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COMPACTNESS PROPERTIES OF THE INTEGRATION MAP ASSOCIATED WITH A VECTOR MEASURE

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The importance of the theory of vector measures in many aspects of modern analysis is by now well established; see, for example, [4, 6, 7] and the references therein. Curiously though, a knowledge of the L^1-spaces of vector measures is somewhat incomplete. Although certain aspects of such spaces (e.g. completeness [7] and lattice properties [1, 7]) are well known there are other aspects (e.g. the dual space, separability) which are not so well understood. For some recent work on the nature of these spaces we refer to [1, 9]. Not surprisingly, these spaces are often very different in nature from the classical L^1-spaces of scalar measures.

A natural operator associated with an X-valued vector measure μ is its integration map $I_\mu : L^1(\mu) \to X$ given by $I_\mu f = \int f\, d\mu$, for every $f \in L^1(\mu)$. The properties of this operator (which is always linear and continuous) are closely related to the nature of $L^1(\mu)$. Our aim is to investigate compactness properties of such operators. We remark that many classical operators (e.g. the Fourier transform, Volterra integral operators, compact scalar-type operators) are integration maps I_μ, for suitable μ, or restrictions of such maps I_μ; see [10], for example.

To be more precise, let X be a Banach space. If X is reflexive, then I_μ is necessarily weakly compact. However, if X is non-reflexive, then to produce examples of (non-trivial) weakly compact integration maps I_μ is not so immediate. One of the problems is the difficulty of identifying the space $L^1(\mu)$ and, in those cases when an identification is actually possible, it often turns out that I_μ is not weakly compact (see [10]). The first simplification of the problem is that attention may be restricted to non-reflexive spaces X that are weakly compactly generated (which includes all separable spaces). This is because the image of I_μ is contained in the closed subspace X_μ, of X, generated by the range of μ (which is always a relatively weakly compact set). The second point is the characterization of weakly compact maps I_μ

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as precisely those arising from measures μ which factor through a reflexive Banach space; see Proposition 2.1.

Combining these two observations provides a method of constructing weakly compact maps I_μ for a certain class of ℓ^1-valued measures μ; see Section 3. Of course, because of the special properties of ℓ^1 such maps I_μ are also compact. It is even possible, via this construction, to determine the subclass of such measures μ which correspond to nuclear integration maps I_μ (cf. Proposition 3.6). Moreover, using the fact that every nuclear map between Banach spaces factors through ℓ^1, it is possible to characterize those nuclear integration maps $I_\mu : L^1(\mu) \to X$ with values in an arbitrary Banach space X (cf. Proposition 3.12).

Using the class of ℓ^1-valued measures constructed in Section 3 it is possible to exhibit non-reflexive spaces X and X-valued measures whose associated integration map is weakly compact but not compact; see Example 3.13. To produce such examples in reflexive spaces is easier: it suffices to note that separable, cyclic Banach spaces X (which include many reflexive spaces) are always isomorphic to $L^1(\mu)$, via the integration map $I_\mu : L^1(\mu) \to X$, for some suitable vector measure μ (see [5; Corollary 1.5]). Finally, it is straightforward to exhibit non-trivial (i.e. X_μ is not finite-dimensional) measures μ in a reflexive space X such that I_μ is both compact and weakly compact. Indeed, it suffices to take any ℓ^1-valued measure ν (of the type constructed in Section 3) for which I_ν is weakly compact (and so, also compact) and consider the measure $\mu = J_p \circ \nu$ with values in $X = \ell^p$, for any $1 < p < \infty$, where $J_p : \ell^1 \to \ell^p$ is the natural inclusion.

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1. Preliminaries. All vector spaces to be considered are over the scalar field, either real or complex. Let X be a Banach space with norm $\|\cdot\|$ and continuous dual space X'. The *dual* of a continuous linear map T from X into a Banach space Y is the linear map $T' : Y' \to X'$ defined by

$$\langle T'y', x\rangle = \langle y', Tx\rangle, \quad y' \in Y',\ x \in X\,,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between a Banach space and its dual space.

A sequence $\{x_n\}_{n=1}^\infty$ in X is said to be *summable* if there exists $x \in X$, called the sum of the sequence, such that $\lim_{N\to\infty} \|x - \sum_{n=1}^N x_n\| = 0$. A sequence $\{x_n\}_{n=1}^\infty$ is called *unconditionally summable* if each of its subsequences is summable in X. Finally, a sequence $\{x_n\}_{n=1}^\infty$ in X is said to be *absolutely summable* if $\sum_{n=1}^\infty \|x_n\| < \infty$.

