FUNDAMENTA
MATHEMATICAE
142 (1993)

A rigid Boolean algebra that admits the elimination of Q_1^2

by

Heike Mildenberger (Bonn)

Abstract. Using $\diamondsuit$, we construct a rigid atomless Boolean algebra that has no uncountable antichain and that admits the elimination of the Malitz quantifier Q_1^2.

1. Introduction. Malitz quantifiers are introduced in [Mag-Mal]. Let us recall the semantics of Q_α^n, $n \geq 1$, $\alpha \in \mathrm{ORD}$: $\mathfrak{A} \models Q_\alpha^n \overset{n}{x}\, \phi(\overline{a}, \overset{n}{x})$ iff there is a subset H of A such that $\mathrm{card}(H) \geq \aleph_\alpha$ and $\mathfrak{A} \models \phi(\overline{a}, \overset{n}{h})$ for all pairwise different $h_0, h_1, \ldots, h_{n-1} \in H$. Such a set H is called a *homogeneous set* for $\phi(\overline{a}, \overset{n}{x})$. Baldwin and Kueker [Bal-Ku], Rothmaler and Tuschik [Ro-Tu], Bürger [Bü] and Koepke [Ko] consider the question of elimination of some of these quantifiers in certain theories or structures. [Ro-Tu] shows that any saturated model allows the elimination of all Q_α^n, $\alpha \in \mathrm{ORD}$, $n \geq 1$.

Saturated models with two elements of the same type are not rigid. On the other hand, there are $\mathcal{L}_{\omega\omega}(Q_1^2)$-sentences ϕ that have only rigid models and that are satisfiable under CH (see [Ot], [Mil]). We consider

$$\begin{aligned}\phi := \ & \text{“the structure is a Boolean algebra with } 0 \neq 1\text{”} \\ & \wedge \forall x (x \neq 0 \to Q_1 y\, y \subseteq x) \wedge \neg Q_1^2 xy\, x \not\subseteq y\,.\end{aligned}$$

[Ba-Ko, Theorem 5(a)] shows that all models of ϕ are rigid. The search for a model of ϕ that contains two different elements of the same $\mathcal{L}_{\omega\omega}(Q_1^2)$-type leads, under $\diamondsuit$, to a model of ϕ that admits the elimination of Q_1^2 and in which therefore any two elements $\neq 0, 1$ have the same $\mathcal{L}_{\omega\omega}(Q_1^2)$-type.

In ZFC + $\diamondsuit$ and even in ZFC + CH there are various constructions of uncountable Boolean algebras with no uncountable antichains and with some other algebraic properties (see [Ba-Ko], [Sh], [Ru], but also [Ba]). In the course of showing that additional tasks may be fulfilled along the way given in [Ba-Ko], we get a partition of all formulas $\phi(\overset{r}{z}, x, y) \in \mathcal{L}_{\omega\omega}(Q_1^2)$, $r \in \omega$, into two classes Φ_1 and Φ_2 such that

1. The methods of [Ba-Ko] are applicable to any $\phi(\overset{r}{z}, x, y) \in \Phi_1$. They will allow us to show that the homogeneous sets for any $\phi(\overset{r}{z}, x, y) \in \Phi_1$ will grow only during countably many steps in the chain which we build in the next section.

2. For any Boolean algebra $\mathfrak{A}$ with $\mathfrak{A} \models \forall x \neq 0\, Q_1 y\, y \subseteq x$ and any $\phi(\overset{r}{z}, x, y) \in \Phi_2$: $\mathfrak{A} \models \exists \overset{r}{z}\, Q_1^2 xy\, \phi(\overset{r}{z}, x, y)$.

"$\phi(\overset{r}{z}, x, y) \in \Phi_1$" will be shown to be equivalent under the first order theory of atomless Boolean algebras to a first order formula with its free variables among $z_0, z_1, \ldots, z_{r-1}$. The consideration of the possible quantifierfree types of the $\overset{r}{z}$ leads to a procedure for eliminating Q_1^2.

2. The construction

Notation. We will use $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{B}_\alpha$ to denote Boolean algebras. Boolean algebras are considered as τ_{BA}-structures with $\tau_{BA} = \{\cap, \cup, -, 0, 1\}$. $x \subseteq y$ is written for $x \cap y = x$, $\subset$ means strict inclusion, $x \setminus y$ is used for $x \cap (-y)$. $\mathcal{P}(\omega)$ denotes the powerset algebra of ω. For $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ we often write A for $\mathfrak{A}$. The interpretations of the τ_{BA}-symbols in $\mathcal{P}(\omega)$ are denoted by the symbols themselves.

$a, b \in A$ are *comparable* (*in* $\mathfrak{A}$) iff $a \subseteq^{\mathfrak{A}} b$ or $b \subseteq^{\mathfrak{A}} a$. $C \subseteq \mathfrak{A}$ is a *chain* (an *antichain*) iff any two distinct elements of C are comparable (not comparable). For $a \subset^{\mathfrak{A}} b \in A$ let $(a, b)_A := \{c \in A \mid a \subset^{\mathfrak{A}} c \subset^{\mathfrak{A}} b\}$.

Using $\diamondsuit$, we shall construct a Boolean algebra $\mathfrak{B}$ such that $\mathfrak{B}$ is a model of the sentence ϕ from the introduction and $\mathfrak{B}$ admits the elimination of Q_1^2. As the construction of our Boolean algebra $\mathfrak{B}$ follows the pattern of [Ba-Ko], we restrict ourselves to a short description, heavily referring to [Ba-Ko].

Inductively on $\alpha \in \omega_1$, we shall build a chain $(\mathfrak{B}_\alpha, M_\alpha)_{\alpha \in \omega_1}$, where the $\mathfrak{B}_\alpha$ are countable atomless subalgebras of $\mathcal{P}(\omega)$ and each $M_{\alpha+1}$ is a countable collection of pairs $(M, \phi(\overline{c}, x, y))$, where $M \subseteq B_\alpha$ and $\phi(\overline{c}, x, y)$ is a quantifierfree (qf) $\mathcal{L}_{\omega\omega}[\tau_{BA}]$-formula with a property that will be defined later on, and $\overline{c}$ are elements of B_α. At limit steps we take unions. $\mathfrak{B}_{\alpha+1}$ will be the Boolean algebra that is generated by $B_\alpha \cup \{x_\alpha\}$ in $\mathcal{P}(\omega)$, where the x_α is chosen by the same forcing $P(B_\alpha)$ as in [Ba-Ko], namely: $P(B_\alpha) = \{(a, b)_{B_\alpha} \mid a \subset b \in B_\alpha\}$, $(a', b')_{B_\alpha} \leq^{P(B_\alpha)} (a, b)_{B_\alpha}$ iff $a \subseteq a' \subset b' \subseteq b$.

We shall define $D_A(M, \phi(\overline{c}, x, y), e, f)$ and $M_{\alpha+1}$. Then we take a $\{D_A(M, \phi(\overline{c}, x, y), e, f) \mid e, f \in B_\alpha,\ (M, \phi(\overline{c}, x, y)) \in M_{\alpha+1}\}$-generic subset $\{(a_n, b_n) \mid n \in \omega\}$ of $P(B_\alpha)$ such that $\{(a_n, b_n) \mid n \in \omega\}$ additionally satisfies the properties described in [Ba-Ko] and set $x_\alpha = \bigcup\{a_n \mid n \in \omega\}$. In [Ba-Ko], $M_{\alpha+1}$ is chosen so that chains and antichains are countable. Our $M_{\alpha+1}$ differs from that of [Ba-Ko], because we also want all homogeneous sets for any $\phi(\overset{r}{z}, x, y) \in \Phi_1$ to be countable. The next items are the generalizations of the corresponding points of [Ba-Ko].

Definition 2.1. Let $A \subseteq \mathcal{P}(\omega)$ and $\overline{c}, e, f \in A$. Let $\phi(\overline{z}, x, y)$ be qf.

(i) $D_A(M, \phi(\overline{c}, x, y), e, f) := \{(a, b)_A \in P(A) \,|\,$ for any $u \in (a, b)_{\mathcal{P}(\omega)}$ one of the following points is true:

1. $(u \cap e) \cup (f \setminus u) \in M$.
2. There is some $y \in M$ such that

$$\mathcal{P}(\omega) \models \neg\phi(\overline{c}, (u \cap e) \cup (f \setminus u), y) \vee \neg\phi(\overline{c}, y, (u \cap e) \cup (f \setminus u))\}\,.$$

(ii) M is called *maximally homogeneous for $\phi(\overline{c}, x, y)$ in $\mathfrak{A}$* iff $M \subseteq A$ is homogeneous for $\phi(\overline{c}, x, y)$ and for all $a \in A \setminus M$ there is some $b \in M$ such that $\mathfrak{A} \models \neg\phi(\overline{c}, a, b) \vee \neg\phi(\overline{c}, b, a)$.

