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# Nonseparability of the quotient space $cabv(\Sigma, m; X)/L^1(m; X)$ for Banach spaces $X$ without the Radon–Nikodym property

by

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**Abstract.** It is shown that if $(S, \Sigma, m)$ is an atomless finite measure space and $X$ is a Banach space without the Radon–Nikodym property, then the quotient space $cabv(\Sigma, m; X)/L^1(m; X)$ is nonseparable.

Throughout, $(S, \Sigma, m)$ is an atomless probability measure space and $X$ is a Banach space. We denote by $cabv(\Sigma, m; X)$ the Banach space of all countably additive $m$-continuous vector measures $\mu : \Sigma \to X$ of bounded variation equipped with the variation norm $\|\mu\|_1 = |\mu|(S)$. As usual, $L^1(m; X) = L^1(S, \Sigma, m; X)$ stands for the Banach space of all (equivalence classes of) Bochner integrable functions $f : S \to X$ under the norm $\|f\|_1 = \int_S \|f(\cdot)\| \, dm$. In what follows we consider $L^1(m; X)$ as a (closed) subspace of $cabv(\Sigma, m; X)$ via the linear isometric embedding that assigns to every function $f$ in $L^1(m; X)$ its indefinite Bochner integral $\int f \, dm$. If $\Sigma_0$ is a sub-$\sigma$-algebra of $\Sigma$ and $m_0 = m|\Sigma_0$, then the corresponding conditional expectation operator from $L^1(m; X)$ onto $L^1(m_0; X)$ is denoted by $\mathbb{E}(\cdot \mid \Sigma_0)$, without any explicit mentioning of $m$ and $X$. In general, the reader is referred to Diestel and Uhl [3] for the theory of vector measures and integration, of which we make free use below. The same book is also our main reference for Banach spaces with the Radon–Nikodym property (RNP); another good reference for this is Bourgin [2].

In a recent paper [5], G. Emmanuele and the present author have formulated the conjecture that $L^1(m; X)$ is uncomplemented in $cabv(\Sigma, m; X)$ whenever $X$ fails the RNP, and showed this to be so in the case of Banach

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spaces $X$ containing an (isomorphic) copy of $c_0$ [1]. Somewhat independently of this problem, one may also raise a more general (and vague) question of how big is the "gap" between $L^1(m;X)$ and $cabv(\Sigma,m;X)$ when $X \notin$ (RNP). One of the results from [5] is relevant to this question: If $X \supset c_0$, then $cabv(\Sigma,m;X)/L^1(m;X)$ contains a complemented copy of $l_\infty/c_0$. The purpose of this note is to prove the following general result.

THEOREM. *If $X$ does not have the Radon–Nikodym property, then the quotient space*

$$cabv(\Sigma,m;X)/L^1(m;X)$$

*is nonseparable.*

At this point let us note that from the results of J. Bourgain [1] it follows that if $X \notin$ (RNP), then $L^1(m;X)$ is a proper closed subspace of $ccabv(\Sigma,m;X)$, where the latter is the closed subspace of $cabv(\Sigma,m;X)$ consisting of measures with relatively norm compact ranges. A quick inspection of the arguments below (in particular, the proof of Proposition 2) shows that also the following stronger version of the Theorem holds true:

*If $X$ does not have the Radon–Nikodym property, then the quotient space*

$$ccabv(\Sigma,m;X)/L^1(m;X)$$

*is nonseparable.*

The proof of the Theorem will be given (and will be short) after we show the following three propositions.

PROPOSITION 1. *Let $\Sigma_0$ be a sub-$\sigma$-algebra of $\Sigma$ and $m_0 = m|\Sigma_0$.*

(a) *The formula*

$$(T\mu_0)(A) = \int_S \mathbb{E}(\chi_A \mid \Sigma_0)\, d\mu_0$$

*defines a linear isometric extension operator*

$$T : cabv(\Sigma_0,m_0;X) \to cabv(\Sigma,m;X)$$

*which is the identity on $L^1(m_0;X)$.*

*In consequence, the operator*

$$P : cabv(\Sigma,m;X) \to cabv(\Sigma,m;X)$$

*given by the equality*

$$P\mu = T(\mu \mid \Sigma_0)$$

[1] In the meantime, the conjecture has been disproved by F. Freniche and L. Rodríguez-Piazza, see the footnote on page 112 of [5]. (Note added November 1992.)