LEMMA 1.1. *Let X be a Banach space and $\mathbf{j}$ be an X-valued continuous linear injection with domain a Banach space Y not containing an isomorphic copy of ℓ^∞. Then a sequence $\{y_n\}_{n=1}^\infty$ in Y is unconditionally summable if and only if every subsequence of $\{\mathbf{j}(y_n)\}_{n=1}^\infty$ is summable, in X, to an element of $\mathbf{j}(Y)$.*

Proof. Let $\{y_{n(k)}\}_{k=1}^\infty$ be a subsequence of $\{y_n\}_{n=1}^\infty$. Then there exists $y \in Y$ such that $\mathbf{j}(y)$ is the sum, in X, of the sequence $\{\mathbf{j}(y_{n(k)})\}_{k=1}^\infty$. In particular,

$$\lim_{N\to\infty} \Big\langle \mathbf{j}'(x'), y - \sum_{k=1}^N \mathbf{j}(y_{n(k)}) \Big\rangle = 0, \qquad x' \in X'.$$

Since $\mathbf{j}$ is injective, the set $\mathbf{j}'(X')$ separates points of Y. Now apply [4; Corollary I.4.7] to conclude that $\{y_n\}_{n=1}^\infty$ is unconditionally summable in Y.

The converse implication is clear. ■

Let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set Ω. Let $\mu : \mathcal{S} \to X$ be a *vector measure*, meaning that $\{\mu(E_n)\}_{n=1}^\infty$ is unconditionally summable in X (with $\sum_{n=1}^\infty \mu(E_n) = \mu(\bigcup_{n=1}^\infty E_n)$) for any sequence of pairwise disjoint sets $E_n \in \mathcal{S}$, $n = 1, 2, \ldots$ For every $x' \in X'$, let $\langle x', \mu\rangle$ denote the scalar measure defined by

$$\langle x', \mu\rangle(E) = \langle x', \mu(E)\rangle, \qquad E \in \mathcal{S},$$

and let $|\langle x', \mu\rangle|$ denote its total variation measure. A scalar-valued, $\mathcal{S}$-measurable function f on Ω is called *μ-integrable* if it is $\langle x', \mu\rangle$-integrable, for every $x' \in X'$, and if there is a unique set function $f\mu : \mathcal{S} \to X$ such that

$$\langle x', (f\mu)(E)\rangle = \int_E f \, d\langle x', \mu\rangle, \qquad x' \in X',\ E \in \mathcal{S}.$$

By the Orlicz–Pettis theorem (see [4; Corollary I.4.4]), $f\mu$ is also a vector measure. The element $(f\mu)(E)$ is also denoted by $\int_E f\, d\mu$, $E \in \mathcal{S}$.

Let $E \in \mathcal{S}$. Then $E \cap \mathcal{S}$ denotes the σ-algebra of sets $\{E \cap F : F \in \mathcal{S}\}$. The characteristic function of a set $F \subseteq \Omega$ is denoted by χ_F.

LEMMA 1.2 ([8; Proposition 8]). *A scalar-valued function f on Ω is μ-integrable if and only if there exist scalars c_n and sets $E(n) \in \mathcal{S}$, $n = 1, 2, \ldots,$ such that*

(i) *the sequence $\{c_n\mu(F_n)\}_{n=1}^\infty$ is unconditionally summable in X, for every choice of $F_n \in E(n) \cap \mathcal{S}$, $n = 1, 2, \ldots,$ and*

(ii) *the identity $f(\omega) = \sum_{n=1}^\infty c_n\chi_{E(n)}(\omega)$ holds, for every $\omega \in \Omega$ for which $\sum_{n=1}^\infty |c_n|\chi_{E(n)}(\omega) < \infty$. In this case*

$$(f\mu)(E) = \int_E f d\mu = \sum_{n=1}^\infty c_n \mu(E(n) \cap E), \qquad E \in \mathcal{S}.$$