(iii) $\phi(\overline{c}, x, y)$ is *small in* $\mathfrak{A}$ iff for any $\emptyset \neq M \subseteq A$ that is maximally homogeneous for $\phi(\overline{c}, x, y)$ in $\mathfrak{A}$, $D_A(M, \phi(\overline{c}, x, y), 1, 0)$ is dense in $P(A)$.

Lemma 2.2. *Let $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ be atomless, $\overline{c} \in A^{<\omega}$, $\phi(\overline{c}, x, y)$ qf and small in $\mathfrak{A}, e, f \in A$ and $M \neq \emptyset$ be maximally homogeneous for $\phi(\overline{c}, x, y)$ in $\mathfrak{A}$. Then $D_A(M, \phi(\overline{c}, x, y), e, f)$ is dense in $P(A)$ for any e, f in A.*

Proof. [Ba-Ko, Lemmas 2.3 and 2.4].

Also the proof of the next lemma can be carried out as in [Ba-Ko]: just take a u for $\mathfrak{A}$ and $\overline{M}$ in the same way as they take x_α for $\mathfrak{B}_\alpha$ and $M_{\alpha+1}$.

Lemma 2.3. *Let $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ be atomless and countable and let $\overline{M}$ be a countable subset of*

$$\{(M, \phi(\overline{c}, x, y)) \,|\, \overline{c} \in A^{<\omega}, \phi(\overline{c}, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}] \text{ qf},\ \phi(\overline{c}, x, y) \text{ small in } A \text{ and } M \text{ is maximally homogeneous for } \phi(\overline{c}, x, y) \text{ in } A\}\,.$$

Then for any $(a, b)_A \in P(A)$ there is a $u \in (a, b)_{\mathcal{P}(\omega)}$ such that:

1. *$u \notin A$.*
2. *$[A \cup \{u\}]^{\mathcal{P}(\omega)}$, the subalgebra generated by $A \cup \{u\}$ in $\mathcal{P}(\omega)$, is atomless.*
3. *For any $(M, \phi(\overline{c}, x, y)) \in \overline{M}$ the set M is maximally homogeneous for $\phi(\overline{c}, x, y)$ also in $[A \cup \{u\}]^{\mathcal{P}(\omega)}$.*

Now using Lemma 2.3 and $\diamondsuit$, we can construct our $\mathfrak{B}$. Let $\langle S_\alpha \,|\, \alpha \in \omega_1\rangle$ be a $\diamondsuit$-sequence. Let $\langle a_\xi \,|\, \xi \in \omega_1\rangle$ be an enumeration of $\mathcal{P}(\omega)$ in which each element of $\mathcal{P}(\omega)$ appears ω_1 times.

In step $\alpha + 1$, let $M_{\alpha+1} = M_\alpha \cup \{(\{a_\xi \,|\, \xi \in S_\alpha\}, \phi(\overline{c}, x, y)) \,|\, \{a_\xi \,|\, \xi \in S_\alpha\}$ is a maximally homogeneous set for $\phi(\overline{c}, x, y)$ in $\mathfrak{B}_\alpha$ and $\phi(\overline{c}, x, y)$ is small in $\mathfrak{B}_\alpha$ and $\overline{c} \in B_\alpha\}$. Apply Lemma 2.3 with $\mathfrak{A} = \mathfrak{B}_\alpha$ and $\overline{M} = M_{\alpha+1}$ to get an x_α. Define $B_{\alpha+1}$ as $[B_\alpha \cup \{x_\alpha\}]^{\mathcal{P}(\omega)}$. Let $\mathfrak{B} = \bigcup\{\mathfrak{B}_\alpha \,|\, \alpha \in \omega_1\}$. Take the x_α so that $\mathfrak{B} \models \forall x(x \neq 0 \to Q_1 y\, y \subseteq x)$. Then it is easy to see that for any $\phi(\overline{c}, x, y)$ which is small in every $\mathfrak{B}_\alpha$ with $\overline{c} \in B_\alpha$, we have

$\mathfrak{B} \models \neg Q_1^2 xy\, \phi(\overline{c}, x, y)$. In particular, $\mathfrak{B}$ is a model of ϕ from the introduction (because "$x \not\subseteq y$" is small), hence $\mathfrak{B}$ is rigid.

3. Large homogeneous sets. The aim of this section is to define a mapping

$$\begin{aligned} \mathrm{big} : \bigcup_{r\in\omega} \mathcal{L}_{\omega\omega}[\tau_{BA}](\overset{r}{z}, x, y) &\to \bigcup_{r\in\omega} \mathcal{L}_{\omega\omega}[\tau_{BA}](\overset{r}{z})\,, \\ \phi(\overset{r}{z}, x, y) &\mapsto \mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{z})\,, \end{aligned}$$

such that for every $\phi(\overset{r}{z}, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}]$

$$(*) \qquad \mathfrak{B} \models \forall \overset{r}{z}\ (Q_1^2 xy\, \phi(\overset{r}{z}, x, y) \leftrightarrow \mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{z}))\,.$$

Then Φ_2 will be

$$\{\phi(\overline{z}, x, y) \mid \mathrm{big}(\phi(\overline{z}, x, y))(\overline{z}) \text{ is valid in any atomless Boolean algebra}\}\,.$$

In order to simplify the notation we tacitly assume that always the variables x and y are intended to be quantified by Q_1^2.

Let $\mathfrak{A}$ be any atomless Boolean algebra. Since $\mathfrak{A}$ admits the elimination of $\exists$ it is enough to define big for quantifierfree $\phi(\overset{r}{z}, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}]$.

For any $\overline{c} \in A$ and qf $\phi(\overline{c}, x, y)$ there is a qf $\psi(\overline{c}', x, y)$ such that $\overline{c}'$ is an (injective) enumeration of the atoms of the subalgebra generated by $\overline{c}$, and $\mathfrak{A} \models \forall xy\, (\psi(\overline{c}', x, y) \leftrightarrow \phi(\overline{c}, x, y))$. Also if $\phi(\overline{z}, x, y)$ is a disjunction $\bigvee_i (\phi(\overline{z}, x, y) \wedge \psi_i(\overline{z}))$ then knowing $\chi_i = \mathrm{big}(\phi(\overline{z}, x, y) \wedge \psi_i(\overline{z}))(\overline{z})$ we can define $\mathrm{big}(\phi(\overline{z}, x, y))(\overline{z})$ to be $\bigvee_i \chi_i$. Hence it suffices to define $\mathrm{big}(\phi(\overline{z}, x, y))(\overline{z})$ only for those qf $\phi(\overline{z}, x, y)$ that imply that $\{z_0, \ldots, z_{r-1}\}$ is the set of atoms in the subalgebra generated by $\{z_0, \ldots, z_{r-1}\}$.

If H is an uncountable homogeneous set for $\phi(\overset{r}{c}, x, y)$, then there is an $\mathcal{L}_{\omega\omega}$-1-type $t(\overset{r}{c}, x)$ over $\overset{r}{c}$ and an uncountable $H_1 \subseteq H$ such that every element of H_1 has the $\mathcal{L}_{\omega\omega}$-1-type $\mathrm{tp}(x/\overset{r}{c}) = t(\overset{r}{c}, x)$ over $\overset{r}{c}$. Hence it is enough to define big for the $\phi(\overset{r}{z}, x, y)$ with the above mentioned property and the additional property that there is an $\mathcal{L}_{\omega\omega}$-1-type $t(\overset{r}{z}, x)$ over $\overset{r}{z}$ (independent of the assignment $\overset{r}{c}$ of $\overset{r}{z}$, because we consider only $\overset{r}{c}$ that are atoms in the subalgebra generated by $\overset{r}{z}$) such that

$$\mathfrak{A} \models \forall xy\overset{r}{z}\ (\phi(\overset{r}{z}, x, y) \leftrightarrow (\phi(\overset{r}{z}, x, y) \wedge t(\overset{r}{z}, x) = \mathrm{tp}(x/\overset{r}{z}) \wedge t(\overset{r}{z}, y) = \mathrm{tp}(y/\overset{r}{z})))\,.$$

We will call such formulas *special*. Finally, note that any $\mathcal{L}_{\omega\omega}$-2-type $t(\overset{r}{c}, x, y)$ over $\overset{r}{c}$ is determined by the corresponding r-tuple of the quantifierfree types of $x \cap c_i$, $y \cap c_i$ in $\{a \in A \mid a \subseteq c_i\}$, $i < r$. For any such type there are 15 possibilities, and under the condition $\mathrm{tp}(x/\overset{r}{z}) = \mathrm{tp}(y/\overset{r}{z})$ there remain the 9 possibilities not marked with an $\bullet$ in the table below.

