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On the LC^1-spaces which are Cantor or arcwise homogeneous

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Abstract. A space X containing a Cantor set (an arc) is *Cantor* (*arcwise*) *homogeneous* iff for any two Cantor sets (arcs) $A, B \subset X$ there is an autohomeomorphism h of X such that $h(A) = B$. It is proved that a continuum (an arcwise connected continuum) X such that either $\dim X = 1$ or $X \in LC^1$ is Cantor (arcwise) homogeneous iff X is a closed manifold of dimension at most 2.

1. Introduction. In the paper [O–P] by K. Omiljanowski and the present author the spaces which are Cantor or arcwise homogeneous have been defined and the problem to describe all such spaces has been proposed.

In this paper we shall prove the following two theorems:

THEOREM 1. *Let X be a continuum such that either* $\dim X = 1$ *or* $X \in LC^1$. *Then X is Cantor homogeneous iff X is a closed manifold of dimension at most* 2.

THEOREM 2. *Let X be an arcwise connected continuum such that either* $\dim X = 1$ *or* $X \in LC^1$. *Then X is arcwise homogeneous iff X is a closed manifold of dimension at most* 2.

Recall that a space X containing a Cantor set (an arc) is *Cantor* (*arcwise*) *homogeneous* iff for any two Cantor sets (arcs) $A, B \subset X$ there is an autohomeomorphism h of X mapping A onto B.

The strategy of the proof of Theorem 1 is the following: Of course it suffices to prove the "only if" part of the theorem. So assume that X is Cantor homogeneous. If $\dim X = 1$ then $X \cong S^1$ by [O–P] (Prop. 2.3 and Cor. 2.9). Thus we can assume that $\dim X \geq 2$ and $X \in LC^1$. As in [O–P] (Th. 2.7) we notice that

$(*)$ X has no locally disconnecting points.

A simple closed curve $S \subset X$ will be called an *NDC-curve* if S does not disconnect X and S is contractible in a proper subset of X. Assume that X is not a 2-manifold. It follows that:

(∗∗) X contains arbitrarily small *NDC*-curves.

Indeed, we infer from (∗) and from Young's characterization of 2-manifolds (see [Y]) that X contains arbitrarily small simple closed curves which do not disconnect X. Thus $X \in LC^1$ implies (∗∗).

Now, to prove that there exists no Cantor homogeneous continuum X with $\dim X \geq 2$, $X \in LC^1$ and which is not a 2-manifold, we define tame and wild Cantor sets similarly to the usual definition and we prove (in Sections 2 and 3) that any space X as above must contain Cantor sets of both kinds.

A Cantor set $C \subset X$ will be called *tame* if each *NDC*-curve $S \subset X \setminus C$ is contractible in $X \setminus C$. Otherwise C will be called *wild*.

The strategy of the proof of Theorem 2 is similar to that of Theorem 1. The proof is given in Section 4.

2. Construction of a tame Cantor set. The existence of a tame Cantor set $C \subset X$ follows from the strong n-homogeneity for all n of a Cantor homogeneous continuum $X \not\cong S^1$ (see [O–P], Th. 2.7) and from the following

LEMMA 1. *Let $X \in LC^1$ be a non-degenerate continuum which is strongly n-homogeneous for all $n = 1, 2, \ldots$ Then there is a tame Cantor set $C \subset X$.*

Proof. First notice that the assumptions of the lemma imply that X satisfies the condition (∗) given before. Indeed, if X contains a locally disconnecting point then by Whyburn's theorem [Wh1] and by homogeneity of X we have $\operatorname{ord}_x X = 2$ for every $x \in X$. This implies $X \cong S^1$, but S^1 is not strongly 3-homogeneous. It follows that:

(1) No subset of X which is the union of an *NDC*-curve and of a finite set disconnects X.

Let $\mathcal{S}$ denote the subspace of the space X^{S^1} (with the "sup" metric) consisting of all homeomorphisms $\phi : S^1 \to \phi(S^1) \subset X$ such that $\phi(S^1)$ is an *NDC*-curve in X. We can assume that $\mathcal{S}$ is non-empty. Then $\mathcal{S}$ is separable, and therefore there exists a sequence $\phi_1, \phi_2, \ldots$ dense in $\mathcal{S}$ such that each ϕ_i occurs in it infinitely often. Let $S_i = \phi_i(S^1)$.

To find a tame Cantor set $C \subset X$ we construct a sequence of closed balls $Q_{i_1 \ldots i_k} \subset X$, $i_j = 0, 1$ for $k = 1, 2, \ldots$, in the given metric of X such that:

$(1)_k$ If $\langle i_1, \ldots, i_k \rangle \neq \langle i'_1, \ldots, i'_k \rangle$, then $Q_{i_1 \ldots i_k}$ and $Q_{i'_1 \ldots i'_k}$ are disjoint.

