,q}) \text{ holds}\} \to \{1, \ldots, k\} \times \{1, \ldots, k-1\}_\infty$$

be defined by the formula

$$\phi(i_{1,q}) = (i_1, i_2\langle i_1\rangle, i_3\langle i_2\rangle, \ldots, i_q\langle i_{q-1}\rangle)\,.$$

Notice that ϕ is a bijection.

The group G acts on $K(\mathbb{F}_k)$ in the following way:

$$(\sigma f)(\mathbf{e}) = f(\mathbf{e}), \qquad (\sigma f)(x_{i_{1,q}}^{h_{1,q}}) = f(x_{\phi^{-1}\sigma\phi(i_{1,q})}^{h_{1,q}})\,,$$

where $f \in K(\mathbb{F}_k)$, $\sigma \in G$, $q \in \mathbb{N}_+$, $W(i_{1,q})$ holds, $h_{1,q} \in (\mathbb{Z}\backslash\{0\})^q$ and $\mathbf{e}$ denotes the unit of $\mathbb{F}_k$.

5. The algebra C. We study a second version of symmetric functions: linear combinations of $S_C(h)$'s (Definition 5.1). This again turns out to be an algebra with a basis e_C. The elements of C are functions invariant under a length preserving action of a group G_C.

5.1. Definition. (a) Let $S_C(h) \in K(\mathbb{F}_k)$ be defined as follows: $S_C(h) = 1 \in K(\mathbb{F}_k)$ if $h = \emptyset$ or h is a finite sequence of zeros, and $S_C(h) = S(h) + S(-h)$ for other $h \in \mathbb{Z}_\infty$. These are analogs of the complete symmetric functions.

(b) The set C of all linear combinations of $S_C(h)$, where $h \in \mathbb{Z}_\infty$, is an analog of the set of symmetric polynomials.

Every element $f \in C$ can be written in a unique way as

$$f = \sum_{h\in\{\emptyset\}\cup\mathbb{N}_+\times(\mathbb{Z}\backslash\{0\})_\infty} a_h S_C(h)\,,$$

where $a_h \in K$. We call a_h the coefficient of $S_C(h)$ in f.

To make the use of the following Lemma 5.2 easier notice that if $h_{q+1-u}+\varepsilon l_u \neq 0$ for $1 \le u < t$ then $L_{\varepsilon,t} = 0$, and in the formula of Lemma 5.2 we actually sum over t until $h_{q+1-t} + \varepsilon l_t \neq 0$.

5.2. Lemma (an application of Lemma 4.2). *Let* $q, s \in \mathbb{N}$, $h_{1,q}, l_{1,s} \in (\mathbb{Z}\backslash\{0\})_\infty$, $l_0 = h_{q+1} = 0$, *and for* $\varepsilon \in \{-1,1\}$, $t \in \{1,2,\ldots,\min(q,s)\}$ *let*

$$L_{\varepsilon,t} = L\Big(t, \sum_{u=1}^{t-1} |h_{q+1-u} + \varepsilon l_u|, h_{q+1-t} + \varepsilon l_t, q-t, s-t\Big)\,,$$
$$S_{\varepsilon,t} = S_C([h_{1,q-t}], h_{q+1-t} + \varepsilon l_t, [\varepsilon l_{t+1,s}])\,.$$

Then

$$S_C(h_{1,q})S_C(l_{1,s}) = \sum_{\varepsilon\in\{-1,1\}} \sum_{t=0}^{\min(q,s)} 2^{\delta(d(S_{\varepsilon,t}))-\delta(q+s)} \cdot (1-\delta(qs+1+\varepsilon))L_{\varepsilon,t}S_{\varepsilon,t}\,. \ \blacksquare$$