The possibilities for the quantifierfree types of $x \cap c_i$, $y \cap c_i$, $i < r$, in $\{a \in A \mid a \subseteq c_i\}$

No.	$x \cap y \cap z_i$	$(-x) \cap (-y) \cap z_i$	$x \cap (-y) \cap z_i$	$(-x) \cap y \cap z_i$	Remarks
0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	
1	$\neq 0$	$\neq 0$	$\neq 0$	0	
2	$\neq 0$	$\neq 0$	0	$\neq 0$	
3	$\neq 0$	$\neq 0$	0	0	$x \cap z_i =$ $y \cap z_i \neq 0, z_i$
4	$\neq 0$	0	$\neq 0$	$\neq 0$	
•5	$\neq 0$	0	$\neq 0$	0	$x \cap z_i = z_i$ $y \cap z_i \neq z_i$
•6	$\neq 0$	0	0	$\neq 0$	$y \cap z_i = z_i$ $x \cap z_i \neq z_i$
7	$\neq 0$	0	0	0	$x \cap z_i =$ $y \cap z_i = z_i$
8	0	$\neq 0$	$\neq 0$	$\neq 0$	
•9	0	$\neq 0$	$\neq 0$	0	$x \cap z_i \neq 0$ $y \cap z_i = 0$
•10	0	$\neq 0$	0	$\neq 0$	$x \cap z_i = 0$ $y \cap z_i \neq 0$
11	0	$\neq 0$	0	0	$x \cap z_i =$ $y \cap z_i = 0$
12	0	0	$\neq 0$	$\neq 0$	$x \cap z_i \neq 0, z_i$ $y \cap z_i = (-x) \cap z_i$
•13	0	0	$\neq 0$	0	$x \cap z_i = z_i$ $y \cap z_i = 0$
•14	0	0	0	$\neq 0$	$x \cap z_i = 0$ $y \cap z_i = z_i$

Let $\phi^k(z_i, x \cap z_i, y \cap z_i)$ say "the $\mathcal{L}_{\omega\omega}$-type of $x \cap c_i, y \cap c_i$ over c_i has number k", $k = 0, \ldots, 14$. The disjunction $\phi^{012}(u,v,w) := \phi^0(u,v,w) \vee \phi^1(u,v,w) \vee \phi^2(u,v,w)$ will play an important role in the following.

Definition 3.1. Let $\phi(\overset{r}{z}, x, y) \in \mathcal{L}_{\omega\omega}[\tau_{BA}]$ be quantifierfree and be of the special form as described above.

$$\mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{z}) =$$
$$\exists a \subset b \, \forall xy \Big(\Big(a \subseteq x, y \subseteq b \wedge \bigwedge_{i<r} ((b \setminus a) \cap z_i \neq 0 \rightarrow \phi^{012}(z_i, x \cap z_i, y \cap z_i)) \Big)$$
$$\rightarrow \phi(\overset{r}{z}, x, y) \Big).$$

Equivalent to $\mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{z})$ is the formula

$$\bigvee_{I_0 \dot\cup I_1 \dot\cup I_2 \dot\cup I_3 = \{0,\ldots,r-1\}, I_0 \neq 0} \forall xy \Big(\Big(\bigwedge_{i \in I_0} \phi^{012}(z_i, x \cap z_i, y \cap z_i)$$
$$\wedge \bigwedge_{i \in I_1} x \cap z_i = y \cap z_i \neq 0, z_i$$

$$\wedge \bigwedge_{i \in I_2} x \cap z_i = y \cap z_i = 0$$

$$\wedge \bigwedge_{i \in I_3} x \cap z_i = y \cap z_i = z_i \Big) \to \phi(\overset{r}{z}, x, y)\Big) ,$$

($\dot\cup$ denotes the disjoint union) which will be useful for the easy direction of $(*)$:

Lemma 3.2. *Let $\mathfrak{A}$ be an atomless Boolean algebra. Let $\mathfrak{A} \models \forall x \neq 0$ $Q_1 y\, y \subseteq x$, and $\phi(\overset{r}{z}, x, y)$ be as above. Then $\mathfrak{A} \models \forall \overset{r}{z}\ (\mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{z}) \to$ $Q_1^2 xy\, \phi(\overset{r}{z}, x, y))$.*

Proof. Let $\mathfrak{A} \models \mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{c})$. For $i \in I_0$ take an uncountable set $H_i \subseteq (0, c_i)_{\mathfrak{A}}$ such that for any $x \in H_i$ the relative complement $c_i \setminus x \notin H_i$. Let $\langle h_{i,\alpha} \mid \alpha \in \omega_1 \rangle$ be an injective enumeration of a subset of H_i. Finally, for $i \in I_1$ let $H_i = \{d_i\}$ for some d_i with $0 \subset d_i \subset c_i$, for $i \in I_2$ let $H_i = \{0\}$, and for $i \in I_3$ let $H_i = \{c_i\}$. Then

$$H := \Big\{ \bigcup \{h_{i,\alpha} \mid i \in I_0\} \cup \bigcup \{d_i \mid i \in I_1\} \cup \bigcup \{c_i \mid i \in I_3\} \Big| \alpha \in \omega_1 \Big\}$$

is an uncountable homogeneous set for $\phi(\overset{r}{c}, x, y)$.

Now for $\mathfrak{B}$ as in Section 2, we shall prove the other direction of $(*)$. By the construction, it would suffice to show:

$(**)$ For any enumeration $\overset{r}{c}$ of the atoms in the subalgebra of $\mathfrak{B}$ generated by $\overset{r}{z}$, if $\mathfrak{B} \models \neg\mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{c})$, then $\phi(\overset{r}{c}, x, y)$ is small in every $\mathfrak{B}_\alpha$ with $\overset{r}{c} \in B_\alpha$.

Unfortunately, this is true only for $\phi(\overset{r}{c}, x, y)$ that do not forbid certain equalities of Boolean terms. We introduce some notation and then give a sketch of our proof of the hard direction of $(*)$.

We say briefly "$\phi(\overset{r}{z}, x, y)$ is valid" or just "ϕ" for "$\phi(\overset{r}{z}, x, y)$ is valid in all atomless Boolean algebras if the assignment of $\overset{r}{z}$ is an enumeration of the atoms in the subalgebra generated by $\overset{r}{z}$". $\phi(\overset{r}{z}, x, y)$ is *satisfiable* or *consistent* if $\neg\phi(\overset{r}{z}, x, y)$ is not valid.

For a given special $\phi(\overset{r}{z}, x, y)$ set

$$R(\phi) := \{i < r \mid \phi \to x \cap z_i = y \cap z_i \text{ is not valid}\} .$$

We will define two mappings s and enl from the set of all special $\phi(\overset{r}{z}, x, y)$ into itself. The mapping s is a technical means used to prove $\mathrm{enl}(\mathrm{enl}(s(\phi)))$ $\to \mathrm{enl}(s(\phi))$ (Lemma 3.7) and $\neg\mathrm{big}(s(\phi)) \to \neg\mathrm{big}(\mathrm{enl}(s(\phi)))$ (Lemma 3.8). Lemma 3.9 says that $(**)$ is true for formulas of the form $\mathrm{enl}(s(\phi))$ for some special ϕ. Hence we get from the construction and from 3.8

$$\mathfrak{B} \vDash \neg\mathrm{big}(s(\phi))(\overset{r}{c}) \to \neg Q_1^2 xy \ \mathrm{enl}(s(\phi))(\overset{r}{c}, x, y)\,,$$

whence $s(\phi) \to \mathrm{enl}(s(\phi))$ and the monotonicity of the quantifier Q_1^2 imply

$$\mathfrak{B} \vDash \neg\mathrm{big}(s(\phi))(\overset{r}{c}) \to \neg Q_1^2 xy \ s(\phi)(\overset{r}{c}, x, y)$$

(Theorem 3.10). Using this result we prove by induction on $\mathrm{card}(R(\phi))$, simultaneously for all special formulas ϕ,

$$\mathfrak{B} \vDash \neg\mathrm{big}(\phi)(\overset{r}{c}) \to \neg Q_1^2 xy \ \phi(\overset{r}{c}, x, y)\,,$$

which will finish the proof of $(*)$.

In order to simplify the notation, we often suppress the free variables $(\overset{r}{z}, x, y)$ or $(z_i, x \cap z_i, y \cap z_i)$.

