Hence, since (y_n) converges to $x^{(2)}$ in the w^*-topology, we conclude from [5] that $x^{(2)} \in B_{1/4}(X)$.

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Isometries of Musielak–Orlicz spaces II

by

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Abstract. A characterization of isometries of complex Musielak–Orlicz spaces L_Φ is given. If L_Φ is not a Hilbert space and $U : L_\Phi \to L_\Phi$ is a surjective isometry, then there exist a regular set isomorphism τ from (T, Σ, μ) onto itself and a measurable function w such that $U(f) = w \cdot (f \circ \tau)$ for all $f \in L_\Phi$. Isometries of real Nakano spaces, a particular case of Musielak–Orlicz spaces, are also studied.

1. Introduction. For any σ-finite atomless measure space (T, Σ, μ), a nonnegative function $\Phi : \mathbb{R}_+ \times T \to \mathbb{R}_+ \cup \{\infty\}$ is said to be a *Young function* if

(1) $\Phi(0, t) = 0$ for all $t \in T$;

(2) for any $t \in T$, $\Phi(\cdot, t)$ is a left continuous nondecreasing convex function;

(3) for any $u \in \mathbb{R}_+$, $\Phi(u, \cdot)$ is a Σ-measurable function;

(4) $\mu(\{t : \Phi(u, t) = 0 \text{ for all } u > 0\}) = 0 = \mu(\{t : \Phi(u, t) = \infty \text{ for all } u > 0\})$.

For any Young function Φ, the *Musielak–Orlicz space* L_Φ associated with Φ is the set of all (complex- or real-valued) measurable functions such that

$$I_\Phi(\lambda f) = \int_T \Phi(|\lambda f(t)|, t)\, d\mu(t) < \infty$$

for some $\lambda > 0$. The space L_Φ is equipped with the Luxemburg norm, that is, the norm of $f \in L_\Phi$ is given by $\|f\|_\Phi = \inf\{\varepsilon > 0 : I_\Phi(\frac{f}{\varepsilon}) \le 1\}$ [10, 13].

If Φ does not depend on t, i.e. $\Phi(u, t) = \varphi(u)$, then we shall call L_Φ the *Orlicz space* L_φ [11]. In [5], Fleming and the first two authors studied the isometries of complex Musielak–Orlicz spaces. They proved that if Φ satisfies the following condition:

$$\text{for almost all } t \in T, \text{ the function } u \to \frac{\Phi'(u, t)}{u} \text{ is monotone},$$

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then for every isometry U on L_Φ there exist a regular set isomorphism τ from (T,Σ,μ) onto (T,Σ,μ) (for definition see Section 2), and a measurable function w such that for every $f\in L_\Phi$,

$$U(f)=w\cdot(f\circ\tau).$$

In this article we continue investigating the surjective isometries of general (real and complex) Musielak–Orlicz spaces.

Let X be a complex Banach space. Recall a bounded linear operator H on X is said to be *Hermitian* if $x^*(Hx)$ is real for every pair of vectors $x\in X$ and $x^*\in X^*$ such that x^* is a support functional of x, i.e. $\|x\|^2=\|x^*\|^2=x^*(x)$. It is known that an operator H is Hermitian if and only if $e^{i\alpha H}=\sum_{k=0}^{\infty} i^k\alpha^kH^k/k!$ is a surjective isometry on X for all $\alpha\in\mathbb{R}$ [3]. The notion of Hermitian operator was introduced by Lumer. Using the technique of Hermitian operators, he [12] gave a representation of the surjective isometries of complex reflexive Orlicz spaces which is a generalization of the classical Banach result for real Lebesgue spaces [1]. Later, Zaĭdenberg [16, 17] showed the assumption of reflexivity can be removed. For more about Hermitian operators, see [2, 4–6, 15–17].

Let

$$C=\left\{t:\frac{\Phi'(u,t)}{u}\text{ is a constant function with respect to }u>0\right\}.$$

It is known that $L_\Phi(C,\mu)=\{f\in L_\Phi:\operatorname{supp}f=\{t:f(t)\neq0\}\subseteq C\}$ is isometrically isomorphic to a Hilbert space. In Section 2, we study Hermitian operators on complex Musielak–Orlicz spaces. We prove that if H is a Hermitian operator on L_Φ, then

(5) if $\operatorname{supp}f\subseteq C$, then $\operatorname{supp}H(f)\subseteq C$;

(6) there exists a bounded real function h on $T\setminus C$ such that if $\operatorname{supp}f\subseteq T\setminus C$, then $H(f)=h\cdot f$.

In Section 3, we use this result to show that if U is a surjective isometry on L_Φ, then

(7) the restriction of U to $L_\Phi(C,\mu)$ is a surjective isometry from $L_\Phi(C,\mu)$ onto itself;

(8) there exist a measurable function w on $T\setminus C$ and a regular set isomorphism τ on $T\setminus C$ such that if $\operatorname{supp}f\subseteq T\setminus C$, then $U(f)=w\cdot(f\circ\tau)$.

This gives a complete characterization of the isometries on L_Φ.

Let $p(t)$ be a measurable function from T into $[1,\infty)$. The *Nakano space* $L^{p(t)}$ [14] associated with $p(t)$ is the Musielak–Orlicz space L_Φ such that $\Phi(u,t)=\frac{u^{p(t)}}{p(t)}$. In Section 4, we study the isometries between two real Nakano spaces $L^{p(t)}$ and $L^{r(t)}$ such that $p(t),r(t)\in(2,\alpha)$ for some $2<\alpha<\infty$. In particular, we prove that if U is a surjective isometry from $L^{p(t)}$ onto itself ($p(t)\in(2,\alpha)$ for some $\alpha<\infty$), then there exist a measurable function w on T and a regular set isomorphism τ such that $U(f)=w\cdot(f\circ\tau)$ and $p(t)=(p\circ\tau)(t)$, $\tau'(t)=|w(t)|^{p(t)}$ for almost all $t\in T$. This agrees with the result for the complex case. But we note that there is no method analogous to Hermitian operators in real Banach spaces. Hence, the method used in Section 4 is different.

The surjective isometries of complex Musielak–Orlicz spaces equipped with the Orlicz norm (another standard norm usually defined on L_Φ [11, 13]) are investigated in [9].