The space of all scalar-valued, μ-integrable functions on Ω will be denoted by $L^1(\mu)$. It is equipped with the mean convergence topology which is given by the seminorm

$$|\!|\!| f |\!|\!|_\mu = \sup\{|\langle x', f\mu\rangle|(\Omega) : x' \in X',\ \|x'\| \le 1\}, \qquad f \in L^1(\mu)\,.$$

The seminormed space $L^1(\mu)$ is complete and the $\mathcal{S}$-simple functions are dense in it (see [7; Chapter II]). A function $f \in L^1(\mu)$ is called *μ-null* if $(f\mu)(E) = 0$, for every $E \in \mathcal{S}$. The space of all μ-null functions is denoted by $\mathcal{N}(\mu)$. The seminormed space $L^1(\mu)$ is identified with its quotient space $L^1(\mu)/\mathcal{N}(\mu)$ so that $L^1(\mu)$ will be regarded as a Banach space. It follows that the integration map $I_\mu : L^1(\mu) \to X$ defined by

$$I_\mu f = (f\mu)(\Omega) = \int_\Omega f\, d\mu, \qquad f \in L^1(\mu)\,,$$

is linear and continuous.

A vector measure $\mu : \mathcal{S} \to X$ is said to *factor* through a Banach space Y if there exist a vector measure $\nu : \mathcal{S} \to Y$ and a continuous linear map $\mathbf{j} : Y \to X$ such that

(F1) $L^1(\mu) = L^1(\nu)$ as vector spaces,
(F2) $\mathcal{N}(\mu) = \mathcal{N}(\nu)$, and
(F3) $I_\mu = \mathbf{j} \circ I_\nu$.

We will also say that μ factors through Y via ν and $\mathbf{j}$. In this case, the continuity of $\mathbf{j}$ implies that the identity map Φ from $L^1(\nu)$ onto $L^1(\mu)$ is continuous. Accordingly, $L^1(\nu)$ and $L^1(\mu)$ are isomorphic Banach spaces (by the open mapping theorem and the injectivity of Φ; see (F2)). If the map $\mathbf{j}$ happens to be injective, then (F3) implies (F2).

2. Weakly compact integration maps. Throughout this section, let X be a Banach space and μ an X-valued vector measure on a σ-algebra $\mathcal{S}$ of subsets of a non-empty set Ω.

According to the Bartle–Dunford–Schwartz theorem (see [4; Corollary I.2.7]), μ has relatively weakly compact range. Accordingly, the range of $I_\mu : L^1(\mu) \to X$ is contained in a weakly compactly generated, closed subspace X_μ, of X, namely that generated by $\mu(\mathcal{S})$. However, as shown in Example 3.8, I_μ may not be weakly compact; for further non-trivial examples, see [10]. The following result characterizes those vector measures μ for which I_μ is weakly compact.

PROPOSITION 2.1. *A vector measure $\mu : \mathcal{S} \to X$ factors through a reflexive Banach space if and only if the associated integration map $I_\mu : L^1(\mu) \to X$ is weakly compact.*

We will require the following

LEMMA 2.2. *Suppose that there exist a Banach space Y containing no copy of ℓ^∞, a vector measure $\nu : \mathcal{S} \to Y$, and a continuous linear injection $\mathbf{j} : Y \to X$ such that $\mu = \mathbf{j} \circ \nu$ and $I_\mu(L^1(\mu)) \subseteq \mathbf{j}(Y)$. Then the conditions* (F1)–(F3) *hold.*

Proof. To establish (F1), let $f \in L^1(\mu)$. Let c_n be scalars and $E_n \in \mathcal{S}$, $n = 1, 2, \ldots$, be sets satisfying the conditions (i) and (ii) in Lemma 1.2. Let $F(n) \in E_n \cap \mathcal{S}$, $n = 1, 2, \ldots$ We claim that, if $\{n(k)\}_{k=1}^\infty$ is a strictly increasing sequence of positive integers, then the sum of the summable sequence $\{c_{n(k)}(\mathbf{j} \circ \nu)(F(n(k)))\}_{k=1}^\infty$ belongs to $\mathbf{j}(Y)$. Indeed, let g be a scalar-valued function on Ω such that $g(\omega) = \sum_{k=1}^\infty c_{n(k)}\chi_{F(n(k))}(\omega)$ for every $\omega \in \Omega$ for which $\sum_{k=1}^\infty |c_{n(k)}|\chi_{F(n(k))}(\omega) < \infty$. By Lemma 1.2, the function g is μ-integrable and