Definition 3.3 (The mapping s). For $R \subseteq r = \{0, 1, \ldots, r-1\}$ and for $\chi(z_i, x \cap z_i, y \cap z_i) \in \mathcal{L}_{\omega\omega}[\tau_{BA}]$ we define

$$s_R(\chi(z_i, x\cap z_i, y\cap z_i)) := \begin{cases} \chi(z_i, x \cap z_i, y \cap z_i) \quad \text{if } i \notin R \text{ or} \\ \quad \phi^{012}(z_i, x \cap z_i, y \cap z_i) \to \chi(z_i, x \cap z_i, y \cap z_i) \\ \quad \text{is valid;} \\ \chi(z_i, x \cap z_i, y \cap z_i) \wedge x \cap z_i \neq y \cap z_i \\ \quad \text{else.} \end{cases}$$

Let $S = \{\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \,|\, w \in W\}$ be a finite set such that for all $w \in W$ the conjunction $\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i)$ is satisfiable and $\bigwedge_{i<r} \chi_{w,i}(z_i, x\cap z_i, y\cap z_i) \to \phi(\overset{r}{z}, x, y)$ is valid, and such that for any satisfiable conjunction $\delta = \bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i)$ such that $\delta \to \phi(\overset{r}{z}, x, y)$ is valid there is a $w \in W$ with $\bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i) \to \bigwedge_{i<r} \chi_{w,i}(z_i, x\cap z_i, y\cap z_i)$. We will call such a set S a *set of representatives* for ϕ. Given such a set, let $R = R(\phi)$ and define

$$s(\phi(\overset{r}{z}, x, y)) = \bigvee_{w\in W} \bigwedge_{i<r} s_R(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i))\,.$$

If $\vDash \neg\exists xy\overset{r}{z} \ \phi(\overset{r}{z}, x, y)$, then let $s(\phi(\overset{r}{z}, x, y))$ be any inconsistent formula.

A brief reflection shows that $s(\phi)$ is well defined up to logical equivalence: Let $S' = \{\bigwedge_{i<r} \chi'_{w',i}(z_i, x \cap z_i, y \cap z_i) \,|\, w' \in W'\}$ be another set of representatives for ϕ.

For $\bigvee_{w'\in W'} \bigwedge_{i<r} s_R(\chi'_{w',i}) \to \bigvee_{w\in W} \bigwedge_{i<r} s_R(\chi_{w,i})$, it suffices to show that for each $w' \in W'$ there is some $w \in W$ such that $\bigwedge_{i<r} s_R(\chi'_{w',i}) \to \bigwedge_{i<r} s_R(\chi_{w,i})$. Let $w' \in W'$ be given. Since S is a set of representatives for ϕ there is a $w \in W$ such that $\bigwedge_{i<r} \chi'_{w',i} \to \bigwedge_{i<r} \chi_{w,i}$, which is equivalent to

$\chi'_{w',i} \to \chi_{w,i}$ for $i < r$. Immediately from the definition of s_R, if $\chi'_{w',i} \to \chi_{w,i}$, then $s_R(\chi'_{w',i}) \to s_R(\chi_{w,i})$. Hence $\bigwedge_{i<r} s_R(\chi'_{w',i}) \to \bigwedge_{i<r} s_R(\chi_{w,i})$.

The other direction follows by symmetry.

Remark. $s(\phi)$ may be unsatisfiable, e.g. for $\phi = (x \cap z_0 = y \cap z_0 \wedge x \cap z_1 \subset y \cap z_1) \vee (x \cap z_0 \subset y \cap z_0 \wedge x \cap z_1 = y \cap z_1) \wedge \bigwedge_{i=0,1} x \cap z_i \neq z_i, 0 \wedge \bigwedge_{i=0,1} y \cap z_i \neq z_i, 0 \wedge z_0 \cap z_1 = 0 \wedge z_0 \cup z_1 = 1$.

Definition 3.4 (The mapping enl). For $\chi(z_i, x \cap z_i, y \cap z_i) \in \mathcal{L}_{\omega\omega}[\tau_{BA}]$ we define

$$\mathrm{enl}(\chi(z_i, x\cap z_i, y\cap z_i)) := \begin{cases} \chi(z_i, x \cap z_i, y \cap z_i) \\ \vee\, (x \cap z_i = (-y) \cap z_i \wedge \exists x\, \chi(z_i, x \cap z_i, y \cap z_i) \\ \wedge\, \exists y\, \chi(z_i, x \cap z_i, y \cap z_i)) \\ \quad \text{if } \phi^{012}(z_i, x \cap z_i, y \cap z_i) \to \chi(z_i, x \cap z_i, y \cap z_i) \\ \quad \text{is not valid;} \\ \chi(z_i, x \cap z_i, y \cap z_i) \vee ((x \cap z_i = (-y) \cap z_i \\ \vee\, x \cap z_i = y \cap z_i) \wedge \exists x\, \chi(z_i, x \cap z_i, y \cap z_i) \\ \wedge\, \exists y\, \chi(z_i, x \cap z_i, y \cap z_i)) \\ \quad \text{otherwise.} \end{cases}$$

Let $\{\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \mid w \in W\}$ be a set of representatives for ϕ. Then set

$$\mathrm{enl}(\phi(\overset{r}{z}, x, y)) = \bigvee_{w \in W} \bigwedge_{i<r} \mathrm{enl}(\chi_{w,i}(z_i, x \cap z_i, y \cap z_i)).$$

If $\models \neg \exists xy\overset{r}{z}\, \phi(\overset{r}{z}, x, y)$, then let $\mathrm{enl}(\phi(\overset{r}{z}, x, y))$ be any inconsistent formula.

From the fact that $\chi'_{w',i} \to \chi_{w,i}$ implies $\mathrm{enl}(\chi'_{w',i}) \to \mathrm{enl}(\chi_{w,i})$, we conclude by an analogous consideration as above that $\mathrm{enl}(\phi)$ is well-defined.

In order to apply Lemmas 2.2 and 2.3 we may replace $\mathrm{enl}(\phi(\overset{r}{z}, x, y))$ by an equivalent (with respect to the theory of atomless Boolean algebras) qf formula.

The next two lemmas collect some properties of s and enl that will be useful in the proofs of 3.7 and of 3.8.

Lemma 3.5. *Let* $\chi_s(z_i, x \cap z_i, y \cap z_i)$, $s = 0, 1$, *be qf and* $R \subseteq r$.

(i) $(\mathrm{enl}(\chi_0) \vee \mathrm{enl}(\chi_1)) \to \mathrm{enl}(\chi_0 \vee \chi_1)$.

(ii) $(s_R(\chi_0) \vee s_R(\chi_1)) \to s_R(\chi_0 \vee \chi_1)$.

For (iii), (iv) *and* (v), *assume additionally that* $\chi_s(z_i, x \cap z_i, y \cap z_i)$, $s = 0, 1$, *determine the same* 1*-type* $t(z_i, x \cap z_i)$ *of* $x \cap z_i$ *over* z_i *and of* $y \cap z_i$ *over* z_i.

(iii) *Assume that, for $s = 0,1$, if not $\phi^{012}(z_i, x\cap z_i, y\cap z_i) \to \chi_s(z_i, x\cap z_i, y\cap z_i)$, then $\chi_s(z_i, x\cap z_i, y\cap z_i) \to x\cap z_i \neq y\cap z_i$. Then* $(\mathrm{enl}(\chi_0) \wedge \mathrm{enl}(\chi_1)) \to \mathrm{enl}(\chi_0 \wedge \chi_1)$.

(iv) $(s_R(\chi_0) \wedge s_R(\chi_1)) \to s_R(\chi_0 \wedge \chi_1)$.

(v) *Assume that $\chi_s \to x\cap z_i = y\cap z_i$ for $s = 0,1$ if $i \notin R$. Then for any $i < r$ the formula*

$$(\mathrm{enl}(s_R(\chi_0))(z_i, x\cap z_i, y\cap z_i) \wedge \mathrm{enl}(s_R(\chi_1))(z_i, x\cap z_i, y\cap z_i)) \\ \to \mathrm{enl}(s_R(\chi_0 \wedge \chi_1))(z_i, x\cap z_i, y\cap z_i)$$

is valid.

Proof. (i), (ii) $\chi_s \to \chi_0 \vee \chi_1$ implies $\mathrm{enl}(\chi_s) \to \mathrm{enl}(\chi_0 \vee \chi_1)$ and $s_R(\chi_s) \to s_R(\chi_0 \vee \chi_1)$.

(iii) Define

$$\phi_=(z_i, x\cap z_i, y\cap z_i) := x\cap z_i = y\cap z_i \wedge t(z_i, x\cap z_i) \quad \text{and}$$
$$\phi_-(z_i, x\cap z_i, y\cap z_i) := x\cap z_i = (-y)\cap z_i \wedge t(z_i, x\cap z_i) \wedge t(z_i, y\cap z_i)\,.$$

Case 1: $\phi^{012} \to \chi_s$ *for* $s = 0,1$. Then $\phi^{012} \to \chi_0 \wedge \chi_1$ and $\mathrm{enl}(\chi_0) \wedge \mathrm{enl}(\chi_1) = (\chi_0 \vee \phi_- \vee \phi_=) \wedge (\chi_1 \vee \phi_- \vee \phi_=) \leftrightarrow (\chi_0 \wedge \chi_1) \vee \phi_- \vee \phi_= = \mathrm{enl}(\chi_0 \wedge \chi_1)$.