*is a norm one projection onto the range of $T$, and $P|L^1(m;X) = \mathbb{E}(\cdot \mid \Sigma_0)$ is a projection onto $L^1(m_0;X)$.*

(b) *The quotient space $cabv(\Sigma_0,m_0;X)/L^1(m_0;X)$ is linearly isometric to a norm one complemented subspace of $cabv(\Sigma,m;X)/L^1(m;X)$.*

Proof. For (a) see [7], [6] and [4]; (b) follows from the second part of (a) by virtue of Lemma 3.5 from [5]. ∎

The next result was obtained independently by A. Michalak [8] in the case of the unit circle $\mathbb{T}$ with Haar measure; his proof is based on the theory of vector-valued harmonic functions. Our proof seems to be more elementary.

PROPOSITION 2. *If $X$ fails the Radon–Nikodym property, then the space $cabv(\Sigma,m;X)$ is nonseparable.*

Proof. In view of Proposition 1 it is enough to show the nonseparability of the space $cabv(\Sigma_0,m_0;X)$, where $\Sigma_0$ is any countably generated sub-$\sigma$-algebra of $\Sigma$ such that the restricted measure $m_0 = m|\Sigma_0$ is still atomless. But, by a well known theorem of Carathéodory, all such measure spaces $(S,\Sigma_0,m_0)$ are measure-algebra isomorphic. In consequence, we may (and will) assume that our measure space $(S,\Sigma,m)$ is $(K,\mathcal{B},m)$, where $K$ is the Cantor cube,

$$K = D^{\mathbb{N}} = \prod_{n=1}^{\infty} D_n$$

($D = D_n = \{0,1\}$ for $n \in \mathbb{N}$), $\mathcal{B}$ is its Borel $\sigma$-algebra, and $m$ is the standard product measure on $K$.

In general, for $M \subset \mathbb{N}$, by the product measure on

$$K(M) = D^M,$$

defined on the Borel $\sigma$-algebra $\mathcal{B}(K(M))$, we mean the one obtained when the two-point space $D$ is equipped with the $(\frac{1}{2},\frac{1}{2})$-measure.

Thus we wish to show that

$$X \notin \text{(RNP)} \Rightarrow cabv(\mathcal{B},m;X) \text{ is nonseparable}.$$

Assume $X \notin$ (RNP) so that there is a measure $\mu \in cabv(\mathcal{B},m;X)$ which is not representable as the indefinite Bochner integral over $(K,\mathcal{B},m)$ of any strongly measurable $X$-valued function. As is well known (see [3; Ch. V]), this can be equivalently expressed as follows:

Given any increasing (in the sense of refinement) sequence $(\pi_n)$ of finite $\mathcal{B}$-partitions of $K$ such that $\mathcal{B} = \sigma(\bigcup_{n=1}^{\infty} \pi_n)$, if

$$f_n = \mathbb{E}(\mu \mid \pi_n) = \sum_{A\in\pi_n} \frac{\mu(A)}{m(A)} \cdot \chi_A\,,$$

then the sequence $(f_n)$ is not Cauchy in $L^1(m;X)$.

In what follows we take

$$\pi_n = \{ A(\varepsilon) : \varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in D^n \},$$

where

$$A(\varepsilon) = \{\varepsilon\} \times \prod_{i=n+1}^{\infty} D_i ,$$

and define $f_n$ by the above formula. By what was said above, we can find a $\delta > 0$ and a sequence $(i_j)_{j=0}^{\infty}$ in $\mathbb{N}$ such that defining

$$k_r = i_0 + \ldots + i_r \quad (r = 0, 1, \ldots),$$

we will have

$$\|f_{k_r} - f_{k_{r-1}}\|_1 > \delta \quad \text{for all } r \geq 1 .$$

Now we are going to construct a family $\{\mu_\eta : \eta \in \{0,1\}^{\mathbb{N}}\}$ of measures in $cabv(\mathcal{B}, m; X)$ so that for any two distinct indices $\eta$, $\gamma$ in $\{0,1\}^{\mathbb{N}}$ there is an $r \geq 1$ for which $\|\mu_\eta - \mu_\gamma\|_1 \geq \|f_{k_r} - f_{k_{r-1}}\|_1$.

We start by writing $\mathbb{N}$ as the union of consecutive (pairs of) blocks

$$M_0,\ M_1^0, M_1^1,\ \ldots,\ M_j^0, M_j^1,\ \ldots$$

so that $|M_0| = i_0$, and $|M_j^0| = |M_j^1| = i_j$ for $j \geq 1$.