$(2)_k$ $\operatorname{diam} Q_{i_1 \ldots i_k} < 1/k$.

$(3)_{k+1}$ $Q_{i_1 \ldots i_k i_{k+1}} \subset \operatorname{Int} Q_{i_1 \ldots i_k}$ for $i_{k+1} = 0, 1$.

$(4)_{k+1}$ If $S_k \subset X \setminus A_k$, where $A_k = \bigcup \{Q_{i_1 \ldots i_k} : i_j = 0, 1 \text{ for } j \leq k\}$, then S_k is contractible in $X \setminus A_{k+1}$.

We construct such a sequence of balls by induction.

Find any two balls Q_0, Q_1 satisfying $(1)_1$ and $(2)_1$ and assume inductively that the balls $Q_{i_1 \ldots i_j}$ for $j \le k$ have been constructed.

To construct the balls $Q_{i_1 \ldots i_k i_{k+1}}$ for $i_j = 0, 1$, $j \le k+1$, we can assume that $S_k \subset X \setminus A_k$. Since S_k is contractible in a proper subset of X, there is a map f of a disc D into X such that $f|\dot{D} : \dot{D} \to S_k$ is a homeomorphism and $f(D) \ne X$. Choose some points $q_{i_1 \ldots i_k} \in \operatorname{Int} Q_{i_1 \ldots i_k} \setminus S_k$ for $i_j = 0, 1$, $j \le k$ and order them into a sequence $p_1, \ldots, p_l$, where $l = 2^k$. We will construct by induction some maps $g_1, \ldots, g_l$: $D \to X$ such that $g_i|\dot{D} = f|\dot{D}$ and $p_j \notin g_i(D)$ for $j \le i$.

To find g_1, choose a point $a \in X \setminus f(D)$ and an arc $I \subset X \setminus S_k = X \setminus f(\dot{D})$ joining a to p_1. Denote by J the subset of I consisting of those points $x \in I$ for which there is a map $g : D \to X$ such that $g|\dot{D} = f|\dot{D}$ and $x \notin g(D)$. Then J is an open and non-empty subset of I, because $a \in J$. Observe that J is also a closed subset of I. Indeed, let $x_0 \in \operatorname{Cl} J$. Since $I \subset X \setminus S_k$ and $X \in LC^1$, there is an $\varepsilon > 0$ such that if h is an autohomeomorphism of X which is ε-close to the identity and ϕ is a homeomorphism of S^1 onto S_k then ϕ and $h\phi : S^1 \to h(S_k)$ are homotopic in $X \setminus I$ (cf. [Hu], p. 160). Since X is homogeneous and $x_0 \in \operatorname{Cl} J$, we infer from the well-known Effros theorem [E] that there are a $y_0 \in J$ and an autohomeomorphism h of X which is ε-close to the identity and which maps x_0 onto y_0. Consequently, there is a map $g' : D \to X$ such that $g'|\dot{D} = f|\dot{D}$ and $x_0 \notin g'(D)$, which proves that $x_0 \in J$. Thus $J = I$ contains p_1, and therefore the desired map g_1 exists.

If $1 \le m < l$ and $g_m : D \to X$ exists, we prove the existence of g_{m+1} in a similar way. By (1) there is an arc $I \subset X \setminus S_k$ joining $a \in X \setminus g_m(D)$ to p_{m+1} and such that $p_i \notin I$ for $i \le m$. We consider the set J of those $x \in I$ for which there exists a map $g : D \to X$ such that $g|\dot{D} = g_m|\dot{D} = f|\dot{D}$ and none of the points $p_1, \ldots, p_m, x$ belongs to $g(D)$. As before we prove that J is non-empty and both open and closed in I. In the proof that J is closed we use the strong $(m+1)$-homogeneity of X and the Effros theorem for the action of the group $H(X)$ of autohomeomorphisms of X on the space

$$F_{m+1}(X) = \{\langle x_1, \ldots, x_{m+1} \rangle \in X^{m+1} : x_i \ne x_j \text{ for } i \ne j\}.$$

This concludes the construction of the maps $g_1, \ldots, g_l$ and therefore there is a map $g_l : D \to X$ such that $g_l|\dot{D}$ is a homeomorphism of $\dot{D}$ onto S_k and some neighborhoods of the points $p_1, \ldots, p_l$ are disjoint from $g_l(D)$.