2. Hermitian operators. Let Φ be any Young function. For any $u>0$ and $t\in T$, let

$$D^+\Phi(u,t)=\lim_{\varepsilon\to0^+}\frac{\Phi(u+\varepsilon,t)-\Phi(u,t)}{\varepsilon},$$
$$D^-\Phi(u,t)=\lim_{\varepsilon\to0^-}\frac{\Phi(u+\varepsilon,t)-\Phi(u,t)}{\varepsilon}.$$

Then for any $t\in T$ and $u>\varepsilon>0$,

$$\Phi(u+\varepsilon,t)-\Phi(u,t)\geq\varepsilon D^+(u,t)\geq\varepsilon D^-(u,t)\geq\Phi(u,t)-\Phi(u-\varepsilon,t).$$

It is known that if $\Phi(u,t)<\infty$ for some $u>0$, then $\Phi(\cdot,t)$ is continuous on $(0,u)$. One can easily prove that

(9) if $\|f\|_\Phi=1$, then $I_\Phi(f)=\lim_{\varepsilon\to1^-}I_\Phi(\varepsilon f)\leq1$;

(10) if $I_\Phi((1+\varepsilon)f)<\infty$ for some $\varepsilon>0$ and $\|f\|_\Phi=1$, then $I_\Phi(f)$ $(=\lim_{\varepsilon\to1^+}I_\Phi(\varepsilon f))=1$.

Hence, if h_f is any measurable function with $D^-\Phi(|f(t)|,t)\leq h_f(t)\leq D^+\Phi(|f(t)|,t)$ and $I_\Phi((1+\varepsilon)f)<\infty$, then

$$\infty>I_\Phi((1+\varepsilon)f)-I_\Phi(f)\geq\int_T\varepsilon h_f(t)|f(t)|\,d\mu(t),$$
$$I_\Phi(g)-I_\Phi(f)\geq\int_T h_f(t)[|g(t)|-|f(t)|]\,d\mu(t),$$

for any measurable function g. For any complex number a, let

$$\operatorname{sgn}a=\begin{cases}a/|a| & \text{if } a\neq0,\\ 0 & \text{otherwise.}\end{cases}$$

The above proof shows that if $\|f\|_\Phi=1=\|g\|_\Phi$ and $I_\Phi((1+\varepsilon)f)<\infty$ for some $\varepsilon>0$, then

$$\int_T D^-\Phi(|f(t)|,t)|g(t)|\,d\mu(t)\leq\int_T D^-\Phi(|f(t)|,t)|f(t)|\,d\mu(t).$$

Hence,

$$F_f(g)=\frac{\int_T D^-\Phi(|f(t)|,t)\overline{\operatorname{sgn} f(t)}g(t)\,d\mu(t)}{\int_T |f(t)|D^-\Phi(|f(t)|,t)\,d\mu(t)}$$

is a support functional of f (see also Proposition 1 in [5]).

The following lemma was proved by Kalton and Wood (Theorems 2.4 and 2.6 of [7]).

LEMMA 2.1. *Let X be a complex Banach space. Let P,Q be any two Hermitian projections on X for which $PQ=0$ and let H be any other Hermitian operator on X. Then PHP, $PHQ+QHP$ and $i(PHQ-QHP)$ are Hermitian on X. Hence, if x^* is a support functional of x, then $x^*(PHQ(x))=x^*(QHP(x))$.*

Remark 2.1. It is known that if A is any measurable subset of T, then the mapping $H_A:f\to\chi_A f$ is a Hermitian projection. Hence, if $\|f\|_\Phi=1$ and $I_\Phi((1+\varepsilon)f)<\infty$ for some $\varepsilon>0$, then for every Hermitian operator H on L_Φ and any two measurable disjoint subsets A,B of T, we have

$$\int_A H(\chi_B f)(t)\overline{\operatorname{sgn} f(t)}D^-\Phi(|f(t)|,t)\,d\mu(t) = \int_B \overline{H(\chi_A f)(t)}\operatorname{sgn} f(t)D^-\Phi(|f(t)|,t)\,d\mu(t)\,.$$

Let (T_1,Σ_1,μ_1) and (T_2,Σ_2,μ_2) be σ-finite measure spaces. Recall a set mapping $\tau:\Sigma_1\to\Sigma_2$, defined modulo null sets, is called a *regular set isomorphism* [15] if it satisfies the following conditions:

(11) $\tau(A^c)=\tau(A)^c$, where A^c is the complement of A;

(12) $\tau(\bigcup_{j=1}^\infty A_j)=\bigcup_{j=1}^\infty \tau(A_j)$ for any pairwise disjoint sets $\{A_j\}$;

(13) $\mu_2(\tau(A))=0$ if and only if $\mu_1(A)=0$.

For any regular set isomorphism $\tau:(T_1,\Sigma_1,\mu_1)\to(T_2,\Sigma_2,\mu_2)$, there is a unique linear transformation from the class of Σ_1-measurable functions to the class of Σ_2-measurable functions, characterized by $\chi_A\to\chi_{\tau(A)}$. We shall denote this transformation by $f\circ\tau$ for any Σ_1-measurable function f; in particular, $\chi_A\circ\tau=\chi_{\tau(A)}$.

Recall a Banach space $(E,\|\cdot\|)$ of measurable functions is called an *ideal* if for any two measurable functions f, g, $|f(t)|\le|g(t)|$ a.e., $g\in E$ implies $f\in E$, and $\|f\|\le\|g\|$.

We need the following lemma which was proved by Zaĭdenberg (Theorem 2 of [16]).

