References

[1] J. Bourgain, *Dunford-Pettis operators on L^1 and the Radon–Nikodym property*, Israel J. Math. 37 (1980), 34–47.
[2] R. D. Bourgin, *Geometric Aspects of Convex Sets with the Radon–Nikodým Property*, Lecture Notes in Math. 993, Springer, Berlin 1983.
[3] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I., 1977.
[4] L. Drewnowski, *Another note on copies of l_∞ and c_0 in $ca(\Sigma, X)$, and the equality $ca(\Sigma, X) = cca(\Sigma, X)$*, preprint, 1990.
[5] L. Drewnowski and G. Emmanuele, *The problem of complementability for some spaces of vector measures of bounded variation with values in Banach spaces containing copies of c_0*, this volume, 111–123.
[6] Z. Lipecki, *Conditional and simultaneous extensions of group-valued quasi-measures*, Glas. Mat. 19 (1984), 49–58.
[7] R. D. Mauldin, *Some effects of set-theoretical assumptions in measure theory*, Adv. in Math. 27 (1978), 45–62.
[8] A. Michalak, in preparation.

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Interpolation of operators when the extreme spaces are L^∞

by

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Abstract. Under some assumptions on the pair (A_0, B_0), we study equivalence between interpolation properties of linear operators and monotonicity conditions for a pair (Y, Z) of rearrangement invariant quasi-Banach spaces when the extreme spaces of the interpolation are L^∞. Weak and restricted weak intermediate spaces fall within our context. Applications to classical Lorentz and Lorentz–Orlicz spaces are given.

0. Introduction. Let $\overline{A} = (A_0, A_1)$, $\overline{B} = (B_0, B_1)$ be two compatible couples of rearrangement invariant Banach function spaces (for short r.i. spaces) over the interval I ($I = [0,1]$ or $[0,\infty)$) (see definitions below). We denote by $\mathcal{A}(\overline{A}, \overline{B})$ the class of linear (or quasilinear or Lipschitz) operators which are bounded from A_0 into B_0 and from A_1 into B_1. If Y, Z are intermediate r.i. spaces with respect to $\overline{A}$, $\overline{B}$ respectively, we say that the pair (Y, Z) has the *linear* (or *quasilinear* or *Lipschitz*) *interpolation property with respect to the class* $\mathcal{A}$ if every member of $\mathcal{A}$ can be extended to a bounded operator from Y into Z.

This interpolation property has been extensively studied in connection with many aspects concerning r.i. spaces, for instance, Boyd or Zippin's indices, monotonicity conditions, boundedness of some suitable "maximal" operators and so on. Here we are concerned with the case $A_1 = B_1 = L^\infty$ and particularly in connection with the monotonicity property $(\mathcal{M})$ given in §1 and the boundedness of only one operator.

In this direction the first result is contained in Calderón's paper [5] where it is shown that both properties, say linear interpolation and monotonicity, are equivalent in the case of $A_0 = B_0 = L^1$. Later on, Lorentz and Shimogaki [11] extended this result to the case $A_0 = B_0 = L^p$ with $p > 1$. The technique used by them consists in a linearization process of the L^p case.

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Sharpley, Maligranda and other authors (see [12] and references quoted there) studied the case $A_0 = \Lambda(X)$, $B_0 = M(X)$ (see definitions in §2) and $A_1 = B_1 = L^\infty$ or $A_1 = \Lambda(Y)$, $B_1 = M(Y)$ relating the interpolation properties with the boundedness of only one "maximal" operator ([19, Theorem 4.7], [12, Theorem 4.5]). On the other hand, Maligranda [12] obtained equivalence of the interpolation property for Lipschitz operators and monotonicity condition in the case $A_0 = \Lambda(X)$, $B_0 = M(X)$ and $A_1 = B_1 = L^\infty$. When $X = L^p$, $p > 1$, then $\Lambda(X) = L^{p,1}$ and $M(X) = L^{p,\infty}$, so Maligranda's result is close to Lorentz–Shimogaki's. The spaces with the interpolation property, when the extreme spaces are $\Lambda(X)$ and $M(X)$, are generally known in the literature as *weak type intermediate spaces*.

These papers leave out the more "natural" case where $A_0 = L^p$, $B_0 = L^{p,\infty}$ or, more generally, $A_0 = X$, $B_0 = M(X)$. In fact, following the usual terminology in Fourier Analysis, the term *weak type intermediate spaces* should be reserved to spaces having the interpolation property in this last setting, while the spaces with the interpolation property in the setting described before should be named *restricted weak type intermediate spaces*.

Our final purpose is to study this "intermediate" case between those of Lorentz–Shimogaki and Maligranda. Our main tool consists in obtaining, in a very general context, equivalence between interpolation properties of linear, quasilinear or Lipschitz type and monotonicity condition $(\mathcal{M})$, which ensures that we are working with Calderón couples. When this result is established it is an easy consequence to reduce the linear interpolation property to the boundedness of only one quasilinear operator.