For every $\eta = (\eta_j)_{j=1}^{\infty}$ in the index set $\{0,1\}^{\mathbb{N}}$ define

$$M_\eta = M_0 \cup \bigcup_{j=1}^{\infty} M_j^{\eta_j}, \quad K_\eta = K(M_\eta), \quad L_\eta = K(\mathbb{N} \setminus M_\eta).$$

Next, writing $M_\eta = \{m_1 < m_2 < \ldots\}$, define the measure $\kappa_\eta : \mathcal{B}(K_\eta) \to X$ to be the image of $\mu$ under the bijection

$$(\varepsilon_1, \varepsilon_2, \ldots) \mapsto (\xi_{m_1}, \xi_{m_2}, \ldots) : K \to K_\eta ,$$

where $\xi_{m_i} = \varepsilon_i$ for every $i$. Thus, in particular, if

$$\xi = (\xi_{m_1}, \ldots, \xi_{m_j}) \in D^{\{m_1, \ldots, m_j\}},$$

then

$$\kappa_\eta(\{\xi\} \times K(M_\eta \setminus \{m_1, \ldots, m_j\})) = \mu\Big(\{\varepsilon\} \times \prod_{n=j+1}^{\infty} D_n\Big),$$

where $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_j)$ with $\varepsilon_i = \xi_{m_i}$ for $i = 1, \ldots, j$.

Also, let $m_\eta$ and $m'_\eta$ be the product measures on $K_\eta$ and $L_\eta$, respectively. Clearly, $m = m_\eta \otimes m'_\eta$.

Finally, set

$$\mu_\eta = \kappa_\eta \otimes m'_\eta,$$

the product measure of $\kappa_\eta$ and $m'_\eta$ on $\mathcal{B}(K_\eta) \otimes \mathcal{B}(L_\eta) = \mathcal{B}$. It is quite obvious that $\mu_\eta \in cabv(\mathcal{B}, m; X)$.



Now let $\eta = (\eta_j)_{j=1}^{\infty}$ and $\gamma = (\gamma_j)_{j=1}^{\infty}$ be two distinct points in our index set $\{0,1\}^{\mathbb{N}}$. Let $r$ be the least $j$ for which $\eta_j \neq \gamma_j$. Thus $\eta_j = \gamma_j$ for $j < r$, and $\eta_r \neq \gamma_r$; without loss of generality we may assume that $\eta_r = 0$ and $\gamma_r = 1$. For $j \geq 1$ write

$$M_j = M_0 \cup M_1^{\eta_1} \cup \ldots \cup M_j^{\eta_j} = \{m_1 < \ldots < m_{k_j}\}.$$

Let $\pi$ be the partition of $K$ consisting of the sets

$$B(\xi) = \{\xi\} \times K(\mathbb{N} \setminus M_r) = C(\xi) \times L_\eta ,$$

where

$$C(\xi) = \{\xi\} \times K(M_\eta \setminus M_r) \in \mathcal{B}(K_\eta), \quad \xi = (\xi_{m_1}, \ldots, \xi_{m_{k_r}}) \in K(M_r).$$

Then

$$\mathbb{E}(\mu_\eta \mid \pi) = \sum_{\xi} \frac{\mu_\eta(B(\xi))}{m(B(\xi))} \cdot \chi_{B(\xi)} = \sum_{\xi} \frac{\kappa_\eta(C(\xi))}{m_\eta(C(\xi))} \cdot \chi_{\{\xi\} \times K(\mathbb{N} \setminus M_r)} .$$

Moreover, if $\xi$ is as above and we write

$$\xi' = (\xi_{m_1}, \ldots, \xi_{m_{k_{r-1}}}) \in K(M_{r-1}),$$
$$\xi'' = (\xi_{m_{k_{r-1}+1}}, \ldots, \xi_{m_{k_r}}) \in K(M_r^0),$$

then

$$B(\xi) = D(\xi') \times \{\xi''\} \times K(\mathbb{N} \setminus M_\gamma \setminus M_r^0),$$

where

$$D(\xi') = \{\xi'\} \times K(M_\gamma \setminus M_{r-1}) \in \mathcal{B}(K_\gamma).$$

Therefore,

$$\begin{aligned}
\mathbb{E}(\mu_\gamma \mid \pi) &= \sum_{\xi} \frac{\mu_\gamma(B(\xi))}{m(B(\xi))} \cdot \chi_{B(\xi)} \\
&= \sum_{\xi'} \sum_{\xi''} \frac{\kappa_\gamma(D(\xi'))}{m_\gamma(D(\xi'))} \cdot \chi_{D(\xi') \times \{\xi''\} \times K(\mathbb{N} \setminus M_\gamma \setminus M_r^0)} \\
&= \sum_{\xi'} \frac{\kappa_\gamma(D(\xi'))}{m_\gamma(D(\xi'))} \cdot \chi_{\{\xi'\} \times K(M_r^0) \times K(\mathbb{N} \setminus M_r)} .
\end{aligned}$$