LEMMA 2.2. *Let E (respectively, F) be any ideal (complex or real) Banach space defined on (T_1,Σ_1,μ_1) (respectively, (T_2,Σ_2,μ_2)). Let $U:E\to F$ be a surjective isometry, and let $\{S_n\}$ be a sequence of pairwise disjoint sets in Σ_1 such that $\bigcup_{n=1}^\infty S_n=T_1$ and $\chi_{S_n}\in E$ for all $n\in\mathbb{N}$. If $\tau:\Sigma_1\to\Sigma_2$ is a regular set isomorphism and w is a measurable function on T_2 such that $U(\chi_A)=w\chi_{\tau(A)}$ for all $A\in\Sigma_1$ contained in S_n, then $U(f)=w\cdot(f\circ\tau)$ for all $f\in E$.*

For a σ-finite measure space (T_1,Σ_1,μ_1) (respectively, (T_2,Σ_2,μ_2)), let Φ_1 (respectively, Φ_2) be any Young function on T_1 (respectively, T_2). We will need the following notations.

$$C_1=\left\{t\in T_1:\frac{\Phi_1'(u,t)}{u}\text{ is a constant function with respect to }u>0\right\},$$
$$C_2=\{t\in T_1:\Phi_1(u,t)=\infty\text{ for some }u>0\}\,,$$
$$C_3=T_1\setminus C_1\setminus C_2\,,$$
$$B_1=\left\{t\in T_2:\frac{\Phi_2'(u,t)}{u}\text{ is a constant function with respect to }u>0\right\},$$
$$B_2=\{t\in T_2:\Phi_2(u,t)=\infty\text{ for some }u>0\}\,,$$
$$B_3=T_2\setminus B_1\setminus B_2,$$
$$\mathfrak{F}_1=\{A\subseteq C_2\cup C_3:A\in\Sigma_1\},\qquad a_1(t)=\sup\{u:\Phi_1(u,t)<\infty\},$$
$$\mathfrak{F}_2=\{A\subseteq B_2\cup B_3:A\in\Sigma_2\},\qquad a_2(t)=\sup\{u:\Phi_2(u,t)<\infty\}.$$

It is known that there exist a partition $\{D_j\}$ of T_1 and a positive sequence $\{N_j\}$ such that

(14) $D_j\subseteq C_k$ for some $1\le k\le 3$ and $\mu_1(D_j)<\infty$ for all $j\in\mathbb{N}$;

(15) if $D_j\subseteq C_1\cup C_3$, then $I_{\Phi_1}(\lambda\chi_{D_j})<\infty$ for all $\lambda>0$ (see [8]);

(16) if $D_j\subseteq C_2$, then $I_{\Phi_1}(\chi_{D_j}a_1)<\frac13$, and $|a_1(t)|>N_j$ for all $t\in D_j$.

Remark 2.2. Let A be a measurable subset of D_j. Suppose that $0<\alpha\le N_j/2$ and f is a bounded function such that $\operatorname{supp} f\subseteq((D_j\cup D_{j'})\cap(C_1\cup C_3))\setminus A$ for some $j'\in\mathbb{N}$. Then there exists $\varepsilon>0$ such that $I_{\Phi_1}((1+\varepsilon)(\alpha\chi_A+f))<\infty$. Hence, if $\|\alpha\chi_A+f\|_{\Phi_1}=1$, then $I_{\Phi_1}(\alpha\chi_A+f)=1$, and

$$F_f(g)=\frac{\int_A D^-\Phi_1(\alpha,t)g(t)\,d\mu_1(t)+\int_{\operatorname{supp} f}D^-\Phi_1(|f(t)|,t)\overline{\operatorname{sgn} f(t)}g(t)\,d\mu_1(t)}{\int_A \alpha D^-\Phi_1(\alpha,t)\,d\mu_1(t)+\int_{\operatorname{supp} f}D^-\Phi_1(|f(t)|,t)|f(t)|\,d\mu_1(t)}$$

is a support functional of $\alpha\chi_A+f$.

LEMMA 2.3. *Let Φ_1 be any Young function on (T_1,Σ_1,μ_1). Let H be any Hermitian operator on L_{Φ_1}. For any $A\subseteq D_j$ for some $j\in\mathbb{N}$, $\operatorname{supp} H(\chi_A)\subseteq A\cup C_1$.*

Proof. For $j' \in \mathbb{N}$, let $H_{D_j \cup D_{j'}}$ denote the Hermitian projection

$$H_{D_j \cup D_{j'}}(f) = \chi_{D_j \cup D_{j'}} \cdot f\,.$$

It is known that $H_{D_j \cup D_{j'}} H H_{D_j \cup D_{j'}}$ is a Hermitian operator on L_{Φ_1}.

Case 1. Suppose that $D_{j'} \subseteq C_1 \cup C_3$. Let A_1 and A_2 be any two disjoint measurable sets such that $A_1 \cup A_2 \subseteq ((D_j \cup D_{j'}) \cap (C_1 \cup C_3)) \setminus A$. Let α, β be any two positive numbers such that $\alpha \le N_j/2$, and $\|\alpha\chi_A + \beta\chi_{A_1}\|_{\Phi_1} = \int_A \Phi_1(\alpha,t)\,d\mu_1 + \int_{A_1} \Phi_1(\beta,t)\,d\mu_1 = 1$. For any $0 \le \gamma < \beta$, $\|\alpha\chi_A + \gamma\chi_{A_1}\|_{\Phi_1} < 1$. So there exists $\delta > 0$ such that

$$\|\alpha\chi_A + \gamma\chi_{A_1} + \delta\chi_{A_2}\|_{\Phi_1} = \int_A \Phi_1(\alpha,t)\,d\mu_1 + \int_{A_1} \Phi_1(\gamma,t)\,d\mu_1 + \int_{A_2} \Phi_1(\delta,t)\,d\mu_1 = 1\,.$$

Let $f = \alpha\chi_A + \beta\chi_{A_1}$ and $g = \alpha\chi_A + \gamma\chi_{A_1} + \delta\chi_{A_2}$. Note that there is an $\varepsilon > 0$ such that $I_{\Phi_1}((1+\varepsilon)f) < \infty$ and $I_{\Phi_1}((1+\varepsilon)g) < \infty$. By Remark 2.1,

$$\int_A H(\beta\chi_{A_1}) D^-\Phi_1(\alpha,t)\,d\mu_1 = \int_{A_1} \overline{H(\alpha\chi_A)} D^-\Phi_1(\beta,t)\,d\mu_1,$$

$$\int_A H(\gamma\chi_{A_1}) D^-\Phi_1(\alpha,t)\,d\mu_1 = \int_{A_1} \overline{H(\alpha\chi_A)} D^-\Phi_1(\gamma,t)\,d\mu_1\,.$$

So if $0 < \gamma' < \gamma \le \beta$, then

$$\int_{A_1} \overline{H(\chi_A)} \Big[D^-\Phi_1(\gamma',t) - \frac{\gamma'}{\gamma} D^-\Phi_1(\gamma,t) \Big]\,d\mu_1 = 0\,.$$