Now, it is easy to construct the balls $Q_{i_1 \ldots i_{k+1}}$ for $i_j = 0, 1$, $j \le k+1$, such that the conditions $(1)_{k+1}$–$(4)_{k+1}$ are satisfied. Having constructed $Q_{i_1 \ldots i_k}$ for all $k = 1, 2, \ldots$, we can define the desired Cantor set:

$$C = \bigcap_{k=1}^{\infty} \bigcup \{Q_{i_1 \ldots i_k} : \ i_j = 0, 1 \text{ for } j \le k\}.$$

Notice now that:

(2) C is a tame Cantor set in X.

Indeed, let $S \subset X \setminus C$ be a simple closed curve such that S is contractible in a proper subset of X. Since $X \in LC^1$, it follows from the properties of the sequence $S_1, S_2, \ldots$ that there is a k such that the curves S and S_k are homotopic in $X \setminus C$, and $S_k \subset X \setminus A_k$, where A_k is defined in $(4)_{k+1}$. By $(4)_{k+1}$, S_k is contractible in $X \setminus A_{k+1} \subset X \setminus C$, which implies that S is contractible in $X \setminus C$, and thus the proof of (2) and of the lemma is complete.

3. Construction of a wild Cantor set. The idea of this construction is the following: It is well known that there exist a wild Cantor set A in the Hilbert cube Q and a simple closed curve $S \subset Q \setminus A$ such that S is not contractible in $Q \setminus A$. We can assume that S is a polygonal curve and therefore a Z-set in Q. The existence of a wild Cantor set $C \subset X$ is a consequence of the following Lemma 2 applied to any *NDC*-curve $S_0 \subset X$.

LEMMA 2. *Let X be a continuum such that $X \not\cong S^1$. Then for any simple closed curve $S_0 \subset X$ there is an imbedding $\phi : X \to Q$ such that $\phi(S_0) = S$ and $A \subset \phi(X)$.*

The reader should compare this lemma with Theorem 5.2 in [L–W], which inspired the author. Also, the author thanks H. Toruńczyk whose remarks brought about a simplification of the proof of the lemma.

P r o o f o f t h e l e m m a. Let $S_0 \subset X$ and let $\phi_0 : S_0 \to S$ be a fixed homeomorphism. Consider the subspace $\mathcal{A}$ of the space Q^X consisting of the maps $f : X \to Q$ such that $f|S_0 = \phi_0$ and $A \subset f(X)$. Then $\mathcal{A}$ is complete as a closed subset of Q^X. Since $X \setminus S_0$ contains a Cantor set and $Q \in \mathrm{AR}$, it follows that $\mathcal{A}$ is not empty. So to find the desired embedding $\phi : X \to Q$ using Baire's theorem, it suffices to prove that:

(1) For every $f \in \mathcal{A}$, $\varepsilon > 0$ and $\delta > 0$ there is an ε-mapping $g \in \mathcal{A}$ which is δ-close to f.

Now, given $f \in \mathcal{A}$, $\varepsilon > 0$ and $\delta > 0$, find a closed neighborhood M of A in Q such that $M \subset Q \setminus S$ and M has a finite number $M_1, \ldots, M_k$ of components such that $\operatorname{diam} M_i < \delta/2$ for $i \le k$. Find a number $0 < \delta' < \delta/2$ such that for any $x \in A$ the closed ball $B(x, \delta')$ is contained in a component M_i. Since $f|S_0 = \phi_0 : S_0 \to S$ is a homeomorphism onto a Z-set in Q, it follows that there is a Z-imbedding $g_0 : X \to Q$ such that $d(f, g_0) < \delta'$ and $g_0|S_0 = \phi_0$ (see [M], p. 279). Since $A \subset f(X)$ as $f \in \mathcal{A}$, we infer that $g_0(X)$ meets the interior of each component M_i of M.

Next, for each $i \le k$ we construct a set $C_i \subset Q$ such that:

(i) $A \cap M_i \subset C_i \subset \operatorname{Int} M_i$.

(ii) C_i is the union of a finite number of disjoint compact AR-sets $D_{i1}, \ldots, D_{ij_i}$.
(iii) $\operatorname{Int} g_0^{-1}(D_{is}) \neq \emptyset$ for $s \le j_i$.
(iv) $\operatorname{diam} g_0^{-1}(D_{is}) < \varepsilon$ for $s \le j_i$.