Case 2: *Not* $\phi^{012} \to \chi_s$ *for* $s = 0,1$. Then not $\phi^{012} \to \chi_0 \wedge \chi_1$ and $\mathrm{enl}(\chi_0) \wedge \mathrm{enl}(\chi_1) = (\chi_0 \vee \phi_-) \wedge (\chi_1 \vee \phi_-) \leftrightarrow (\chi_0 \wedge \chi_1) \vee \phi_- = \mathrm{enl}(\chi_0 \wedge \chi_1)$.

Case 3: $\phi^{012} \to \chi_0$ *and not* $\phi^{012} \to \chi_1$. Then not $\phi^{012} \to \chi_0 \wedge \chi_1$ and $\mathrm{enl}(\chi_0) \wedge \mathrm{enl}(\chi_1) = (\chi_0 \vee \phi_- \vee \phi_=) \wedge (\chi_1 \vee \phi_-) \leftrightarrow (\chi_0 \wedge \chi_1) \vee \phi_- \vee (\phi_= \wedge \chi_1)$. Since by the assumption of (iii), $\phi_= \wedge \chi_1$ is not satisfiable, the latter formula is equivalent to $(\chi_0 \wedge \chi_1) \vee \phi_- = \mathrm{enl}(\chi_0 \wedge \chi_1)$.

(iv) Assume $i \in R$, otherwise s_R does not change $\chi_0, \chi_1, \chi_0 \wedge \chi_1$.

Case 1: $\phi^{012} \to \chi_s$ *for* $s = 0,1$. Then $\phi^{012} \to \chi_0 \wedge \chi_1$ and $s_R(\chi_0) \wedge s_R(\chi_1) = \chi_0 \wedge \chi_1 = s_R(\chi_0 \wedge \chi_1)$.

Case 2: *E.g. not* $\phi^{012} \to \chi_0$. Then not $\phi^{012} \to \chi_0 \wedge \chi_1$ and $s_R(\chi_0) \wedge s_R(\chi_1) = (\chi_0 \wedge x\cap z_i \neq y\cap z_i) \wedge s_R(\chi_1) \leftrightarrow (\chi_0 \wedge \chi_1) \wedge x\cap z_i \neq y\cap z_i = s_R(\chi_0 \wedge \chi_1)$.

(v) For $i \in R$, the assumptions for (iii) are true for $\psi_s = s_R(\chi_s)$. Hence by (iii) and (iv),

$$(\mathrm{enl}(s_R(\chi_0))(z_i, x\cap z_i, y\cap z_i) \wedge \mathrm{enl}(s_R(\chi_1))(z_i, x\cap z_i, y\cap z_i)) \\ \to \mathrm{enl}(s_R(\chi_0 \wedge \chi_1))(z_i, x\cap z_i, y\cap z_i)\,.$$

For $i \notin R$, we have $\chi_s \to x\cap z_i = y\cap z_i$ for $s = 0,1$ and hence $\mathrm{enl}(s_R(\chi_0)) \wedge \mathrm{enl}(s_R(\chi_1)) = (\chi_0 \vee \phi_-) \wedge (\chi_1 \vee \phi_-) \leftrightarrow (\chi_0 \wedge \chi_1) \vee \phi_- = \mathrm{enl}(s_R(\chi_0 \wedge \chi_1))$.

Lemma 3.6. *Let ϕ be special and satisfiable, $R = R(\phi)$, and let $\{\bigwedge_{i<r} \chi_{w,i} \mid w \in W\}$ be a set of representatives for ϕ.*

(i) *For any $\bigwedge_{i<r} \chi'_i \to \bigvee_{w\in W} \bigwedge_{i<r} s_R(\chi_{w,i})$, there is a $w \in W$ such that $\bigwedge_{i<r} \chi'_i \to \bigwedge_{i<r} s_R(\chi_{w,i})$.*

(ii) $\mathrm{enl}(s(\phi)) \leftrightarrow \bigvee_{w\in W} \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}))$.

(iii) *For any* $\bigwedge_{i<r} \chi'_i \rightarrow \bigvee_{w\in W} \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}))$, *there is a* $w \in W$ *such that* $\bigwedge_{i<r} \chi'_i \rightarrow \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}))$.

Proof. We will first prove (iii). Then the proof of (i) which is similar but easier will be clear. Let $\bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i)$ be consistent, otherwise one can take any $w \in W$.

For $i < r$ there is an n_i, $0 < n_i < 15$, and there are $\widehat{\chi}_{i,0}, \ldots, \widehat{\chi}_{i,n_i-1} \in \{\phi^0, \ldots, \phi^{14}\}$ such that

$$\bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i) \leftrightarrow \bigwedge_{i<r} (\widehat{\chi}_{i,0} \vee \ldots \vee \widehat{\chi}_{i,n_i-1})(z_i, x\cap z_i, y\cap z_i)\,.$$

We will show the claim by induction on $\prod_{i<r} n_i$.

Case $\prod_{i<r} n_i = 1$. Take an atomless Boolean algebra $\mathfrak{A}$ and $\overset{r}{c} \in A$ such that $\overset{r}{c}$ is an enumeration of all the atoms in the generated subalgebra. Take $a, b \in A$ such that $\mathfrak{A} \models \bigwedge_{i<r} \chi'_i(c_i, a\cap c_i, b\cap c_i)$. Then there is some $w \in W$ with $\mathfrak{A} \models \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}(c_i, a\cap c_i, b\cap c_i)))$. Since $\bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i)$ defines an $\mathcal{L}_{\omega\omega}$-2-type of (x, y) over $\overset{r}{z}$, we have $\bigwedge_{i<r} \chi'_i(z_i, x\cap z_i, y\cap z_i) \rightarrow \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}(z_i, x\cap z_i, y\cap z_i)))$.

Induction step. We consider the step from $\prod_{i<r} n_i$ to $(n_0 + 1) \times \prod_{0<i<r} n_i$, the other cases are similar.

$$(\widehat{\chi}_{0,0}\vee\ldots\vee\widehat{\chi}_{0,n_0})\wedge \bigwedge_{0<i<r} \chi'_i \leftrightarrow \Big(\widehat{\chi}_{0,0}\wedge \bigwedge_{0<i<r} \chi'_i\Big)\vee\Big((\widehat{\chi}_{0,1}\vee\ldots\vee\widehat{\chi}_{0,n_0})\wedge \bigwedge_{0<i<r} \chi'_i\Big).$$

By induction hypothesis there are $w', w'' \in W$ such that

$$\widehat{\chi}_{0,0} \wedge \bigwedge_{0<i<r} \chi'_i \rightarrow \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w',i}))\,,$$
$$(\widehat{\chi}_{0,1} \vee \ldots \vee \widehat{\chi}_{0,n_0}) \wedge \bigwedge_{0<i<r} \chi'_i \rightarrow \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w'',i}))\,.$$

Thus we have

$$\Big(\Big(\widehat{\chi}_{0,0} \wedge \bigwedge_{0<i<r} \chi'_i\Big) \vee \Big((\widehat{\chi}_{0,1} \vee \ldots \vee \widehat{\chi}_{0,n_0}) \wedge \bigwedge_{0<i<r} \chi'_i\Big)\Big) \rightarrow$$
$$(\mathrm{enl}(s_R(\chi_{w',0})) \vee \mathrm{enl}(s_R(\chi_{w'',0}))) \wedge \bigwedge_{0<i<r} (\mathrm{enl}(s_R(\chi_{w',i})) \wedge \mathrm{enl}(s_R(\chi_{w'',i})))\,.$$

Note that in the last conjunction we get "and" and not only "or", because

$$\bigwedge_{0<i<r} \chi'_i \rightarrow \bigwedge_{0<i<r} \mathrm{enl}(s_R(\chi_{w',i})) \wedge \bigwedge_{0<i<r} \mathrm{enl}(s_R(\chi_{w'',i}))\,,$$

as the situation below any z_i is independent of the situation below the other z_j.