Let B be any subset of A_1 such that $\mu_1(B) > 0$. Then there exists $\beta' > 0$ such that $\|\alpha\chi_A + \beta'\chi_B\|_{\Phi_1} = 1$. The above proof shows that if $0 < \gamma' < \gamma < \beta'$, then

$$\int_B \overline{H(\chi_A)} \Big[D^-\Phi_1(\gamma',t) - \frac{\gamma'}{\gamma} D^-\Phi_1(\gamma,t) \Big]\,d\mu_1 = 0\,.$$

Note that for any $N > 0$, there exists a partition $\{E_n\}$ of $(D_j \cup D_{j'}) \cap (C_1 \cup C_3)$ such that $\|\alpha\chi_A + N\chi_{E_n}\|_{\Phi_1} \le 1$ for all $n \in \mathbb{N}$. So for any $0 < \gamma' < \gamma < \infty$,

$$H(\chi_A) \Big[D^-\Phi_1(\gamma',t) - \frac{\gamma'}{\gamma} D^-\Phi_1(\gamma,t) \Big] = 0\,.$$

This implies that $\operatorname{supp} H(\chi_A) \cap (C_1 \cup C_3) \subseteq A \cup C_1$.

Case 2. Suppose that $A_1 \subseteq [(D_j \cup D_{j'}) \setminus A] \cap C_2$. Let α, $0 < \alpha < N_j/2$, be any positive number such that $I_{\Phi_1}(\alpha\chi_A) \le \frac{1}{3}$. Then $I_{\Phi_1}(\alpha\chi_A + a_1\chi_{A_1}) \le 1$ and $\|\alpha\chi_A + a_1\chi_{A_1}\|_{\Phi_1} = 1$. Hence, for any measurable subset B of A_1 with

$\mu_1(B) > 0$,

$$F_B(g) = \int_B \frac{g(t)}{a_1(t)\mu_1(B)}\,d\mu_1(t)$$

is a support functional of $f = \alpha\chi_A + a_1\chi_{A_1}$. By Remark 2.1, we have

$$0 = \int_B \frac{\chi_A H(f\chi_B)}{a\mu_1(B)}\,d\mu_1 = \int_B \frac{\overline{\chi_B H(f\chi_A)}}{a\mu_1(B)}\,d\mu_1 = \alpha \int_B \frac{\overline{H(\chi_A)}}{a\mu_1(B)}\,d\mu_1\,.$$

So $\mu_1(\operatorname{supp} H(\chi_A) \cap (C_2 \setminus A)) = 0$. ∎

Now, we can give a characterization of Hermitian operators on L_Φ.

THEOREM 2.4. *Let H be a Hermitian operator on L_{Φ_1}. Then*

(5) *if $\operatorname{supp} f \subseteq C_1$, then $\operatorname{supp} H(f) \subseteq C_1$;*

(6) *there exists a bounded measurable real function h on $T_1 \setminus C_1$ such that if $\operatorname{supp} f \subseteq T_1 \setminus C_1$, then $H(f) = h \cdot f$.*

Proof. By Lemma 2.3, for any $\alpha \in \mathbb{R}$ and $A \subseteq D_j$, $\operatorname{supp} e^{i\alpha H}(\chi_A) \subseteq A \cup C_1$. But $L_{\Phi_1}(C_1,\mu_1)$ is separable. Hence, $e^{i\alpha H}$ maps $L_{\Phi_1}(C_1,\mu_1)$ into itself. But $(e^{i\alpha H})^{-1} = e^{-i\alpha H}$. So $e^{i\alpha H}$ maps $L_{\Phi_1}(C_1,\mu_1)$ onto itself.

We claim that if $A \subseteq D_j \subseteq C_2 \cup C_3$, then $\operatorname{supp} e^{i\alpha H}(\chi_A) \subseteq A$.

Suppose this is not true. Then there exist f_1, f_2, g_1 and g_2 such that

(17) $\operatorname{supp} f_1 \subseteq C_2 \cup C_3$, $\operatorname{supp} g_1 \subseteq C_2 \cup C_3$, $\operatorname{supp} f_2 \subseteq C_1$, $\operatorname{supp} g_2 \subseteq C_1$;

(18) $$\|g_2\|_{\Phi_1} > 0\,;$$

(19) $$e^{i\alpha H}(f_1) = g_1 + g_2 \quad\text{and}\quad e^{i\alpha H}(f_2) = g_2\,.$$

Then (we omit the subscript Φ_1 at the norm below for simplicity)

$$\|f_1\| = \|g_1 + g_2\| \ge \|g_1\| = \|f_1 - f_2\| \ge \|f_1\|\,.$$

So $\|f_1\| = \|f_1 + f_2\|$. Let $\nu = \sup\{\varrho : \|f_1 + \varrho f_2\| = \|f_1\|\}$. Then $\nu < \infty$. But

$$\begin{aligned}\|f_1\| &= \|f_1 + \nu f_2\| = \|g_1 + (\nu+1)g_2\| \\ &= \|g_1 - (\nu+1)g_2\| = \|f_1 - (\nu+2)f_2\| = \|f_1 + (\nu+2)f_2\|\,.\end{aligned}$$

We get a contradiction.

Since $\lim_{\alpha\to 0}(e^{i\alpha H} - I)/\alpha = H$, the above proof shows that

(20) $$H(L_{\Phi_1}(C_1,\mu_1)) \subseteq L_{\Phi_1}(C_1,\mu_1)\,;$$

(21) if $A \subseteq D_j \subseteq C_2 \cup C_3$, then $\operatorname{supp} H(\chi_A) \subseteq A$.

Let h be the function on $C_2 \cup C_3$ such that if $t \in D_j \subseteq C_2 \cup C_3$, then $h(t) = H(\chi_{D_j})(t)$. By Lemma 2.2, $H(f) = h \cdot f$ for every $f \in L_{\Phi_1}$ with $\operatorname{supp} f \subseteq C_2 \cup C_3$. But H is Hermitian, so h must be a real function. ∎

3. Isometries between Musielak–Orlicz spaces. For any σ-finite measure space (T_1, Σ_1, μ_1) (respectively, (T_2, Σ_2, μ_2)), let Φ_1 (respectively,

Φ_2) be any Young function on T_1 (respectively, T_2). Recall an isometry U is said to have the *disjoint support property* if for any f, g with $fg = 0$, $Uf \cdot Ug = 0$ holds for almost all $t \in T$. The following theorem shows that if U is a surjective isometry from L_{Φ_1} onto L_{Φ_2}, then the restriction of U to $L_{\Phi_1}(C_2 \cup C_3)$ has the disjoint support property.