Now

$$\begin{aligned}
\|\mu_\eta - \mu_\gamma\|_1 &\geq \|\mathbb{E}(\mu_\eta \mid \pi) - \mathbb{E}(\mu_\gamma \mid \pi)\|_1 \\
&= \int_K \Big\| \sum_{\xi} \frac{\kappa_\eta(C(\xi))}{m_\eta(C(\xi))} \cdot \chi_{\{\xi\} \times K(\mathbb{N} \setminus M_r)} \\
&\qquad - \sum_{\xi'} \frac{\kappa_\gamma(D(\xi'))}{m_\gamma(D(\xi'))} \cdot \chi_{\{\xi'\} \times K(M_r^0) \times K(\mathbb{N} \setminus M_r)} \Big\| \, dm
\end{aligned}$$

$$= \int_{K(M_r)} \Big\| \sum_{\xi} \frac{\kappa_\eta(C(\xi))}{m_\eta(C(\xi))} \cdot \chi_{\{\xi\}} - \sum_{\xi'} \frac{\kappa_\gamma(D(\xi'))}{m_\gamma(D(\xi'))} \cdot \chi_{\{\xi'\}\times K(M_r^0)} \Big\| \, dm'$$

($m'$ = the product measure on $K(M_r)$)

$$= \int_{D^{k_r}} \Big\| \sum_{\varepsilon} \frac{\mu(A(\varepsilon))}{m(A(\varepsilon))} \cdot \chi_{\{\varepsilon\}} - \sum_{\varepsilon'} \frac{\mu(A(\varepsilon'))}{m(A(\varepsilon'))} \cdot \chi_{\{\varepsilon'\}\times K(\{k_{r-1}+1,\ldots,k_r\})} \Big\| \, dm''$$

($m''$ = the product measure on $D^{k_r}$; $\varepsilon \in D^{k_r}$, $\varepsilon' \in D^{k_{r-1}}$)

$$= \int_{K} \Big\| \sum_{\varepsilon} \frac{\mu(A(\varepsilon))}{m(A(\varepsilon))} \cdot \chi_{A(\varepsilon)} - \sum_{\varepsilon'} \frac{\mu(A(\varepsilon'))}{m(A(\varepsilon'))} \cdot \chi_{A(\varepsilon')} \Big\| \, dm$$

$$= \|f_{k_r} - f_{k_{r-1}}\|_1 > \delta .$$

We have thus shown that $\|\mu_\eta - \mu_\gamma\|_1 > \delta$ whenever $\eta, \gamma \in \{0,1\}^{\mathbb{N}}$ and $\eta \neq \gamma$, from which the nonseparability of $cabv(\mathcal{B}, m; X)$ is immediate. ∎

In the proof of our third proposition it will be convenient to have the following elementary lemma at hand.

LEMMA. *Let $W$ be a Banach space and $V$ be a closed subspace of $W$. Suppose we have an upward directed family $(V_i)_{i\in I}$ of closed subspaces of $V$ whose union is dense in $V$, and a uniformly bounded family of onto projections $P_i : W \to V_i$ $(i \in I)$; let $K = \sup_i \|P_i\|$. Then for every $w \in W$,*

$$d(w, V) \le \inf_i \|w - P_i(w)\| \le \limsup_i \|w - P_i(w)\| \le (1 + K) d(w, V),$$

*where $d(w, V) = \inf\{\|w - v\| : v \in V\}$ $(= \|w + V\|$ in $W/V)$.*

Proof. Let $w \in W$ and $i \in I$. If $v \in V_i$, then

$$\|w - v\| \ge \|w - P_i w\| - \|P_i w - v\| = \|w - P_i w\| - \|P_i(w - v)\| \ge \|w - P_i w\| - K\|w - v\| .$$

Hence

$$(1 + K) \inf_{v\in V_i} \|w - v\| \ge \|w - P_i w\| \ge \inf_{v\in V_i} \|w - v\|,$$

from which the assertion of the lemma follows easily. ∎

PROPOSITION 3. *Let $Y$ be a closed subspace of the Banach space $X$. Then the mapping*

$$J : \mu + L^1(m; Y) \mapsto \mu + L^1(m; X)$$



*is an isomorphic embedding of the quotient space $cabv(\Sigma, m; Y)/L^1(m; Y)$ into the quotient space $cabv(\Sigma, m; X)/L^1(m; X)$.*