From 3.5(i), (ii) and (v) we get

$$\Big(\widehat{\chi}_{0,0} \wedge \bigwedge_{0<i<r} \chi'_i\Big) \vee \Big((\widehat{\chi}_{0,1} \vee \ldots \vee \widehat{\chi}_{0,n_0}) \wedge \bigwedge_{0<i<r} \chi'_i\Big)$$
$$\to \operatorname{enl}(s_R(\chi_{w',0} \vee \chi_{w'',0})) \wedge \bigwedge_{0<i<r} \operatorname{enl}(s_R(\chi_{w',i} \wedge \chi_{w'',i}))\,.$$

Since $\{\bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i) \,|\, w \in W\}$ is a set of representatives for $\phi(\overset{r}{z}, x, y)$ and since $w', w'' \in W$, we have $(\chi_{w',0} \vee \chi_{w'',0}) \wedge \bigwedge_{0<i<r}(\chi_{w',i} \wedge \chi_{w'',i}) \to \phi$ and there is a $w \in W$ such that

$$(\chi_{w',0} \vee \chi_{w'',0}) \wedge \bigwedge_{0<i<r} (\chi_{w',i} \wedge \chi_{w'',i}) \to \bigwedge_{i<r} \chi_{w,i}\,.$$

For such a w we have

$$\operatorname{enl}(s_R(\chi_{w',0} \vee \chi_{w'',0})) \wedge \bigwedge_{0<i<r} \operatorname{enl}(s_R(\chi_{w',i} \wedge \chi_{w'',i})) \to \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))\,,$$

and thus the induction step is complete and (iii) is shown.

(ii) Assume $s(\phi)$ is satisfiable, otherwise both sides are not satisfiable. Let $S = \{\bigwedge_{i<r} \chi_{w,i} \,|\, w \in W\}$ be a set of representatives for ϕ, and $S' = \{\bigwedge_{i<r} \chi'_{w',i} \,|\, w' \in W'\}$ be a set of representatives for $s(\phi) = \bigvee_{w\in W} \bigwedge_{i<r} s_R(\chi_{w,i})$ such that $W' \supseteq \widehat{W} := \{w \in W \mid \bigwedge_{i<r} s_R(\chi_{w,i})$ is satisfiable$\}$ and $\chi'_{w',i} = s_R(\chi_{w,i})$ for $w \in \widehat{W}$.

By definition, $\operatorname{enl}(s_R(\phi)) = \bigvee_{w'\in W'} \bigwedge_{i<r} \operatorname{enl}(\chi'_{w',i})$. By (i), for any $w' \in W'$ there is some $w \in W$ such that $\bigwedge_{i<r} \chi'_{w',i} \to \bigwedge_{i<r} s_R(\chi_{w,i})$ and hence $\bigwedge_{i<r} \operatorname{enl}(\chi'_{w',i}) \to \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$. Thus $\operatorname{enl}(s(\phi)) \to \bigvee_{w\in W} \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$. The other direction follows immediately from the choice of S' and the definition of enl.

LEMMA 3.7. *Let ϕ be a special formula. Then* $\operatorname{enl}(\operatorname{enl}(s(\phi))) \leftrightarrow \operatorname{enl}(s(\phi))$.

Proof. Assume $s(\phi)$ is satisfiable, otherwise both sides are not satisfiable. Let S, W be as above and $S'' = \{\bigwedge_{i<r} \chi''_{w'',i} \,|\, w'' \in W''\}$ be a set of representatives for $\operatorname{enl}(s(\phi))$. By definition, $\operatorname{enl}(\operatorname{enl}(s(\phi))) = \bigvee_{w''\in W''} \bigwedge_{i<r} \operatorname{enl}(\chi''_{w'',i})$. For $w'' \in W''$ we have $\bigwedge_{i<r} \chi''_{w'',i} \to \operatorname{enl}(s(\phi))$, hence by 3.6(ii), $\bigwedge_{i<r} \chi''_{w'',i} \to \bigvee_{w\in W} \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$. By 3.6(iii) there is some $w \in W$ such that $\bigwedge_{i<r} \chi''_{w'',i} \to \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$, whence $\bigwedge_{i<r} \operatorname{enl}(\chi''_{w'',i}) \to \bigwedge_{i<r} \operatorname{enl}(\operatorname{enl}(s_R(\chi_{w,i})))$. It is easy to check that for qf $\chi(z_i, x \cap z_i, y \cap z_i)$ by definition

$$\operatorname{enl}(\operatorname{enl}(\chi(z_i, x \cap z_i, y \cap z_i))) \to \operatorname{enl}(\chi(z_i, x \cap z_i, y \cap z_i))\,.$$

Therefore $\bigwedge_{i<r} \operatorname{enl}(\chi''_{w'',i}) \to \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$, and putting things together yields $\bigvee_{w''\in W''} \bigwedge_{i<r} \operatorname{enl}(\chi''_{w'',i}) \to \bigvee_{w\in W} \bigwedge_{i<r} \operatorname{enl}(s_R(\chi_{w,i}))$, and, by 3.6(ii), $\bigvee_{w''\in W''} \bigwedge_{i<r} \operatorname{enl}(\chi''_{w'',i}) \to \operatorname{enl}(s(\phi))$.

The other direction is obvious.

Lemma 3.8. $\neg\mathrm{big}(s(\phi)) \to \neg\mathrm{big}(\mathrm{enl}(s(\phi)))$ *is valid for special* ϕ.

Proof. Let $\mathfrak{A}$ be any atomless Boolean algebra. Assume $\mathfrak{A} \models \mathrm{big}(\mathrm{enl}(s(\phi(\overset{r}{z}, x, y))))(\overset{r}{c})$. We show that $\mathfrak{A} \models \mathrm{big}(s(\phi(\overset{r}{z}, x, y)))(\overset{r}{c})$. Since the 1-types of x and of y over $\overset{r}{c}$ are determined by $\mathfrak{A} \models \exists y\, \mathrm{enl}(s(\phi(\overset{r}{c}, x, y)))$ and $\mathfrak{A} \models \exists x\, \mathrm{enl}(s(\phi(\overset{r}{c}, x, y)))$, there is just one pair (I_2, I_3) such that

$$\mathfrak{A} \models \bigvee_{\{(I_0,I_1) \mid I_0 \dot\cup I_1 \dot\cup I_2 \dot\cup I_3 = \{0,\ldots,r-1\}, I_0 \neq 0\}} \forall xy$$
$$\Big(\Big(\bigwedge_{i\in I_0} \phi^{012}(c_i, x\cap c_i, y \cap c_i) \wedge \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i$$
$$\wedge \bigwedge_{i\in I_2} x\cap c_i = y \cap c_i = 0 \wedge \bigwedge_{i\in I_3} x \cap c_i = y\cap c_i = c_i\Big) \to \mathrm{enl}(s(\phi(\overset{r}{c}, x, y)))\Big).$$

Take I_0 $\subseteq$-maximal such that

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{i\in I_0} \phi^{012}(c_i, x\cap c_i, y \cap c_i) \wedge \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i$$
$$\wedge \bigwedge_{i\in I_2} x\cap c_i = y \cap c_i = 0 \wedge \bigwedge_{i\in I_3} x \cap c_i = y\cap c_i = c_i\Big) \to \mathrm{enl}(s(\phi(\overset{r}{c}, x, y)))\Big).$$

Let $R = R(\phi)$ and $\{\bigwedge_{i<r} \chi_{w,i} \mid w \in W\}$ be a set of representatives for ϕ. By 3.6(ii) and (iii) there is a $w \in W$ such that

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{i\in I_0} \phi^{012}(c_i, x\cap c_i, y \cap c_i) \wedge \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i$$
$$\wedge \bigwedge_{i\in I_2} x\cap c_i = y \cap c_i = 0 \wedge \bigwedge_{i\in I_3} x \cap c_i = y\cap c_i = c_i\Big)$$
$$\to \bigwedge_{i<r} \mathrm{enl}(s_R(\chi_{w,i}(c_i, x\cap c_i, y\cap c_i)))\Big).$$

We claim that also

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{i\in I_0} \phi^{012}(c_i, x\cap c_i, y \cap c_i) \wedge \bigwedge_{i \in I_1} x \cap c_i = y \cap c_i \neq 0, c_i$$
$$\wedge \bigwedge_{i\in I_2} x\cap c_i = y \cap c_i = 0 \wedge \bigwedge_{i\in I_3} x \cap c_i = y\cap c_i = c_i\Big)$$
$$\to \bigwedge_{i<r} s_R(\chi_{w,i}(c_i, x\cap c_i, y\cap c_i))\Big).$$

Indeed, by the definition of enl we have for any $s_R(\chi_{w,i}(z_i, x\cap z_i, y\cap z_i))$: For $i \in I_0$, if $\phi^{012} \to \mathrm{enl}(s_R(\chi_{w,i}))$, then $\phi^{012} \to s_R(\chi_{w,i})$. For $i \in I_2$, if $x \cap z_i = y \cap z_i = 0 \to \mathrm{enl}(s_R(\chi_{w,i}))$, then $x\cap z_i = y\cap z_i = 0 \to s_R(\chi_{w,i})$.

For $i \in I_3$, if $x \cap z_i = y \cap z_i = z_i \to \mathrm{enl}(s_R(\chi_{w,i}))$, then $x \cap z_i = y \cap z_i = z_i \to s_R(\chi_{w,i})$.