5.3. Corollary. $C < K(\mathbb{F}_k)$. ■

5.4. Definition. The set $e_C \subseteq C$ we now define is the analog of the set of elementary symmetric polynomials. Let

$$e_C = \{S_C(h) : h \in \{-1,1\}_\infty\}\,.$$

5.5. Proposition (an application of Lemma 5.2). *Let* $q \in \mathbb{N}_+$, $h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty\backslash\{-1,1\}_\infty$ *and* $v = \min\{t \in \{1,\ldots,q\} : h_t \notin \{-1,1\}\}$. *Then*

$$\begin{aligned} S_C(h_{1,q}) &= S_C([h_{1,v-1}], \operatorname{sgn}(h_v))S_C(h_v - \operatorname{sgn}(h_v), [h_{v+1,q}]) \\ &\quad - \sum_{\varepsilon\in\{-1,1\}} S_C([h_{1,v-1}], \operatorname{sgn}(h_v), \varepsilon(h_v - \operatorname{sgn}(h_v)), \varepsilon[h_{v+1,q}]) \\ &\quad - 2^{\delta(|h_v|+q-3)} L(1, 0, 2\operatorname{sgn}(h_v) - h_v, v-1, q-v) \\ &\quad \cdot S_C([h_{1,v-1}], 2\operatorname{sgn}(h_v) - h_v, -[h_{v+1,q}]) \\ &\quad - \sum_{\varepsilon\in\{-1,1\}} \sum_{t=2}^{\min(v,q+1-v)} L'_{\varepsilon,t} S'_{\varepsilon,t}, \end{aligned}$$

where $L'_{\varepsilon,t}$ *and* $S'_{\varepsilon,t}$ *are as in Lemma* 5.2. ■

5.6. Theorem. (a) $\mathrm{Alg}(\{f \in e_C : d(f) \le i\}) = \mathrm{Alg}(\{g \in C : d(g) \le i\})$ *for every* $i \in \mathbb{N}$.

(b) $\mathrm{Alg}(e_C) = C$.

Proof. First $\mathrm{Alg}(\{f \in e_C : d(f) \le 0\}) = K = \mathrm{Alg}(\{g \in C : d(g) \le 0\})$.

Let (a) hold for $i < j$, where $j > 0$. To show that if $h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty$ and $d(S_C(h_{1,q})) \le j$ then $S_C(h_{1,q}) \in \mathrm{Alg}(\{f \in e_C : d(f) \le j\})$, apply Proposition 5.5 and induction in $(\mathbb{Z}\backslash\{0\})_\infty$ with respect to $<_1$. ∎

5.7. THEOREM. *The set e_C is algebraically independent. Thus it is an algebraic basis of C.*

Proof. Let

$$f = \sum_{h_{1,q} \in (\{-1,1\}_\infty)_\infty} a_{h_{1,q}} \prod_{t=1}^{q} S_C(1, [h_t]) = 0\,,$$

where $a_{h_{1,q}} \in K$ and all but finitely many $a_{h_{1,q}}$ are 0.

To show that every $a_{h_{1,q}} = 0$ it is enough to apply induction in $(\{-1,1\}_\infty)_\infty$ with respect to $<_4$ showing that $a_{h_{1,q}}$ is the coefficient of $S_C(I_2(h_{1,q}))$ in f and therefore $a_{h_{1,q}} = 0$, similarly to Theorems 4.6 and 4.10. ∎

5.8. Remark. The algebra C consists of functions invariant under the following length preserving action of the group $G_C = G \times \mathbb{Z}_2$ on $K(\mathbb{F}_k)$, where $\mathbb{Z}_2 = (\{-1,1\}, \cdot)$ (compare Remark 4.11): if $\sigma \in G$, $\varepsilon \in \{-1,1\}$, $W(i_{1,q})$ holds, $h_{1,q} \in (\mathbb{Z}\backslash\{0\})^q$ and $f \in K(\mathbb{F}_k)$ then

$$((\sigma,\varepsilon)f)(x_{i_{1,q}}^{h_{1,q}}) = (\sigma f)(x_{i_{1,q}}^{\varepsilon h_{1,q}})\,.$$

6. The algebra M. We give a third version of analogs: an algebra M with bases N and E. Elements of M are invariant under an action of a group G_M.

6.1. DEFINITION. (a) We now define functions $S_M(h) \in K(\mathbb{F}_k)$ which are analogs of the complete symmetric functions. Let $S_M(h) = 1$ if either $h = \emptyset$ or h is a finite sequence of zeros, and

$$S_M(h) = \sum_{\varepsilon_{1,s} \in \{-1,1\}^s} S(\varepsilon_1 l_1, \ldots, \varepsilon_s l_s)$$

for other $h \in \mathbb{Z}_\infty$, where the sequence $l_{1,s}$ is obtained from h by omission of zeros.