Proof. Let $\Pi$ be the (directed upward by refinement) family of all finite $\Sigma$-partitions $\pi$ of $S$ such that $m(A) > 0$ for every $A \in \pi$. For every $\pi \in \Pi$ the conditional expectation operator $E_\pi : cabv(\Sigma, m; X) \to L^1(m; X)$ defined by

$$E_\pi(\mu) = \mathbb{E}(\mu \mid \pi) = \sum_{A\in\pi} \frac{\mu(A)}{m(A)} \cdot \chi_A$$

is a norm one projection onto the subspace of $L^1(m; X)$ consisting of $\pi$-simple functions. Obviously, the union of all these subspaces is simply the subspace of all $\Sigma$-simple functions and so it is dense in $L^1(m; X)$. Applying the above Lemma to this situation, we get for every $\mu \in cabv(\Sigma, m; X)$,

$$d(\mu, L^1(m; X)) \le \limsup_{\pi} \|\mu - E_\pi(\mu)\| \le 2d(\mu, L^1(m; X)) .$$

Evidently, since $E_\pi$ maps $Y$-valued measures to $Y$-valued functions, for the same reason we have for every $\mu \in cabv(\Sigma, m; Y)$,

$$d(\mu, L^1(m; Y)) \le \limsup_{\pi} \|\mu - E_\pi(\mu)\| \le 2d(\mu, L^1(m; Y)) .$$

It follows that for every $\mu \in cabv(\Sigma, m; Y)$,

$$d(\mu, L^1(m; X)) \le d(\mu, L^1(m; Y)) \le 2d(\mu, L^1(m; X)),$$

and the assertion follows. ∎

Proof of the Theorem. Let $Y$ be a separable closed subspace of $X$ without the Radon–Nikodym property (see [3; III.3.2]). Moreover, choose a countably generated sub-$\sigma$-algebra $\Sigma_0$ of $\Sigma$ such that the measure $m_0 = m|\Sigma_0$ is atomless. By Proposition 2, $cabv(\Sigma_0, m_0; Y)$ is nonseparable. Since $L^1(m_0; Y)$ is separable, the quotient space $cabv(\Sigma_0, m_0; Y)/L^1(m_0; Y)$ must be nonseparable. By Proposition 1(b), it embeds isometrically into the quotient space $cabv(\Sigma, m; Y)/L^1(m; Y)$, and to conclude the proof it suffices to apply Proposition 3. ∎

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# Interpolation of operators when the extreme spaces are $L^\infty$

by

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**Abstract.** Under some assumptions on the pair $(A_0, B_0)$, we study equivalence between interpolation properties of linear operators and monotonicity conditions for a pair $(Y, Z)$ of rearrangement invariant quasi-Banach spaces when the extreme spaces of the interpolation are $L^\infty$. Weak and restricted weak intermediate spaces fall within our context. Applications to classical Lorentz and Lorentz–Orlicz spaces are given.


**0. Introduction.** Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be two compatible couples of rearrangement invariant Banach function spaces (for short r.i. spaces) over the interval $I$ ($I = [0,1]$ or $[0,\infty)$) (see definitions below). We denote by $\mathcal{A}(\overline{A}, \overline{B})$ the class of linear (or quasilinear or Lipschitz) operators which are bounded from $A_0$ into $B_0$ and from $A_1$ into $B_1$. If $Y$, $Z$ are intermediate r.i. spaces with respect to $\overline{A}$, $\overline{B}$ respectively, we say that the pair $(Y, Z)$ has the *linear* (or *quasilinear* or *Lipschitz*) *interpolation property with respect to the class* $\mathcal{A}$ if every member of $\mathcal{A}$ can be extended to a bounded operator from $Y$ into $Z$.

This interpolation property has been extensively studied in connection with many aspects concerning r.i. spaces, for instance, Boyd or Zippin's indices, monotonicity conditions, boundedness of some suitable "maximal" operators and so on. Here we are concerned with the case $A_1 = B_1 = L^\infty$ and particularly in connection with the monotonicity property $(\mathcal{M})$ given in §1 and the boundedness of only one operator.

In this direction the first result is contained in Calderón's paper [5] where it is shown that both properties, say linear interpolation and monotonicity, are equivalent in the case of $A_0 = B_0 = L^1$. Later on, Lorentz and Shimogaki [11] extended this result to the case $A_0 = B_0 = L^p$ with $p > 1$. The technique used by them consists in a linearization process of the $L^p$ case.

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