THEOREM 3.1. *Let U be a surjective isometry from L_{Φ_1} onto L_{Φ_2}. Then*

(7) *the restriction of U to $L_{\Phi_1}(C_1, \mu_1)$ is a surjective isometry from $L_{\Phi_1}(C_1, \mu_1)$ onto $L_{\Phi_2}(B_1, \mu_2)$;*

(8) *the restriction of U to $L_{\Phi_1}(C_2 \cup C_3)$ has the disjoint support property; and so there exist a measurable function w on $T_2 \setminus B_1$ and a regular set isomorphism τ of $(T_1 \setminus C_1, \mathfrak{F}_1)$ onto $(T_2 \setminus B_2, \mathfrak{F}_2)$ such that if $\operatorname{supp} f \subseteq T_1 \setminus C_1$, then $U(f) = w \cdot (f \circ \tau)$.*

Proof. For any measurable subset A of T_1, H_A denotes the Hermitian operator defined by

$$H_A(f) = \chi_A \cdot f .$$

It is known that UH_AU^{-1} is a Hermitian operator on L_{Φ_2} and $UH_AU^{-1}(L_{\Phi_2})$ is isometric to $H_A(L_{\Phi_1})$. If $A = C_1$, then $UH_AU^{-1}(L_{\Phi_2})$ is isometric to a Hilbert space. By Theorem 2.4, we must have $U(L_{\Phi_1}(C_1, \mu_1)) \subseteq L_{\Phi_2}(B_1, \mu_2)$. Similarly, $U^{-1}(L_{\Phi_2}(B_1, \mu_2)) \subseteq L_{\Phi_1}(C_1, \mu_1)$. This implies $U(L_{\Phi_1}(C_1, \mu_1)) = L_{\Phi_2}(B_1, \mu_2)$. We have proved (7).

Let A be any measurable subset of $T_1 \setminus C_1$. Then $H_A(L_{\Phi_1})$ is isometric to $UH_AU^{-1}(L_{\Phi_2})$. But $U(L_{\Phi_1}(C_1, \mu_1)) = L_{\Phi_2}(B_1, \mu_2)$. By Theorem 2.4, we have

$$UH_AU^{-1}(L_{\Phi_2}(T_2 \setminus B_1)) \subseteq L_{\Phi_2}(T_2 \setminus B_1) .$$

By Theorem 2.4 again, there is a real function h_A on T_2 such that $\operatorname{supp} h_A \subseteq T_2 \setminus B_1$ and for every $f \in L_{\Phi_2}$,

$$UH_AU^{-1}(f) = h_A \cdot f .$$

Since UH_AU^{-1} is a projection, h_A must be a characteristic function. Let

$$\tau(A) = \{t \in T_2 : h_A(t) = 1\} \subseteq T_2 \setminus B_1 .$$

Let $f, g \in L_{\Phi_1}$ be any functions such that $\operatorname{supp} f \cup \operatorname{supp} g \subseteq T_1 \setminus C_1$ and $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$. Let $A = \operatorname{supp} f$. Then

$$\chi_{\tau(A)} U(f) = (UH_AU^{-1})(U(f)) = U(\chi_A f) = U(f),$$
$$\chi_{\tau(A)} U(g) = (UH_AU^{-1})(U(g)) = U(\chi_A g) = 0 .$$

So $\operatorname{supp} U(f) \cap (T_2 \setminus \operatorname{supp} U(g)) = \emptyset$. Similarly, $\operatorname{supp} U(g) \cap (T_2 \setminus \operatorname{supp} U(f)) = \emptyset$. This implies that U has the disjoint support property. By Theorem 2.4, there exists a measurable function w on $T_2 \setminus B_1$ such that if $\operatorname{supp} f \subseteq T_1 \setminus C_1$, then $U(f) = w \cdot (f \circ \tau)$. ■

LEMMA 3.2. *Let U be an isometry from L_{Φ_1} into L_{Φ_2} with the disjoint support property. If $\|f\|_{\Phi_1} = 1$ and $I_{\Phi_1}((1+\varepsilon)f) < \infty$ for some $\varepsilon > 0$, then $I_{\Phi_2}((1+\varepsilon)U(f)) < \infty$ and $I_{\Phi_1}(f) = 1 = I_{\Phi_2}(U(f))$. Hence, if $I_{\Phi_1}(a_1) > 1$ and $I_{\Phi_1}(f) < \infty$, then $I_{\Phi_1}(f) = I_{\Phi_2}(U(f))$.*

Proof. Let $f \in L_{\Phi_1}$ be such that $\|f\|_{\Phi_1} = 1$ and $I_{\Phi_1}((1+\varepsilon)f) < \infty$ for some $\varepsilon > 0$. Let $\{D_1, \ldots, D_n\}$ be measurable subsets of T_1 such that $\bigcup_{j=1}^{n} D_j = T_1$ and $I_{\Phi_1}((1+\varepsilon)\chi_{D_j} f) \le 1$. This implies

$$I_{\Phi_2}((1+\varepsilon)f) \le \sum_{j=1}^{n} I_{\Phi_2}((1+\varepsilon)U(\chi_{D_j} f)) \le \sum_{j=1}^{n} \|(1+\varepsilon)U(\chi_{D_j} f)\|_{\Phi_2} = n .$$

Hence $I_{\Phi_1}(f) = 1 = I_{\Phi_2}(U(f))$.

Let f be any function in L_{Φ_1} such that $I_{\Phi_1}(f) > 1$. Since $I_{\Phi_1}(f) = \lim_{\lambda \to 1^-} I_{\Phi_1}(\lambda f)$, we may assume that $I_{\Phi_1}((1+\varepsilon)f) < \infty$, and $I_{\Phi_1}(f)$ is a rational number. For convenience, we assume that $I_{\Phi_1}(f) = 1 + 1/n$.

Let $\{D_1, \ldots, D_{n+1}\}$ be a partition of T_1 such that $I_{\Phi_1}(\chi_{D_j} f) = \frac{1}{n}$ for all j, $1 \le j \le n+1$.

By the above proof, for all j with $1 \le j \le n+1$,

$$\sum_{k \ne j} I_{\Phi_2}(U(\chi_{D_k} f)) = I_{\Phi_2}\Big(U\Big(\sum_{k \ne j} \chi_{D_k} f\Big)\Big) = 1.$$

So $I_{\Phi_2}(U(\chi_{D_j} f))$ must be $\frac{1}{n}$ for all j with $1 \le j \le n+1$ and $I_{\Phi_2}(U(f)) = 1 + \frac{1}{n}$.