$$\sum_{k=1}^{\infty} c_{n(k)}(\mathbf{j} \circ \nu)(F(n(k))) = \sum_{k=1}^{\infty} c_{n(k)}\mu(F(n(k))) = I_\mu g\,,$$

which is clearly an element of $I_\mu(L^1(\mu)) \subseteq \mathbf{j}(Y)$. It now follows from Lemma 1.1 that the sequence $\{c_n\nu(F(n))\}_{n=1}^\infty$ is unconditionally summable in Y. Hence, $f \in L^1(\nu)$ by Lemma 1.2. Thus $L^1(\mu) \subseteq L^1(\nu)$. Since the continuity of $\mathbf{j}$ implies that $L^1(\nu) \subseteq L^1(\mu)$ we obtain (F1). Now (F3) is clear. The property (F2) is a consequence of (F3) and the injectivity of $\mathbf{j}$. ∎

Proof of Proposition 2.1. If μ factors through a reflexive Banach space, then clearly I_μ is weakly compact.

Suppose that I_μ is weakly compact. By [2; Corollary 1, p. 314], there exists a reflexive Banach space Y such that

(i) Y is a linear subspace of X and the natural injection $\mathbf{j} : Y \to X$ is continuous, and

(ii) $I_\mu(L^1(\mu)) \subseteq Y$.

It follows from (ii) that there is a unique set function $\nu : \mathcal{S} \to Y$ satisfying $\mu = \mathbf{j} \circ \nu$. The σ-additivity of ν is again a consequence of Lemma 1.1 because Y is reflexive. Lemma 2.2 now implies that μ factors through the reflexive space Y. ∎

We remark that there may be more than one reflexive Banach space through which a vector measure factors (see Examples 3.10 and 3.11).

3. Measures with values in ℓ^1. The main aim of this section is to give a systematic way of constructing ℓ^1-valued measures whose associated integration map is compact or nuclear. Throughout, let $\mathcal{S}$ be a σ-algebra of subsets of a non-empty set Ω and $\lambda : \mathcal{S} \to [0, \infty)$ be a finite, non-trivial measure. This means that there exist infinitely many pairwise disjoint, non-λ-null sets in $\mathcal{S}$.

Let ℓ^1 denote the usual Banach space of scalar-valued functions ϕ on $\mathbb{N} = \{1, 2, \ldots\}$ such that $\|\phi\|_1 = \sum_{n=1}^{\infty} |\phi(n)| < \infty$. The standard unit vectors in ℓ^1 are denoted by e_n, $n \in \mathbb{N}$. The same notation will be used if we wish to regard them as elements of $\ell^p, 1 < p \le \infty$.

The following result is known (see [3; Exercise VII 3]); it is a consequence of the uniform boundedness principle.

LEMMA 3.1. *A linear map $T : L^1(\lambda) \to l^1$ is continuous if and only if there exist functions $g_n \in L^\infty(\lambda)$, $n \in \mathbb{N}$, satisfying*

$$\sum_{n=1}^{\infty} |\langle g_n, f\rangle| < \infty, \qquad f \in L^1(\lambda)\,, \tag{1}$$

such that

$$Tf = \sum_{n=1}^{\infty} \langle g_n, f\rangle e_n, \qquad f \in L^1(\lambda)\,. \tag{2}$$

Remark 3.2. A sequence $\{g_n\}_{n=1}^{\infty}$ in $L^\infty(\lambda)$ satisfies (1) if and only if it is conditionally summable in $L^\infty(\lambda)$ with respect to the weak-$*$ topology.

LEMMA 3.3. *Let $g_n \in L^\infty(\lambda)$, $n \in \mathbb{N}$, be functions satisfying* (1)*. Define a set function $\mu : \mathcal{S} \to \ell^1$ by*

$$\mu(E) = \sum_{n=1}^{\infty} \Big(\int_E g_n \, d\lambda \Big) e_n, \qquad E \in \mathcal{S}\,. \tag{3}$$

Then μ is a vector measure, $L^1(\lambda) \subseteq L^1(\mu)$ and $\mathcal{N}(\lambda) \subseteq \mathcal{N}(\mu)$.