(b) The set of all linear combinations of the $S_M(h)$, where $h \in \mathbb{Z}_\infty$, is denoted by M. This is an analog of the algebra of symmetric polynomials.

Applying Lemma 4.2 we have

6.2. LEMMA. *Let $q, s \in \mathbb{N}$, $h_{1,q}, l_{1,s} \in (\mathbb{Z}\backslash\{0\})_\infty$ and $h_{q+1} = l_0 = \varepsilon_0 = 0$. Then*

$$S_M(h_{1,q})S_M(l_{1,s}) = \sum_{t=0}^{\min(q,s)} \sum_{\varepsilon_{1,t} \in \{-1,1\}^t} 2^{t-1+\delta(h_{q+1-t}+\varepsilon_t l_t)}$$

$$\cdot L\Big(t,\sum_{u=1}^{t-1}|h_{q+1-u}+\varepsilon_u l_u|, h_{q+1-t}+\varepsilon_t l_t, q-t, s-t\Big)$$
$$\cdot S_M([h_{1,q-t}], h_{q+1-t}+\varepsilon_t l_t, [l_{t+1,s}])\,. \ \blacksquare$$

Similarly to Lemmas 4.2 and 5.2 we can stop the summation in Lemma 6.2 when $h_{q+1-t}+\varepsilon_t l_t \neq 0$ (compare remarks before 4.2 and 5.2).

6.3. Corollary. *$M < K(\mathbb{F}_k)$.* $\blacksquare$

6.4. Definition. We now define a set $N \subseteq M$ which is an analog of the set of the polynomials $\sum_i x_i^l$. We put $N = \{S_M(i) : i \in \mathbb{N}_+\}$.

6.5. Theorem. (a) $\mathrm{Alg}(\{f \in N : d(f) \le i\}) = \mathrm{Alg}(\{g \in M : d(g) \le i\})$ *for every $i \in \mathbb{N}$.*

(b) $\mathrm{Alg}(N) = M$.

Proof. Lemma 6.2 implies that

$$S_M(h_{1,q}) = S_M(h_1)S_M(h_{2,q}) - \sum_{\varepsilon\in\{-1,1\}} 2^{\delta(h_1+\varepsilon h_2)}L(1,0,h_1+\varepsilon h_2,0,q-2)S_M(h_1+\varepsilon h_2,[h_{3,q}])$$

for $2 \le q \in \mathbb{N}$ and $h_{1,q} \in (\mathbb{Z}\backslash\{0\})_\infty$. To prove that $S_M(h_{1,q}) \in \mathrm{Alg}(\{f \in N : d(f) \le i\})$ if $d(S_M(h_{1,q})) \le i$, use induction on q. $\blacksquare$

6.6. Theorem. *The set N is an algebraic basis of M.*

Proof. To show the algebraic independence apply induction with respect to $<_1$ considered in $(\mathbb{N}_+)_\infty$ (compare Theorem 4.6). $\blacksquare$

6.7. Definition. We define a set $E \subseteq M$ which is an analog of the elementary symmetric polynomials. We put $E = \{S_M(1_{1,q}) : q \in \mathbb{N}_+\}$.

The connection between elements of E and N is given in Proposition 6.8 which follows from Lemma 6.2.

6.8. Proposition (Euler formula). *Let $i \in \mathbb{N}_+$ and $\varepsilon \in \{-1,1\}$. Then*

$$\sum_{t=0}^{i}(-1)^t S_M(1_{1,t})S_M(i-t) = \sum_{t=1}^{i-1}(-1)^t 2^{\delta(i-1-t)}L(1,0,i-1-t,t-1,0)S_M(1_{1,t-1},i-1-t)$$

and

$$\sum_{t=0}^{i}(-1)^t S_M(t)S_M(1_{1,i-t})$$

$$= \sum_{t=1}^{i-1} (-1)^t 2^{\delta(i-1-t)} L(1,0,t-1,0,i-t-1) S_M(t-1, 1_{1,i-t-1}) . \ \blacksquare$$

6.9. THEOREM. (a) $\mathrm{Alg}(\{f \in E : d(f) \le i\}) = \mathrm{Alg}(\{g \in M : d(g) \le i\})$ *for every* $i \in \mathbb{N}$.