Suppose that $I_{\Phi_1}(a_1) > 1$. We claim that for any $f \in L_{\Phi_1}$, $I_{\Phi_1}(f) = I_{\Phi_2}(U(f))$.

Let $\{D_1, \ldots, D_k\}$ be a partition of T_1 such that for any $1 \le j \le k$, there is a $g_j \in L_{\Phi_1}$ such that $\operatorname{supp} g_j \cap D_j = \emptyset$ and $I_{\Phi_1}(g_j) = 1$. The above proof shows that $I_{\Phi_1}(f\chi_{D_j} + g_j) = I_{\Phi_2}(U(f\chi_{D_j} + g_j))$ and $I_{\Phi_2}(U(g_j)) = 1 = I_{\Phi_1}(g_j)$. So $I_{\Phi_1}(f\chi_{D_j}) = I_{\Phi_2}(U(f\chi_{D_j}))$. But $U(f\chi_{D_j})$'s have disjoint supports. This implies $I_{\Phi_1}(f) = I_{\Phi_2}(U(f))$. ■

Remark 3.1. Let U be a surjective isometry from L_{Φ_1} onto L_{Φ_2}. By Theorem 3.1, $I_{\Phi_1}(f) = I_{\Phi_2}(U(f))$ for every $f \in L_{\Phi_1}(C_1, \mu_1)$. Hence, Lemma 3.2 is still true if U is a surjective isometry.

COROLLARY 3.3. *L_{Φ_1} is isometric to L_∞ if and only if $I_{\Phi_1}(a_1) \le 1$.*

Proof. If $I_{\Phi_1}(a_1) \le 1$, then for every $f \in L_{\Phi_1}$,

$$\|f\|_{\Phi_1} = \sup\{\nu : \mu\{t : |f(t)| \ge \nu a_1(t)\} > 0\} .$$

The mapping $U : f \to \frac{f}{a_1}$ is then an isometry from L_{Φ_1} onto $L_\infty(T_1, \mu_1)$.

On the other hand, if $I_{\Phi_1}(a_1) > 1$, then there exist two disjoint subsets D_1 and D_2 of T_1 such that $I_{\Phi_1}(\chi_{D_1} a_1) = 1 \ge I_{\Phi_1}(\chi_{D_2} a_1) > 0$. Then

$$I_{\Phi_1}(\chi_{D_1 \cup D_2} a_1) > 1 = \max(I_{\Phi_1}(\chi_{D_1} a_1), I_{\Phi_1}(\chi_{D_2} a_1)) .$$

So L_{Φ_1} is not isometric to L_∞. ∎

THEOREM 3.4. *Let U be a surjective isometry from L_{Φ_1} onto L_{Φ_2}. Let w and τ be the function and regular set isomorphism defined in Theorem 3.1. Then:*

(22) *If $I_{\Phi_1}(a_1) \le 1$, then $w(t) = \frac{a_2(t)}{a_1 \circ \tau(t)}$.*

(23) *If $I_{\Phi_1}(a_1) > 1$, then*
$$\Phi_2(\lambda|w(t)|, t) = \tau'(t)[\Phi_1(\lambda,\cdot)\circ\tau](t)$$
for almost all $t \in B_2 \cup B_3$, where $\tau' = \frac{d(\mu_1\circ\tau^{-1})}{d\mu_2}$.

Proof. By Corollary 3.3, we only need to prove (23).

We note that τ is a regular set isomorphism from $(T_1 \setminus C_1, \mathfrak{F}_1, \mu_1)$ onto $(T_2 \setminus B_1, \mathfrak{F}_2, \mu_2)$. So τ^{-1} is well-defined. By the chain rule, $\tau' = \left(\frac{1}{(\tau^{-1})'}\right)\circ\tau$.

Suppose that $I_{\Phi_1}(a_1) > 1$. By Lemma 3.2, for any measurable subset $D \subseteq T_1 \setminus C_1$ and for any $\lambda > 0$,
$$\int_D \Phi_1(\lambda, t)\,d\mu_1 = \int_{\tau(D)} \Phi_2(\lambda|w(t)|, t)\,d\mu_2$$
$$= \int_D [\Phi_2(\lambda|w|,\cdot)]\circ\tau^{-1}(t)\cdot(\tau^{-1})'(t)\,d\mu_1\,.$$
So we must have
$$[(\tau')^{-1}\Phi_2(\lambda|w|,\cdot)]\circ\tau^{-1}(t) - \Phi_1(\lambda, t) = 0$$
for almost all $t \in T_1 \setminus C_1$, which is equivalent to (23). ∎

Remark 3.2. Let (T, Σ, μ) be a σ-finite measure space and let $\Phi(u,t)$ be any Young function on T. Then the above theorem provides a complete characterization of the surjective isometries of the complex Musielak–Orlicz space L_Φ. A similar characterization was obtained in [5] under the assumption that the function $u \mapsto \frac{\Phi'(u,t)}{u}$ is monotone for almost all $t \in T$.

Remark 3.3. If $\Phi_i(u,t) \equiv \varphi_i(u)$ and if φ_i's are continuous, then the condition (23) becomes
$$\varphi_2(|w(t)|u) = \tau'(t)\varphi_1(u)$$
for all $u \ge 0$ and almost all $t \in T_2$. This implies that L_φ is isometric to L^p for $1 \le p < \infty$, $p \ne 2$, if and only if $\varphi(u) = cu^p$ for some $c > 0$.

Remark 3.4. Suppose that $\Phi_1(u,t) = \frac{u^{p(t)}}{p(t)}$ (respectively, $\Phi_2(u,t) = \frac{u^{r(t)}}{r(t)}$) for some Σ_1-measurable (respectively, Σ_2-measurable) function $p : T_1 \to [1,\infty)$ (respectively, $r : T_2 \to [1,\infty)$). Then (23) becomes

(*) $$r(t) = (p\circ\tau)(t)\,, \qquad \tau'(t) = |w(t)|^{r(t)}$$

for almost all $t \in T_2$. Thus, $U : L^{p(t)} \to L^{r(t)}$ is a surjective isometry if and only if $U(f) = w\cdot(f\circ\tau)$ for some regular set isomorphism τ from (T_1, Σ_1, μ_1) onto itself and the condition (*) is satisfied with $|w(t)| \ne 0$ a.e.