Proof. The σ-additivity of μ follows from the continuity of the map T specified by (2). Let $\xi \in \ell^\infty$ (which is identified with the dual space of ℓ^1 in the usual way). Then

$$\sum_{n=1}^{\infty} |\langle \xi(n) g_n, f\rangle| < \infty, \qquad f \in L^1(\lambda)\,.$$

Hence, the sequence $\{\xi(n) g_n\}_{n=1}^{\infty}$ is unconditionally summable to an element $\sum_{n=1}^{\infty} \xi(n) g_n$ in $L^\infty(\lambda)$ with respect to the weak-$*$ topology (cf. Remark 3.2). In other words,

$$\lim_{N\to\infty} \Big\langle \sum_{n=1}^{\infty} \xi(n) g_n - \sum_{n=1}^{N} \xi(n) g_n, f \Big\rangle = 0, \qquad f \in L^1(\lambda)\,.$$

Therefore we obtain

$$\langle \xi, \mu\rangle = \Big(\sum_{n=1}^{\infty} \xi(n) g_n \Big) \lambda\,, \tag{4}$$

that is, $\langle \xi, \mu \rangle$ is absolutely continuous with respect to λ and has $\sum_{n=1}^{\infty} \xi(n) g_n$ as its Radon–Nikodym derivative.

To prove the inclusion $L^1(\lambda) \subseteq L^1(\mu)$, let $f \in L^1(\lambda)$. Then, because of (4), f is $\langle \xi, \mu \rangle$-integrable, for every $\xi \in l^\infty$. Furthermore

$$\Big\langle \xi, \sum_{n=1}^{\infty} \Big(\int_E g_n f \, d\lambda \Big) e_n \Big\rangle = \int_E f \, d\langle \xi, \mu \rangle ,$$

for every $\xi \in \ell^\infty$ and $E \in \mathcal{S}$. Accordingly, $f \in L^1(\mu)$ and

$$(f\mu)(E) = \int_E f \, d\mu = \sum_{n=1}^{\infty} \Big(\int_E g_n f \, d\lambda \Big) e_n, \quad E \in \mathcal{S} . \tag{5}$$

So, $L^1(\lambda) \subseteq L^1(\mu)$. The containment $\mathcal{N}(\lambda) \subseteq \mathcal{N}(\mu)$ is clear. ∎

Remark 3.4. Let μ be the vector measure specified in Lemma 3.3. Then $L^1(\mu)$ consists of those scalar-valued, $\mathcal{S}$-measurable functions f on Ω such that $g_n f \in L^1(\lambda)$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} | \int_\Omega g_n f \, d\lambda| < \infty$. Hence, (5) holds. Accordingly, it is always possible to choose functions $g_n \in L^\infty(\lambda)$, $n \in \mathbb{N}$, such that $L^1(\lambda) \neq L^1(\mu)$.

Proposition 3.5. *Let g_1 be the constant function $\mathbf{1}$ and $g_n \in L^\infty(\lambda)$, $n = 2, 3, \ldots$, be functions such that* (1) *holds. Let μ be the vector measure defined by* (3). *Then*

$$L^1(\mu) = L^1(\lambda) \quad \textit{and} \quad \mathcal{N}(\mu) = \mathcal{N}(\lambda) . \tag{6}$$

In particular, $L^1(\mu)$ and $L^1(\lambda)$ are isomorphic Banach spaces. Moreover, the measure $f\mu$ is given by (5) *and*

$$I_\mu f = \sum_{n=1}^{\infty} \langle g_n, f \rangle e_n, \quad f \in L^1(\mu) .$$

Proof. We have

$$\langle e_1, \mu \rangle = \mathbf{1}\lambda = \lambda , \tag{7}$$

from which the inclusions $L^1(\mu) \subseteq L^1(\langle e_1, \mu \rangle) = L^1(\lambda)$ and $\mathcal{N}(\mu) \subseteq \mathcal{N}(\langle e_1, \mu \rangle) = \mathcal{N}(\lambda)$ follow. By Lemma 3.3 we now have (6). The conditions (6) and (7) jointly imply that the identity map from $L^1(\mu)$ onto $L^1(\lambda)$ is continuous and injective; hence that map is an isomorphism by the open mapping theorem.

The rest of the statement of Proposition 3.5 is now clear from the proof of Lemma 3.3. ∎

By the Schur theorem, a sequence in ℓ^1 is weakly convergent if and only if it is norm convergent (see [3; p. 85]). It follows that every weakly compact operator from a Banach space into ℓ^1 is also compact.