For $i \in I_1$ the formula $x \cap z_i = y \cap z_i \neq 0, z_i \wedge \mathrm{enl}(s_R(\chi_{w,i})) \wedge \neg s_R(\chi_{w,i})$ is consistent only if $\phi^{012} \to s_R(\chi_{w,i})$. But then we could take $I_0' := I_0 \cup \{i\}$ and $I_1' = I_1 \setminus \{i\}$ and replace (I_0, I_1) by (I_0', I_1'), which contradicts the maximality of I_0.

Now we are ready to prove $(**)$ for special formulas of the form $s(\phi)$.

LEMMA 3.9. *Let ϕ be special and $\overset{r}{c} \in B$ be an r-tuple that consists of atoms in the generated subalgebra.*

(i) *If $\neg\mathrm{big}(\phi)$ and $\mathrm{enl}(\phi) \to \phi$ are valid, then for any α with $\overset{r}{c} \in B_\alpha$ the relation $\phi(\overset{r}{c}, x, y)$ is small in $\mathfrak{B}_\alpha$.*

(ii) *If $\neg\mathrm{big}(s(\phi))$ is valid, then for any α with $\overset{r}{c} \in B_\alpha$ the relation $\mathrm{enl}(s(\phi(\overset{r}{c}, x, y)))$ is small in $\mathfrak{B}_\alpha$.*

Proof. (i) Let $\mathfrak{B} \models \neg\mathrm{big}(\phi(\overset{r}{z}, x, y))(\overset{r}{c})$ and $\overset{r}{c} \in B_\alpha$ be atoms in the generated subalgebra. Set $\mathfrak{B}_\alpha =: \mathfrak{A}$, and let $M \neq \emptyset$ be a maximally homogeneous set for $\phi(\overset{r}{c}, x, y)$ in $\mathfrak{A}$, and $(a, b)_A \in P(A)$, i.e. $(a, b)_A$ is an interval in $\mathfrak{A}$. Take $(a', b')_A \le (a, b)_A$ such that there is just one $i \in r$, say i_0, with $(b' \setminus a') \subseteq c_i$ and $c_i \cap a' \neq 0$ and $b' \cap c_i \neq c_i$. We assume $\mathfrak{B}$ (and also $\mathfrak{A}$ and $\mathcal{P}(\omega)$) satisfy

$$\forall x \in (a', b')(\exists y\, \phi(\overset{r}{z}, x, y) \wedge \exists y\, \phi(\overset{r}{z}, y, x))(\overset{r}{c}),$$

for otherwise $(a', b')_A \in D_A(M, \phi(\overset{r}{c}, x, y), 1, 0)$.

Since $\mathfrak{B} \models \neg\mathrm{big}(\phi)(\overset{r}{c})$, we have $(a', b')_A \cap M \neq (a', b')_A$. We fix a $d \in (a', b')_A \setminus M$ and an $m \in M$ such that $\mathfrak{A} \models \neg\phi(\overset{r}{c}, d, m) \vee \neg\phi(\overset{r}{c}, m, d)$, say $\mathfrak{A} \models \neg\phi(\overset{r}{c}, d, m)$, and show that there is an $(a'', b'')_A \le (a', b')_A$ such that for any $x \in (a'', b'')_{\mathcal{P}(\omega)}$ we have $x \in M$ or $\mathcal{P}(\omega) \models \neg\phi(\overset{r}{c}, x, m)$.

Then (i) will be proved, because such an $(a'', b'')_A$ is in $D_A(M, \phi(\overset{r}{c}, x, y), 1, 0)$. Fix a set $\{\bigwedge_{i<r} \chi_{w,i} \mid w \in W\}$ of representatives for ϕ.

CLAIM. $d \cap c_{i_0} \neq c_{i_0} \setminus m$.

Proof. $\phi(\overset{r}{z}, x, y) = \bigvee_{w \in W} \bigwedge_{i<r} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i)$, w.l.o.g. $W = \{0, 1, \ldots, s-1\}$. Hence $\mathfrak{A} \models \bigwedge_{w \in W} \bigvee_{i<r} \neg\chi_{w,i}(c_i, d \cap c_i, m \cap c_i)$, say for $w = 0, 1, \ldots, s'-1$

$$\mathfrak{A} \models \bigvee_{i<r, i \neq i_0} \neg\chi_{w,i}(c_i, d \cap c_i, m \cap c_i),$$

and for $w = s', s'+1, \ldots, s-1$

$$\mathfrak{A} \models \bigwedge_{i<r, i \neq i_0} \chi_{w,i}(c_i, d \cap c_i, m \cap c_i) \wedge \neg\chi_{w,i_0}(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0}).$$

We may assume $s > 0$ and $s' \leq s-1$, because otherwise $(a',b')_A \in D_A(M, \phi(\overset{r}{c}, x, y), 1, 0)$. Since

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{s' \leq w < s} \bigwedge_{i<r, i \neq i_0} \chi_{w,i}(c_i, x \cap c_i, y \cap c_i) \\ \wedge \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})\Big) \rightarrow \phi(\overset{r}{c}, y, x)\Big)\,,$$

we have

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{s' \leq w < s} \bigwedge_{i<r, i \neq i_0} \chi_{w,i}(c_i, x \cap c_i, y \cap c_i) \\ \wedge \Big(\bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \vee (x \cap c_{i_0} = (-y) \cap c_{i_0} \\ \wedge \exists x\, \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \wedge \exists y\, \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}))\Big)\Big) \\ \rightarrow \mathrm{enl}(\phi(\overset{r}{c}, x, y))\Big)\,.$$

By the assumptions on $\phi(\overset{r}{z}, x, y)$ and on $\overset{r}{c}$ there is just one 1-type of $x \cap c_{i_0}$ over c_{i_0} consistent with $\phi(\overset{r}{c}, x, y)$ such that for every $w \in W$ the formula $\exists y\, \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$ is implied by this type. The same holds for the 1-type of $y \cap c_{i_0}$ over c_{i_0}, which coincides with the 1-type of $x \cap c_{i_0}$ over c_{i_0}, and the formula $\exists x\, \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$. Since $m \cap c_{i_0}$ and $d \cap c_{i_0}$ have this 1-type, we get

$$\mathfrak{A} \models \exists x \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \\ \wedge \exists y \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, d \cap c_{i_0}, y \cap c_{i_0})\,.$$

Note that $\mathfrak{A} \models \neg\phi(\overset{r}{c}, d, m)$ and ϕ is equivalent to $\mathrm{enl}(\phi)$. Therefore $d \cap c_{i_0} \neq c_{i_0} \setminus m$ and the claim is proved.

We now give $(a'', b'')_A$ case by case.

Case 1: $d \cap c_{i_0} \neq m \cap c_{i_0}$. Then

$$\mathfrak{A} \models \bigvee_{i=0,1,2,4,8} \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0})\,.$$

Assume that $\mathfrak{A} \models \phi^i(c_{i_0}, d \cap c_{i_0}, m \cap c_{i_0})$.

If $i = 0$ or $i = 2$, take an e' such that $0 \subset e' \subset c_{i_0} \cap m \cap (-d)$, and $(a'', b'')_A = (d, b' \setminus e')_A$. If $i = 1$ or $i = 8$, take $(a'', b'')_A = (a', d)_A$. Finally, if $i = 4$, take $(a'', b'')_A = (d, b')_A$.

Then, in each subcase, for any $x \in (a'', b'')_{\mathcal{P}(\omega)}$ we have

$$\mathcal{P}(\omega) \models \mathrm{tp}(x, m/\overset{r}{c}) = \mathrm{tp}(d, m/\overset{r}{c}) \text{ and hence } \mathcal{P}(\omega) \models \neg\phi(\overset{r}{c}, x, m)\,.$$

Case 2: $d \cap c_{i_0} = m \cap c_{i_0}$.

Subcase 2.1:

$$\mathfrak{A} \models \exists xy\Big(\phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \wedge \neg \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})\Big)\,.$$

Since $\phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})$ determines the $\mathcal{L}_{\omega\omega}$-1-type of $y \cap c_{i_0}$ over c_{i_0}, and m has the same one, we have

$$\mathfrak{A} \models \exists x\Big(\phi^{012}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0}) \wedge \neg \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, m \cap c_{i_0})\Big)\,.$$

There is an example d' for x with $d' \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$, because $m \cap c_{i_0} = d \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$ and hence within the given 1-type of $x \cap c_{i_0}$ over c_{i_0} the formula $\phi^i(c_i, x \cap c_i, m \cap c_i)$ can be realized with some $x \cap c_{i_0} \in (a' \cap c_{i_0}, b' \cap c_{i_0})_A$ for $i = 0, 1, 2$. We can argue with $(d' \cap c_{i_0}) \cup (d \setminus c_{i_0})$ as with d in case 1 for $i = 0, 1, 2$.