For this purpose find an arc $L \subset Q$ containing the Cantor set A (cf. [Wh2]). Let $\eta > 0$ be such that for any $W \subset g_0(X)$ with $\operatorname{diam} W < \eta$ we have $\operatorname{diam} g_0^{-1}(W) < \varepsilon$. For each $i \le k$ find a set

$$L_i = L_{i1} \cup \ldots \cup L_{ij_i} \subset L \cap \operatorname{Int} M_i$$

such that $L_i \supset A \cap M_i$ and L_{is} for $s \le j_i$ are disjoint subarcs of L of diameter less than η. If $g_0^{-1}(L_{is}) \neq \emptyset$, then we enlarge the arc L_{is} to an AR-set D_{is} satisfying (iii) and (iv) by adding to it a small ball in Q. If $g_0^{-1}(L_{is}) = \emptyset$, then we enlarge the arc L_{is} to D_{is} by adding the union of a small ball (whose interior intersects $g_0(X)$) and of an arc. Since $g_0^{-1}(\operatorname{Int} M_i) \neq \emptyset$, it is clear that the construction can be done so that the conditions (i)–(iv) are satisfied.

Now, we can define the desired ε-mapping $g \in \mathcal{A}$ satisfying (1) by modifying g_0 only on the set

$$\bigcup_{i=1}^{k} g_0^{-1}(C_i) = \bigcup_{i=1}^{k} \bigcup \{g_0^{-1}(D_{is}) : s \le j_i\}\,.$$

Namely, for each $s \le j_i$ we find a Cantor set $C_{is} \subset \operatorname{Int} g_0^{-1}(D_{is})$ and a map g_{is} of C_{is} onto $A \cap D_{is}$. Let $g_{is}^* : g_0^{-1}(D_{is}) \to D_{is}$ be any map which is equal to g_{is} on C_{is} and to g_0 on $\operatorname{Bd} g_0^{-1}(D_{is})$ (recall that $D_{is} \in \mathrm{AR}$).

It is clear from the construction that the map g obtained by modification of g_0 by means of g_{is}^*'s, $i \le k, s \le j_i$, belongs to $\mathcal{A}$, and it is an ε-mapping by (ii), (iv) and since g_{is}^* maps $g_0^{-1}(D_{is})$ into D_{is}.

Observe that $d(g_0, g) < \delta/2$, because if $g(x) \neq g_0(x)$ then $g(x)$ and $g_0(x)$ belong to the same component M_i of M and $\operatorname{diam} M_i < \delta/2$. Consequently, $d(f, g) < \delta$, because $d(f, g_0) < \delta' < \delta/2$. Thus the proof of (1) and of the lemma is complete.

4. Proof of Theorem 2. Of course it suffices to prove the "only if" part of the theorem. First assume that X is an arcwise connected and arcwise homogeneous continuum such that $\dim X = 1$. It follows from Proposition 3.3 in [O–P] that X is locally connected and homogeneous and therefore we infer from Anderson's theorem [A] that either $X \cong S^1$ or $X \cong M_1^3$. Since the universal curve M_1^3 is not arcwise homogeneous we conclude that $X \cong S^1$.

Thus we can assume that X is an arcwise homogeneous continuum such that $\dim X > 1$, $X \in LC^1$ and X is not a 2-manifold and we shall find a contradiction.

In the same way as in the Introduction we define *NDC-curves* in X, and using Proposition 3.3 in [O–P] analogously to the case of Cantor sets we observe that:

(∗) X contains arbitrarily small *NDC*-curves.

Next, we define *tame* and *wild arcs* in X as for Cantor sets, but—as noticed by the referee—not equivalently to the usual definition.

Observe that the space X must contain a wild arc. Indeed, applying Lemma 2 to any *NDC*-curve $S_0 \subset X$ we obtain a Cantor set $C \subset X \setminus S_0$ such that S_0 is not contractible in $X \setminus C$. By Whyburn's Theorem [Wh2] and Proposition 3.3 in [O–P] there is an arc $I \subset X \setminus S_0$ containing C. Thus I is a wild arc in X.

To obtain a contradiction proving Theorem 2 we shall prove the following

LEMMA 3. *There is no arcwise homogeneous continuum $X \in LC^1$ containing a wild arc.*

Proof. On the contrary, assume that $X \in LC^1$ is an arcwise homogeneous continuum containing a wild arc J. For any arc $I = pq \subset X$ and $x \in I \setminus \{p\}$ let I_x denote the subarc of I with end-points p and x. We shall first prove that:

(1) There are an arc $I = pq \subset X$ and an *NDC*-curve $S \subset X \setminus I$ such that S is not contractible in $X \setminus I$ but S is contractible in $S \setminus I_x$ for every $x \in \mathring{I}$.