PROPOSITION 3.6. *Let* $g_n \in L^\infty(\lambda)$, $n \in \mathbb{N}$, *be functions such that* $g_1 = \mathbf{1}$ *and* (1) *holds. Let* $\mu : \mathcal{S} \to \ell^1$ *be the vector measure defined by* (3) *and* $I_\mu : L^1(\mu) \to l^1$ *be its associated integration map.*

(i) *The operator* I_μ *is compact if and only if the sequence* $\{g_n\}_{n=1}^\infty$ *is unconditionally norm summable in the Banach space* $L^\infty(\lambda)$.

(ii) *The operator* I_μ *is nuclear if and only if* $\{g_n\}_{n=1}^\infty$ *is absolutely summable in* $L^\infty(\lambda)$.

Proof. Statement (i) is a particular case of [3; Exercise VII 3(ii)], and (ii) follows from the definition of a nuclear map (see [4; Definition VI 4.1]). ■

EXAMPLE 3.7. Let $g_1 = \mathbf{1}$. Let $g_n = \chi_{E(n)}, n = 2, 3, \ldots$, where $\{E(n)\}_{n=2}^\infty$ is a sequence of pairwise disjoint, non-λ-null sets in $\mathcal{S}$. Then (1) holds and the set function μ defined by (3) is a vector measure; see Lemma 3.3. By Proposition 3.6(i) the integration map $I_\mu : L^1(\mu) \to \ell^1$ is not compact because $\{g_n\}_{n=1}^\infty$ is not unconditionally (norm) summable in $L^\infty(\lambda)$. Alternatively, I_μ is surjective, which also implies that I_μ is not compact by Proposition 2.1. Clearly I_μ is not injective. ■

EXAMPLE 3.8. Let $g_1 = \mathbf{1}$. Take a sequence of sets $E(n) \in \mathcal{S}$, $n = 2, 3, \ldots$, which are pairwise disjoint and non-λ-null. Let $g_n = n^{-1}\chi_{E(n)}$, $n = 2, 3, \ldots$, in which case $\{g_n\}_{n=1}^\infty$ is unconditionally but not absolutely summable in $L^\infty(\lambda)$. Define a vector measure $\mu : \mathcal{S} \to \ell^1$ by (3). Then the integration map $I_\mu : L^1(\mu) \to \ell^1$ is compact but not nuclear (cf. Proposition 3.6). ■

EXAMPLE 3.9. Let $g_1 = \mathbf{1}$ and $\{E(n)\}_{n=2}^\infty$ be a sequence of sets in $\mathcal{S}$ as in Example 3.8. Let $g_n = 2^{-n}\chi_{E(n)}$, $n = 2, 3, \ldots$ For the vector measure μ defined by (3), Proposition 3.6(ii) implies that $I_\mu : L^1(\mu) \to \ell^1$ is nuclear. ■

The following example provides an ℓ^1-valued measure which factors through more than one reflexive Banach space.

EXAMPLE 3.10. Let $\mu : \mathcal{S} \to \ell^1$ be the vector measure defined in Example 3.7. Let ξ belong to $\bigcap_{1<p<\infty} \ell^p$ and satisfy $\xi(n) \neq 0$, for every $n \in \mathbb{N}$. Then the linear map $\Lambda : \ell^1 \to \ell^1$ defined by $\Lambda(\kappa) = \eta$, for every $\kappa \in \ell^1$, where $\eta(n) = \xi(n)\kappa(n)$, $n \in \mathbb{N}$, is injective and compact. Moreover, the set function $\varrho = \Lambda \circ \mu$ is again an ℓ^1-valued measure on $\mathcal{S}$.

Now fix $p \in (1, \infty)$. Then ϱ factors through the reflexive space ℓ^p. Indeed, let $V_p : \ell^1 \to \ell^p$ denote the natural injection and $\mathbf{j}_p : \ell^p \to \ell^1$ be the linear map given by $\mathbf{j}_p(\kappa) = \eta$, for every $\kappa \in \ell^p$, where $\eta(n) = \xi(n)\kappa(n)$, $n \in \mathbb{N}$. Then both V_p and $\mathbf{j}_p$ are continuous injections and $\Lambda = \mathbf{j}_p \circ V_p$. The set function $\nu_p = V_p \circ \mu : \mathcal{S} \to \ell^p$ is σ-additive. Furthermore,