Subcase 2.2:

$$\mathfrak{A} \models \forall xy\Big(\phi^{012}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0}) \to \bigvee_{s' \leq w < s} \chi_{w,i_0}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})\Big)\,.$$

Again we have

$$\mathfrak{A} \models \forall xy\Big(\Big(\bigwedge_{s' \leq w < s} \ \bigwedge_{i < r, i \neq i_0} \chi_{w,i}(c_i, x \cap c_i, y \cap c_i) \\ \wedge \bigvee_{s' \leq w < s} \chi_{w,i}(c_{i_0}, x \cap c_{i_0}, y \cap c_{i_0})\Big) \to \phi(\overset{r}{c}, x, y)\Big)\,.$$

Since

$$\phi^{012}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \to \bigvee_{s' \leq w < s} \chi_{w,i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0})\,,$$

by the definition of enl we have

$$\forall xy\Big(\Big(\bigwedge_{i < r, i \neq i_0} \mathrm{enl}\Big(\bigwedge_{s' \leq w < s} \chi_{w,i}(z_i, x \cap z_i, y \cap z_i)\Big) \\ \wedge \Big(\bigvee_{s' \leq w < s} \chi_{w,i}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \vee \Big(x \cap z_{i_0} = y \cap z_{i_0} \\ \wedge \exists x \bigvee_{s' \leq w < s} \chi_{w,i_0}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0}) \\ \wedge \exists y \bigvee_{s' \leq w < s} \chi_{w,i_0}(z_{i_0}, x \cap z_{i_0}, y \cap z_{i_0})\Big)\Big)\Big) \to \mathrm{enl}(\phi(\overset{r}{z}, x, y))\Big)\,.$$

In $\mathfrak{A}$ we get

$$\begin{aligned}\mathfrak{A} \models \forall xy\Big(\Big(&\bigwedge_{s'\le w<s}\ \bigwedge_{i<r, i\ne i_0} \mathrm{enl}(\chi_{w,i}(c_i, x\cap c_i, y\cap c_i))\\ &\wedge \Big(\bigvee_{s'\le w<s} \chi_{w,i}(c_{i_0}, x\cap c_{i_0}, y\cap c_{i_0}) \vee \Big(x\cap c_{i_0} = y\cap c_{i_0}\\ &\wedge \exists x \bigvee_{s'\le w<s} \chi_{w,i_0}(c_{i_0}, x\cap c_{i_0}, y\cap c_{i_0})\\ &\wedge \exists y \bigvee_{s'\le w<s} \chi_{w,i_0}(c_{i_0}, x\cap c_{i_0}, y\cap c_{i_0})\Big)\Big)\Big) \to \mathrm{enl}(\phi(\overset{r}{c}, x, y))\Big)\,.\end{aligned}$$

As in the first subcase, we get

$$\begin{aligned}\mathfrak{A} \models \exists x &\bigvee_{s'\le w<s} \chi_{w,i_0}(c_{i_0}, x\cap c_{i_0}, m\cap c_{i_0})\\ &\wedge \exists y \bigvee_{s'\le w<s} \chi_{w,i_0}(c_{i_0}, d\cap c_{i_0}, y\cap c_{i_0}) \wedge d\cap c_{i_0} = m\cap c_{i_0}\,.\end{aligned}$$

Putting things together yields $\mathfrak{A} \models \mathrm{enl}(\phi(\overset{r}{c}, d, m))$ and hence $\mathfrak{A} \models \phi(\overset{r}{c}, d, m)$, a contradiction to the choice of d and m.

(ii) By 3.8, $\neg\mathrm{big}(s(\phi)) \to \neg\mathrm{big}(\mathrm{enl}(s(\phi)))$, and, by 3.7, $\mathrm{enl}(\mathrm{enl}(s(\phi))) \to \mathrm{enl}(s(\phi))$ is valid. Therefore (ii) follows from (i) applied to $\mathrm{enl}(s(\phi))$.

Lemma 3.9, the construction and the monotonicity of Q_1^2 yield:

THEOREM 3.10. *For any special* ϕ,

$$\mathfrak{B} \models \forall\overset{r}{z}\ ((\text{“}\overset{r}{z}\ \textit{are the atoms in the generated subalgebra}\text{”} \wedge \neg\mathrm{big}(s(\phi))(\overset{r}{z})) \to \neg Q_1^2 xy\ s(\phi(\overset{r}{z}, x, y)))\,.$$

Finally, we show how to get Theorem 3.10 for ϕ instead of $s(\phi)$.

THEOREM 3.11. *For any special* ϕ

$$\mathfrak{B} \models \forall\overset{r}{z}\ ((\text{“}\overset{r}{z}\ \textit{are the atoms in the generated subalgebra}\text{”} \wedge \neg\mathrm{big}(\phi)(\overset{r}{z})) \to \neg Q_1^2 xy\, \phi(\overset{r}{z}, x, y))\,.$$

P r o o f (by induction on $\mathrm{card}(R(\phi))$). If $R(\phi) = \emptyset$, then $\phi(\overset{r}{z}, x, y) \to x = y$, and hence $\mathfrak{B} \models \neg Q_1^2 xy\, \phi(\overset{r}{c}, x, y)$.

Now assume $\mathfrak{B} \models \forall\overset{r}{z}$ ((“$\overset{r}{z}$ are the atoms in the generated subalgebra” $\wedge\neg\mathrm{big}(\psi)(\overset{r}{z})) \to \neg Q_1^2 xy\, \psi(\overset{r}{z}, x, y))$ for all ψ with $R(\psi) \subset R(\phi)$. We show $\mathfrak{B} \models Q_1^2 xy\, \phi(\overset{r}{c}, x, y) \to \mathrm{big}(\phi)(\overset{r}{c})$ for any r-tuple $\overset{r}{c}$ that consists of atoms in the generated subalgebra. Assume $\mathfrak{B} \models Q_1^2 xy\, \phi(\overset{r}{c}, x, y)$ and let H be an uncountable homogeneous set for $\phi(\overset{r}{c}, x, y)$ in $\mathfrak{B}$. By recursion on $i \le r$ we define uncountable subsets $H^{(i)}$, $0 \le i \le r$.

Set $H^{(0)} := H$. Assume $H^{(i)}$ is defined. We distinguish two cases:

Case 1: $\{x \cap c_i \mid x \in H^{(i)}\}$ *is uncountable*. Then take $H^{(i+1)} \subseteq H^{(i)}$ such that $H^{(i+1)}$ is uncountable and for any $x, y \in H^{(i+1)}$, if $x \neq y$ then $x \cap c_i \neq y \cap c_i$.

Case 2: $\{x \cap c_i \mid x \in H^{(i)}\}$ *is countable*. Then there is some $x \in H^{(i)}$ such that $\{y \in H^{(i)} \mid x \cap c_i = y \cap c_i\}$ is uncountable. Let $H^{(i+1)}$ be such a set.

For $i \notin R$, $\{x \cap c_i \mid x \in H^{(i)}\}$ is a singleton, and we are in case 2. Now consider $H^{(0)}, H^{(1)}, \ldots, H^{(r)}$. If for all $i \in R$ case 1 is true, then $H^{(r)}$ shows $\mathfrak{B} \models Q_1^2 xy\, s(\phi(\overset{r}{c}, x, y))$. By 3.10, $\mathfrak{B} \models \mathrm{big}(s(\phi(\overset{r}{c})))$. Since $s(\phi) \to \phi$, $\mathfrak{B} \models \mathrm{big}(\phi(\overset{r}{c}))$.

If there is some $i \in R$ with case 2 being true, fix such an i. Then $H^{(i+1)}$ shows $\mathfrak{B} \models Q_1^2 xy\, (\phi \wedge x \cap z_i = y \cap z_i)(\overset{r}{c}, x, y)$. Take $\psi = \phi \wedge x \cap z_i = y \cap z_i$. Then ψ is also special. Since $\psi \to \phi$ and $i \in R(\phi) \setminus R(\psi)$, we have $R(\psi) \subset R(\phi)$. By induction hypothesis, we conclude from $\mathfrak{B} \models Q_1^2 xy\, (\phi \wedge x \cap z_i = y \cap z_i)(\overset{r}{c}, x, y)$ that $\mathfrak{B} \models \mathrm{big}(\psi(\overset{r}{c}))$ and hence $\mathfrak{B} \models \mathrm{big}(\phi(\overset{r}{c}))$.

Acknowledgement. The results are a part of the author's doctoral thesis. I would like to thank my thesis advisor H.-D. Ebbinghaus for his friendly support.

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MATHEMATISCHES INSTITUT
UNIVERSITÄT BONN
BERINGSTR. 4
D-5300 BONN 1, GERMANY

Received 8 April 1991;
in revised form 24 July 1991, 18 February 1992 and 14 September 1992