4. Isometries of real Nakano spaces $L^{p(t)}$. Let (T_1, Σ_1, μ_1) and (T_2, Σ_2, μ_2) be any two σ-finite atomless measure spaces. For any α, $\infty > \alpha > 2$, let $p : T_1 \to (2,\alpha)$ (respectively, $r : T_2 \to (2,\alpha)$) be a Σ_1-measurable (respectively, Σ_2-measurable) function. In this section, we study the isometries from the real Nakano space $L^{p(t)}$ into the real Nakano space $L^{r(t)}$. Recall an isometry U is said to have the disjoint support property if for any f, g with $fg = 0$, $Uf\cdot Ug = 0$ holds almost everywhere. The following proposition shows that every isometry from $L^{p(t)}$ into $L^{r(t)}$ possesses the disjoint support property.

PROPOSITION 4.1. *Let $2 < \alpha < \infty$ and let p (respectively, r) be any measurable function from T_1 (respectively, T_2) into $(2,\alpha)$. Then any isometry $U : L^{p(t)} \to L^{r(t)}$ has the disjoint support property.*

Proof. Let Φ_1 and Φ_2 denote the Young functions
$$\Phi_1(u,t) = \frac{u^{p(t)}}{p(t)}, \qquad t \in T_1,$$
$$\Phi_2(u,t) = \frac{u^{r(t)}}{r(t)}, \qquad t \in T_2\,.$$
We will need the following inequalities:

(24) $$(1+x)^p + (1-x)^p \ge 2 + \frac{p(p-1)}{2}x^2 \qquad \text{for all } p \ge 2 \text{ and } |x| \le 1\,,$$

(25) $$(1+x)^{-m} \ge 1 - mx \qquad \text{for all } x \ge 0 \text{ and } m > 0\,.$$

Since $p(t) > 2$ for almost all $t \in T_1$ and the measure space (T_1, Σ_1, μ_1) is σ-finite, we can find a sequence $\{C_n\}$ of measurable sets and a sequence $\{\beta_n\}$ of numbers such that

(a) for each n, $C_n \subseteq C_{n+1}$ and $\chi_{C_n} \in L^{p(t)}$;

(b) $\bigcup_{n=1}^\infty C_n = T_1$;

(c) $\alpha \ge p(t) \ge \beta_n > 2$ for all $t \in C_n$.

Then any $f \in L^{p(t)}$ can be approximated by simple functions vanishing outside of some C_n. Thus, it is enough to prove the disjoint support property for characteristic functions of sets contained in C_n. Without loss of generality, we assume that $p(t) \ge \beta > 2$ for all $t \in T_1$ and $\chi_A \in L^{p(t)}$ for any $A \in \Sigma_1$.

Let A and B be any two disjoint measurable subsets of T_1, and let $c > 0$

be such that

$$\|c\chi_A\|_{\Phi_1} = \int_A \frac{c^{p(t)}}{p(t)}\,d\mu_1 = 1.$$

For any $|\lambda| \le 1$, we have $1 \le I_{\Phi_1}(c\chi_A + \lambda\chi_B) \le 1 + \beta^{-1}|\lambda|^\beta \mu_1(B)$. This implies

$$\left\| \frac{c\chi_A + \lambda\chi_B}{1+\beta^{-1}|\lambda|^\beta \mu_1(B)} \right\|_{\Phi_1} \le 1.$$

Let $f = U(c\chi_A)$ and $g = U(\chi_B)$. We have

$$\left\| \frac{f + \lambda g}{1+\beta^{-1}|\lambda|^\beta \mu_1(B)} \right\|_{\Phi_2} \le 1.$$

So

$$(26)\quad 2 \ge I_{\Phi_2}\left(\frac{f+\lambda g}{1+\beta^{-1}|\lambda|^\beta\mu_1(B)}\right) + I_{\Phi_2}\left(\frac{f-\lambda g}{1+\beta^{-1}|\lambda|^\beta\mu_1(B)}\right)$$
$$\ge \int \frac{|f+\lambda g|^{r(t)}}{(1+\beta^{-1}|\lambda|^\beta\mu_1(B))^\alpha}\,d\mu_2 + \int \frac{|f-\lambda g|^{r(t)}}{(1+\beta^{-1}|\lambda|^\beta\mu_1(B))^\alpha}\,d\mu_2.$$

For a contradiction, suppose that $fg > 0$ on a set with positive measure. Then there is a set $C \subsetneq T_2$ and $0 < \lambda_0 < 1$ such that $\mu_2(C) > 0$ and $0 < |\lambda_0 g(t)| < |f(t)|$ for all $t \in C$. By (24), we have

$$(27)\qquad \int_{C^c} (|f+\lambda g|^{r(t)} + |f-\lambda g|^{r(t)})\,d\mu_2 \ge 2\int_{C^c} |f|^{r(t)}\,d\mu_2.$$

For any $|\lambda| < \lambda_0$ and for $t \in C$, $\left|\frac{\lambda g(t)}{f(t)}\right| < 1$. Thus, by (24), we have

$$(28)\quad \int_C (|f+\lambda g|^{r(t)} + |f-\lambda g|^{r(t)})\,d\mu_2$$
$$\ge \int_C |f|^{r(t)}\left(2 + \frac{r(t)(r(t)-1)}{2}\lambda^2\left|\frac{g(t)}{f(t)}\right|^2\right)d\mu_2$$
$$= \int_C 2|f|^{r(t)}\,d\mu_2 + \lambda^2 \int_C \frac{r(t)(r(t)-1)}{2}|f|^{r(t)-2}|g|^2\,d\mu_2.$$

Writing γ for the last integral in (28), we get

$$2 \ge \frac{2+\lambda^2\gamma}{(1+\beta^{-1}|\lambda|^\beta\mu_1(B))^\alpha} \ge (2+\lambda^2\gamma)(1-\alpha\beta^{-1}|\lambda|^\beta\mu_1(B)),$$

for all $|\lambda| < \lambda_0$. However, $(2+\lambda^2\gamma)(1-\alpha\beta^{-1}\lambda^\beta\mu_1(B)) > 2$ for sufficiently small λ and this contradiction finishes the proof. ■