$$L^1(\lambda) \subseteq L^1(\mu) \subseteq L^1(\nu_p) \subseteq L^1(\varrho) \subseteq L^1(\langle e_1, \varrho\rangle) \subseteq L^1(\lambda)$$

because $\langle e_1, \varrho\rangle = \xi(1)^{-1}\langle e_1, \mu\rangle = \xi(1)^{-1}\lambda$. Hence, $L^1(\nu_p) = L^1(\varrho)$. Since $\varrho = \mathbf{j}_p \circ \nu_p$ and since the $\mathcal{S}$-simple functions are dense in both $L^1(\nu_p)$ and $L^1(\varrho)$, for their respective topologies, the Lebesgue dominated convergence theorem (see [7; Theorem II 4.2]) implies that $I_\varrho = \mathbf{j}_p \circ I_{\nu_p}$. Accordingly, ϱ factors through ℓ^p via ν_p and $\mathbf{j}_p$ (as $\mathbf{j}_p$ is injective). ∎

Example 3.10 gives a vector measure ϱ which factors through every $\ell^p, 1 < p < \infty$. However, the associated integration map I_ϱ is not injective. It is possible to modify that example so that the integration map is injective. Instead of doing so we present a further example, of a different nature, which exhibits the same phenomenon.

EXAMPLE 3.11. Let η be normalized Haar measure on the Borel σ-algebra $\mathcal{B}$ of the circle group. Let $\mathbb{Z}$ be the set of all integers. For each $E \in \mathcal{B}$, let $\mu(E)$ denote the Fourier transform of χ_E. The so defined set function $\mu : \mathcal{B} \to c_0(\mathbb{Z})$ is a vector measure such that $L^1(\mu) = L^1(\eta)$ and $\mathcal{N}(\mu) = \mathcal{N}(\eta)$. It turns out that the integration map I_μ coincides with the Fourier transform operator on $L^1(\eta)$ and that I_μ is injective but not weakly compact; for the details see [10]. Let $\gamma \in \ell^1(\mathbb{Z})$ satisfy $\gamma(n) \neq 0$, $n \in \mathbb{Z}$. Define $W : c_0(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ by $W\kappa = \xi$, for every $\kappa \in c_0(\mathbb{Z})$, where $\xi(n) = \gamma(n)\kappa(n)$, $n \in \mathbb{Z}$. Then W is a continuous linear injection. By arguments similar to those in Example 3.10, the vector measure $\varrho = W \circ \mu : \mathcal{B} \to \ell^1(\mathbb{Z})$ factors through every space $\ell^p(\mathbb{Z}), 1 < p < \infty$, and the integration map I_ϱ is injective. ∎

Each vector measure for which the associated integration map is nuclear factors through the quotient space of ℓ^1 with respect to some closed subspace. Hence, the statement of Proposition 3.6(ii) has some generality. This is made precise in the following result whose proof is based on the fact that every nuclear map from one Banach space into another factors through ℓ^1 (see [4; Proposition VI 4.2]).

PROPOSITION 3.12. *Let μ be a vector measure on $\mathcal{S}$ with values in a Banach space X. Then the integration map $I_\mu : L^1(\mu) \to X$ is nuclear if and only if there exist a closed subspace M of ℓ^1, a vector measure $\nu : \mathcal{S} \to \ell^1/M$, and a continuous linear injection $\mathbf{j} : \ell^1/M \to X$ such that μ factors through the quotient space ℓ^1/M via ν and $\mathbf{j}$ and such that the integration map $I_\nu : L^1(\nu) \to \ell^1/M$ is nuclear.*

Proof. Suppose that I_μ is nuclear. Then there exist unit vectors θ_n, $n \in \mathbb{N}$, in $L^1(\mu)'$ and an absolutely summable sequence $\{x_n\}_{n=1}^\infty$ of non-zero vectors in X such that

$$I_\mu f = \sum_{n=1}^{\infty} \langle \theta_n, f\rangle x_n, \qquad f \in L^1(\mu)\,.$$