(b) $\mathrm{Alg}(E) = M$.

Proof. (a) We apply Theorem 6.5, Proposition 6.8 and induction on i. $\blacksquare$

6.10. THEOREM. *The set E is algebraically independent. Thus it is an algebraic basis of M.*

Proof. This follows from Theorem 5.7 because

$$S_M(1_{1,q}) = \sum_{\varepsilon \in \{-1,1\}^{q-1}} S_C(1, [\varepsilon]) \qquad \text{for } q \in \mathbb{N}_+ . \ \blacksquare$$

6.11. Remark. The algebra M consists of functions invariant under the following length preserving action of the group $G_M = G \times (\mathbb{Z}_2)^\infty$ on $K(\mathbb{F}_k)$ (compare Remarks 4.10 and 5.8): if $\sigma \in G$, $\varepsilon = (\varepsilon_t)_{t=1}^\infty \in (\mathbb{Z}_2)^\infty$, $W(i_{1,q})$ holds, $h_{1,q} \in (\mathbb{Z}\backslash\{0\})^q$ and $f \in K(\mathbb{F}_k)$ then

$$((\sigma,\varepsilon)f)(x_{i_{1,q}}^{h_{1,q}}) = (\sigma f)(x_{i_{1,q}}^{(\varepsilon_1 h_1, \ldots, \varepsilon_q h_q)}) .$$

7. The algebra λ. We give a version of analogs in the case of the algebra $K(\mathbb{P}_k)$ which consists of noncommutative polynomials. We introduce an algebra λ with algebraic bases e_λ and n_λ; the Euler formula also holds in this setting. The algebra λ consists of polynomials invariant under the action of the group G on $K(\mathbb{P}_k)$. We show, in the case of $k > 2$ generators, that the algebra Λ_{perm} of noncommutative polynomials invariant under permutations of generators cannot be generated by a sum of λ and a finite number of elements of Λ_{perm}.

7.1. DEFINITION. (a) We introduce analogs λ, e_λ and n_λ of the sets of symmetric polynomials, elementary symmetric polynomials and the polynomials $\sum_i x_i^l$ respectively:

$$\lambda = m \cap K(\mathbb{P}_k), \quad e_\lambda = e \cap K(\mathbb{P}_k) \quad \text{and} \quad n_\lambda = n \cap K(\mathbb{P}_k) .$$

(b) The functions $S(h) \in K(\mathbb{P}_k)$ for $h \in \mathbb{N}_\infty$ are analogs of the complete symmetric functions.

7.2. LEMMA (a special case of Lemma 4.2). *Let $q, s \in \mathbb{N}$, $h_{1,q}, l_{1,s} \in (\mathbb{N}_+)_\infty$ and $l_0 = h_{q+1} = 0$. Then*

$$S(h_{1,q})S(l_{1,s}) = S([h_{1,q}],[l_{1,s}]) + (1-\delta(qs))S([h_{1,q-1}], h_q + l_1, [l_{2,s}]) . \ \blacksquare$$

7.3. COROLLARY. $\lambda < K(\mathbb{P}_k)$. $\blacksquare$

7.4. PROPOSITION (follows from Lemma 7.2). *Let* $q \in \mathbb{N}_+$ *and* $h_{1,q} \in (\mathbb{N}_+)_\infty$. *Then*

$$\begin{aligned} S(h_{1,q}) &= \sum_{t=1}^{q} (-1)^{t+1} S(h_1 + h_2 + \ldots + h_t) S(h_{t+1,q}) \\ &= \sum_{t=0}^{q-1} (-1)^{q-t-1} S(h_{1,t}) S(h_{t+1} + h_{t+2} + \ldots + h_q) \,. \ \blacksquare \end{aligned}$$

7.5. THEOREM. (a) (an application of Proposition 7.4 and induction on q). $\mathrm{Alg}(\{f \in n_\lambda : d(f) \le i\}) = \mathrm{Alg}(\{g \in \lambda : d(g) \le i\})$ *for every* $i \in \mathbb{N}$.