Let $\{A_k : k \in \mathbb{N}\}$ be any partition of T_1 such that $\mu_1(A_k) < \infty$ for all $k \in \mathbb{N}$. For an isometry U from $L^{p(t)}$ into $L^{r(t)}$, let Σ_3 be the σ-ring generated by

$$\{\operatorname{supp} U(\chi_A) : A \in \Sigma_1 \text{ and } \mu_1(A) < \infty\},$$

let

$$T_3 = \bigcup_{k=1}^\infty \operatorname{supp} U(\chi_{A_k}),$$

and let w be the function from T_3 into $\mathbb{R}$ defined by

$$w(t) = U(\chi_{A_k})(t) \quad \text{if } t \in \operatorname{supp} U(\chi_{A_k}).$$

Then Σ_3 is a σ-algebra of subsets of T_3 and w is nonzero almost everywhere on T_3. Let τ be the mapping from Σ_1 into Σ_3 defined by

$$\tau(A) = \bigcup_{k=1}^\infty \operatorname{supp} U(\chi_{A\cap A_k}).$$

By Proposition 4.1, τ is a regular set isomorphism from Σ_1 onto Σ_3. So τ^{-1} is well-defined.

Let f be an integrable Σ_2-measurable function. A Σ_3-measurable function $g = \mathcal{E}(f \mid \Sigma_3)$ is said to be the *conditional expectation* of f relative to Σ_3 if for every $A \in \Sigma_3$,

$$\int_A f\,d\mu_2 = \int_A g\,d\mu_2.$$

It is known that if h is a Σ_3-measurable function and if fh is integrable, then $\mathcal{E}(fh \mid \Sigma_3) = h\mathcal{E}(f \mid \Sigma_3)$. We claim that $r|_{T_3} = p \circ \tau$.

Suppose the claim is proved. Then r is a Σ_3-measurable function. By Theorem 3.1 and Lemma 3.2, for any Σ_3-measurable subset D of A_j and any $\lambda > 0$,

$$\int_D \frac{\lambda^{p(t)}}{p(t)}\,d\mu_1(t) = \int_{\tau(D)} \frac{(\lambda|w(t)|)^{r(t)}}{r(t)}\,d\mu_2(t)$$
$$= \int_{\tau(D)} \mathcal{E}\left(\frac{(\lambda|w|)^r}{r}\,\middle|\,\Sigma_3\right)(t)\,d\mu_2(t)$$
$$= \int_D \left[\mathcal{E}\left(\frac{(\lambda|w|)^r}{r}\,\middle|\,\Sigma_3\right)\right]\circ\tau^{-1}(t)\cdot(\tau^{-1})'(t)\,d\mu_1(t)$$
$$= \int_D [\mathcal{E}(|w|^r \mid \Sigma_3)]\circ\tau^{-1}(t)\cdot\left(\frac{\lambda^r}{r\tau'}\right)\circ\tau^{-1}(t)\,d\mu_1(t).$$

So we have

$$[\mathcal{E}(|w|^r \mid \Sigma_3)]\circ\tau^{-1}(t)\cdot\left(\frac{\lambda^r}{r\tau'}\right)\circ\tau^{-1}(t) = \frac{\lambda^{p(t)}}{p(t)}.$$

But $r(t) = p \circ \tau(t)$. This implies
$$\tau' = \mathcal{E}(|w|^r \mid \Sigma_3).$$

Suppose the claim is not true. Without loss of generality, we assume that there exist $\lambda_1 < \lambda_2$, $\varepsilon > 0$, $N \in \mathbb{N}$, and a Σ_3-measurable subset C of T_3 such that

(d) $\tau^{-1}(C) \subseteq p^{-1}([\lambda_1, \lambda_2])$;
(e) $\mu_2(r^{-1}([\lambda_2+\varepsilon, \infty)) \cap \{t : |w(t)| \geq 1/N\} \cap C) > 0$;
(f) $\mu_1(\tau^{-1}(C)) < \infty$.

Let $D = r^{-1}([\lambda_2+\varepsilon, \infty)) \cap w^{-1}([1/N, \infty)) \cap C$. By Theorem 3.2, if $u > \max\{N, 1\}$, then
$$\begin{aligned}(u/N)^{\lambda_2+\varepsilon}\mu_2(D) &\leq \int_D \Phi_2(u|w(t)|, t)\, d\mu_2(t) \leq \int_C \Phi_2(u|w(t)|, t)\, d\mu_2(t)\\ &= \int_C \tau'(t)[\Phi_1(u, \cdot) \circ \tau](t)\, d\mu_2(t)\\ &= \int_{\tau^{-1}(C)} \Phi_1(u, t) d\mu_1 \leq u^{\lambda_2}\mu_1(\tau^{-1}(C)).\end{aligned}$$
But this is impossible for u large enough. So our claim is proved.

Now we can formulate a criterion for U to be an isometry between two real Nakano spaces.

THEOREM 4.2. *Let $L^{p(t)}$ (respectively, $L^{r(t)}$) be a real Nakano space with $2 < p(t) \leq \alpha$ (respectively, $2 < r(t) \leq \alpha$) for almost all $t \in T_1$ (respectively, $t \in T_2$) and some $\alpha > 2$. Let U be an isometry from $L^{p(t)}$ into $L^{r(t)}$. Then there exist a Σ_3-measurable set $T_3 \subseteq T_2$, a Σ_2-measurable function w on T_3, and a regular set isomorphism $\tau : (T_1, \Sigma_1) \to (T_3, \Sigma_3)$ such that for all $f \in L^{p(t)}$,*
$$Uf = w \cdot (f \circ \tau), \tag{29}$$
$$r = p \circ \tau, \qquad \tau' = \mathcal{E}(|w|^r \mid \Sigma_3). \tag{30}$$

Conversely, if the equalities (30) are satisfied, then the operator U given by (29) is an isometry from $L^{p(t)}$ into $L^{r(t)}$.

Proof. We only need to prove (29). Since every $f \in L^{p(t)}$ can be approximated by step functions and $U(\chi_A) = w \cdot \chi_{\tau(A)}$ for every A with $\mu_1(A) < \infty$, U is of the form
$$U(f) = w \cdot (f \circ \tau).$$

Suppose U is a mapping which satisfies (29) and (30). By a change of variable, $\|f\|_{\Phi_1} = \|U(f)\|_{\Phi_2}$ for every $f \in L^{p(t)}$. So U is an isometry. ∎

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