The linear map $S : L^1(\mu) \to \ell^1$ defined by

$$Sf = \sum_{n=1}^{\infty} \langle \theta_n, f\rangle \|x_n\| e_n, \qquad f \in L^1(\mu)\,,$$

is nuclear. Define $J : \ell^1 \to X$ by

$$J\xi = \sum_{n=1}^{\infty} \xi(n)\|x_n\|^{-1} x_n, \qquad \xi \in l^1\,.$$

Then J is a continuous linear map such that $I_\mu = J \circ S$. Let $M = \{\xi \in \ell^1 : J(\xi) = 0\}$ and $\pi : \ell^1 \to \ell^1/M$ be the quotient map. Then there is a unique continuous linear injection $\mathbf{j} : \ell^1/M \to X$ such that $J = \mathbf{j} \circ \pi$. Accordingly, there exists a vector measure $\nu : \mathcal{S} \to \ell^1/M$ satisfying $\mu = \mathbf{j} \circ \nu$. Since $I_\mu(L^1(\mu)) = \mathbf{j} \circ \pi \circ S(L^1(\mu)) \subseteq \mathbf{j}(\ell^1/M)$ and since the separable space ℓ^1/M does not contain a copy of ℓ^∞, it follows from Lemma 2.2 that μ factors through ℓ^1/M via ν and $\mathbf{j}$. Since $\mathbf{j} \circ I_\nu = I_\mu = \mathbf{j} \circ (\pi \circ S)$, the injectivity of $\mathbf{j}$ implies that $I_\nu = \pi \circ S$ and so I_ν is nuclear.

The converse implication is clear. ■

We conclude with an example of a non-reflexive Banach space-valued measure for which the associated integration map is weakly compact but not compact. By the Schur theorem there are no such ℓ^1-valued measures.

EXAMPLE 3.13. Let $E(1) = \Omega$ and $\{E(n)\}_{n=2}^{\infty}$ be a sequence in $\mathcal{S}$ of pairwise disjoint, non-λ-null sets. The ℓ^1-valued set functions μ and ν_0 defined by

$$\mu(E) = \sum_{n=1}^{\infty} n^{-2}\lambda(E(n)\cap E)e_n \quad \text{and} \quad \nu_0(E) = \sum_{n=1}^{\infty} \lambda(E(n)\cap E)e_n\,,$$

for every $E \in \mathcal{S}$, satisfy

$$L^1(\lambda) = L^1(\mu) = L^1(\nu_0) \quad \text{and} \quad \mathcal{N}(\lambda) = \mathcal{N}(\mu) = \mathcal{N}(\nu_0); \tag{8}$$

see Lemma 3.3 and Proposition 3.5.

Let $\mathbf{j} : \ell^1 \to \ell^2$ be the natural injection. Then the measure $\nu = j \circ \nu_0 : \mathcal{S} \to \ell^2$ factors through ℓ^1 via ν_0 and $\mathbf{j}$. Moreover,

$$L^1(\nu) = L^1(\nu_0)\,. \tag{9}$$

Indeed, this follows from (8) by continuity of $\mathbf{j}$, because $L^1(\nu_0) \subseteq L^1(\nu)$, and because $L^1(\nu) \subseteq L^1(\lambda)$, as $\langle e_1, \nu\rangle = \lambda$.

The set function $\eta : \mathcal{S} \to \ell^1 \times \ell^2$ defined by $\eta(E) = (\mu(E), \nu(E))$, for every $E \in \mathcal{S}$, is σ-additive. Direct computation shows that $L^1(\eta) = L^1(\lambda)$ and $I_\eta f = (I_\mu f, I_\nu f)$, for every $f \in L^1(\eta)$. Accordingly, the integration map $I_\eta : L^1(\eta) \to \ell^1 \times \ell^2$ is weakly compact because the component map

I_μ (respectively, I_ν) is weakly compact by Proposition 3.6 (respectively, by the reflexivity of ℓ^2). However, I_η is not compact. To see this, let $f_m = m(m+1)\chi_{E(m)}, m = 2, 3, \ldots$ Then $I_\nu f_m = e_1 + e_m$ while

$$|||f_m|||_\nu \leq \int_\Omega f_m \, d|\langle e_1, \nu\rangle| + \int_\Omega f_m \, d|\langle e_m, \nu\rangle| = 2\,,$$

for every $m = 2, 3, \ldots$ So, I_ν maps the bounded sequence $\{f_m\}_{m=2}^\infty$ to the sequence $\{e_1 + e_m\}_{m=2}^\infty$ which does not contain any convergent subsequence in ℓ^2. Thus I_ν is not compact and hence, neither is I_η. ■

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