For this purpose consider the wild arc $J = pr \subset X$ and let $S \subset X \setminus J$ be an *NDC*-curve such that S is not contractible in $X \setminus J$. We shall first show that there is a subarc J_x of J such that S is contractible in $X \setminus J_x$.

Indeed, since S is an *NDC*-curve, there is a map f of a disc D into X such that $f|\dot{D} : \dot{D} \to S$ is a homeomorphism and $f(D) \neq X$. Let $a \in X \setminus f(D)$. Find an arc $K \subset X \setminus S$ joining a to p. Denote by K' the set of points $y \in K$ such that there is a map $g : D \to X$ satisfying $g|\dot{D} = f|\dot{D}$ and $y \notin g(D)$. As in the proof of Lemma 1 we notice that K' is a non-empty and both open and closed subset of K. In the proof that K' is closed we use the assumptions that $X \in LC^1$ and X is homogeneous (as it is arcwise connected and arcwise homogeneous). Thus we can apply the Effros theorem [E] which implies that any point $y \in \overline{K'}$ can be mapped to a point $z \in K'$ by an autohomeomorphism of X which is sufficiently close to the identity, and therefore $y \in K'$. Consequently, $K \subset K'$, whence we infer that $p \in K'$. Thus there is a subarc J_x of J such that S is contractible in $X \setminus J_x$.

Now, since S is not contractible in $X \setminus J$, there exists the least $q \in J \setminus \{p\}$ (in the ordering of J from p to r) such that S is not contractible in $X \setminus J_q$. Consequently, the arc $I = J_q$ satisfies (1).

It follows from the arcwise homogeneity of X that:

(2) For every arc $I = pq \subset X$ there is an *NDC*-curve $S \subset X \setminus I$ such that the condition described in (1) is satisfied.

Now, let $\mathcal{S}$ denote the subspace of the space X^{S^1} (with the "sup" metric) consisting of all homeomorphisms $\phi : S^1 \to \phi(S^1) \subset X$ such that $\phi(S^1)$ is an *NDC*-curve in X. Let $\phi_1, \phi_2, \ldots$ be a sequence dense in X and let $S_i = \phi_i(S^1)$ for $i = 1, 2, \ldots$

Choose any arc $I = pq \subset X$. Observe that:

(3) For any $x \in I \setminus \{p\}$ there is an $i(x) = 1, 2, \ldots$ such that the arc I_x and the curve $S_{i(x)}$ satisfy the condition described in (1).

Indeed, if $S \subset X \setminus I_x$ is an *NDC*-curve satisfying that condition then there exists a curve $S_i \subset X \setminus I_x$ so close to S that S is homotopic to S_i in $X \setminus I_x$. It follows that S_i satisfies (3).

Now observe that if $x, y \in I \setminus \{p\}$ and $x \neq y$ then $i(x) \neq i(y)$. Indeed, assume that x precedes y in the ordering of I from p to q and $i(x) = i(y)$. Then $S_{i(x)} \subset X \setminus I_y \subset X \setminus I_x$ would be both contractible and non-contractible in $X \setminus I_y$, which is impossible. Thus (3) yields a contradiction, which proves the lemma.

5. Final remarks. The author does not know whether the assumption that $X \in LC^1$ (when $\dim X \geq 2$) is essential to Theorems 1 and 2, and not only to the presented proofs.

It follows from Lemma 2 that any continuum $X \in LC^1$ such that $X \not\cong S^1$ satisfies the condition: For every simple closed curve $S_0 \subset X$ there is a Cantor set $A \subset X \setminus S_0$ which is wild with respect to S_0 in the sense that S_0 is not contractible in $X \setminus A$. It would be of interest to know whether there exists a (homogeneous) continuum $X \in LC^1$ such that for each Cantor set $C \subset X$ there is a simple closed curve $S \subset X \setminus C$ which is not contractible in $X \setminus C$. Since for $X \not\cong S^1$, n-homogeneity implies strong n-homogeneity (cf. [U]), it follows from Lemma 1 that X cannot be n-homogeneous for all $n = 1, 2, \ldots$

Observe that our proof of Theorem 2 does not yield a construction of a tame arc in X, in contrast to the proof of Lemma 1 which does give a construction of a tame Cantor set. It would be interesting to have such a construction of a tame arc for instance in a homogeneous continuum $X \in LC^1$ of dimension at least 2.

Other problems concern arcwise homogeneous continua X which are not arcwise connected but such that for any $x \in X$ there is an arc $I \subset X$ containing x. Such a continuum must be homogeneous and a solenoid is an example (cf. [O–P]). It would be of interest to know whether it is the only example of dimension 1. When $\dim X > 1$, must X be a bundle space over a 2-manifold with fiber being a Cantor set?

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