(b) $\mathrm{Alg}(n_\lambda) = \mathrm{Alg}(\lambda)$. ■

7.6. THEOREM. *The set* n_λ *is an algebraic basis of* λ.

Proof. The fact that n_λ is algebraically independent follows from Theorem 4.6 because $n_\lambda \subseteq n$. ■

7.7. PROPOSITION (Euler formula). *If* $i \in \mathbb{N}_+$ *then*

$$\sum_{t=0}^{i} (-1)^t S(1_{1,t}) S(i-t) = \sum_{t=0}^{i} (-1)^t S(i-t) S(1_{1,t}) = 0 \,.$$

Proof. This is a special case of Proposition 4.8 if $\varepsilon = 1$. ■

7.8. THEOREM. (a) $\mathrm{Alg}(\{f \in e_\lambda : d(f) \le i\}) = \mathrm{Alg}(\{g \in \lambda : d(g) \le i\})$ *for every* $i \in \mathbb{N}$.

(b) $\mathrm{Alg}(e_\lambda) = \mathrm{Alg}(\lambda)$.

Proof. (a) We apply Theorem 7.5, Proposition 7.7 and induction on i. ■

7.9. THEOREM. *The set* e_λ *is an algebraic basis of* λ.

Proof. The algebraic independence of e_λ follows from Theorem 4.10 because $e_\lambda \subseteq e$. ■

7.10. Remarks. (a) The algebra λ consists of functions invariant under the length preserving action of the group G on $K(\mathbb{P}_k)$ (compare Remark 4.11).

(b) $M < C < m$ and $\lambda < m$.

We denote by $\Lambda_{\mathrm{perm}} < K(\mathbb{P}_k)$ the subalgebra composed of functions invariant under permutations of the generators $x_1, \ldots, x_k$ of $\mathbb{P}_k$. It is clear from the definitions that $\lambda < \Lambda_{\mathrm{perm}}$. Moreover, $\lambda = \Lambda_{\mathrm{perm}}$ for $k = 2$, which is essential in the proof of the following theorem.

7.11. THEOREM. *If* $k > 2$ *then the algebra* Λ_{perm} *cannot be obtained as an algebra generated by* λ *and a finite number of elements of* Λ_{perm}.

Proof. Let $t \in \mathbb{N}_+$, $f_1, \ldots, f_t \in \Lambda_{\mathrm{perm}} \backslash \{0\}$, $T = \lambda \cup \{f_1, \ldots, f_t\}$ and $r = \max\{d(f_u) : u \in \{1, \ldots, t\}\} + 1$, where the degree $d(f_u)$ is defined in Section 2.

In order to show that $\Lambda_{\mathrm{perm}} \backslash \operatorname{Alg}(T) \neq \emptyset$ we consider

$$h = \sum_{i \neq j} x_i x_j^r x_i \in \Lambda_{\mathrm{perm}} .$$

It is clear that $h(x_1 x_2^r x_1) = 1 \neq 0 = h(x_1 x_2^r x_3)$. We prove that $h \notin \operatorname{Alg}(T)$.

Let $p \in \mathbb{N}_+$ and let $g_v \in T \backslash K$ for $v \in \{1, \ldots, p\}$. For every $s \in \mathbb{N}$ we obtain

$$g_p(x_2^s x_1) = g_p(x_2^s x_3)$$

because $g_p \in \Lambda_{\mathrm{perm}}$ and

$$g_p(x_1 x_2^r x_1) = g_p(x_1 x_2^r x_3)$$

because if one of $g_p(x_1 x_2^r x_1)$, $g_p(x_1 x_2^r x_3)$ is nonzero, then $d(g_p) > r$ and $g_p \in \lambda$. Therefore,

$$(g_1 g_2 \ldots g_p)(x_1 x_2^r x_1) = (g_1 g_2 \ldots g_p)(x_1 x_2^r x_3) .$$

This yields that the function h cannot be a linear combination of such products $g_1 g_2 \ldots g_p$, which means that $h \notin \operatorname{Alg}(T)$. ■

REFERENCES

[1] C. Faith, *Algebra*: *Rings*, *Modules and Categories I*, Springer, Berlin, 1973.

[2] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[3] W. Magnus, A. Karras and D. Solitar, *Combinatorial Group Theory*, Pure and Appl. Math., Interscience Publishers, New York, 1966.

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