This general result can be applied in the both cases stated before, namely, weak and restricted weak intermediate. So, on the one hand, we obtain some generalizations of Maligranda's results, and on the other hand, we get several results in the case of $A_0 = X$, $B_0 = M(X)$. When $A_0 = L^p$, the quasilinear operator can be iterated and, as a consequence, we deduce that the weak type intermediate spaces are exactly the restricted weak intermediate spaces.

Moreover, by using a characterization of the boundedness of the Hardy operator in Lorentz spaces due to Ariño and Muckenhoupt, we can characterize those Lorentz spaces which are intermediate in terms of handy conditions on the weights. Finally, the last part of the paper is devoted to extending some of the previous results to the more general case of Lorentz–Orlicz spaces.

The paper is organized in two sections: the first one contains notations and the general results, and the second one the applications.

1. General results. A Banach space $(X, \|\cdot\|)$ of real-valued, locally integrable, Lebesgue measurable functions on I ($I = [0,1]$ or $[0,\infty)$) is said to be a *rearrangement invariant Banach function space* over I (for short *r.i. space*) if it satisfies the following conditions:

(i) If $|g| \le |f|$ a.e. and $f \in X$, then $g \in X$ and $\|g\| \le \|f\|$.

(ii) $0 \le f_n \uparrow$, $\sup_{n\in\mathbb{N}} \|f_n\| \le M$ imply that $f = \sup f_n \in X$ and $\|f\| = \sup_{n\in\mathbb{N}} \|f_n\|$.

(iii) X contains the simple integrable functions.

(iv) $f \in X \Leftrightarrow f^* \in X$, and $\|f\| = \|f^*\|$, where f^* denotes the nonincreasing rearrangement of the function f.

(ii) is known in the literature as the *Fatou property* (cf. [10]). It is quite clear that if X is r.i. then $L^1 \cap L^\infty \hookrightarrow X \hookrightarrow L^1 + L^\infty$ (where the symbol $\hookrightarrow$ signifies continuously embedded).

A classical result by Lorentz and Luxemburg ensures that for these spaces

(v) $\|f\| = \sup_{\|g\|_{X'} \le 1} |\int_I fg|$, where X' is the associated space of X which is also r.i. In particular, $X = X''$ isometrically.

The *fundamental function* ϕ_X of a r.i. space is defined by

$$\phi_X(t) = \|\chi_{[0,t]}\|, \qquad t \in I.$$

There is no loss of generality if we assume ϕ_X to be positive, nondecreasing, absolutely continuous far from the origin and concave and to satisfy (see [19], [22]):

(vi) $\phi_X(t)\phi_{X'}(t) = t$ for all $t \in I$;

(vii) $d\phi_X(t)/dt \le \phi_X(t)/t$ a.e. on I.

In what follows it may be convenient to let X be a r.i. quasi-Banach space. The main difference occurs in the triangle inequality satisfied in X, i.e. $\|f+g\| \le C(\|f\| + \|g\|)$ for some constant $C \ge 1$. In this case we suppose that a quasi-Banach space X satisfies properties (i)–(iv) but, in general, no other conditions will be assumed. We say that a quasi-Banach function space is *σ-order continuous* if every order bounded nondecreasing sequence converges in the quasi-norm topology (cf. [10, Proposition 1.a.8]).

Let (A_0, L_∞), (B_0, L_∞), be two compatible couples of r.i. quasi-Banach spaces on I. Let Y and Z be intermediate r.i. spaces on I with respect to (A_0, L^∞) and (B_0, L^∞) respectively. We introduce the following notation:

- $(Y, Z) \in \mathcal{LI}(A_0, B_0; L^\infty)$ if any linear operator $T : A_0 + L^\infty \to B_0 + L^\infty$ which is bounded from A_0 into B_0 and from L^∞ into L^∞ is also bounded from Y into Z. The closed graph theorem implies that there exists a constant $C \ge 1$ such that

$$\|T\|_{Y\to Z} \le C \max\{\|T\|_{A_0\to B_0}, \|T\|_{L^\infty\to L^\infty}\}.$$

- $(Y, Z) \in \mathcal{QLI}(A_0, B_0; L^\infty)$ if any quasilinear operator $T : A_0 + L^\infty \to B_0 + L^\infty$ which is bounded from A_0 into B_0 and from L^∞ into L^∞ is also bounded from Y into Z.

- $(Y,Z) \in \mathcal{LPI}(A_0,B_0;L^\infty)$ if any operator $T: A_0+L^\infty \to B_0+L^\infty$ that is a Lipschitz operator from A_0 into B_0 and from L^∞ into L^∞ also maps Y into Z.

Recall that a map $T: A_0 \to B_0$ is *bounded quasilinear* if there are constants $K, C \geq 1$ such that $|T(\lambda f)| = |\lambda||T(f)|$, $|T(f+g)| \leq K(|T(f)| + |T(g)|)$ and $\|T(f)\| \leq C\|f\|$. We define $\|T\|_{A_0\to B_0} = \inf C$. In the same way, a map $T: A_0 \to B_0$ is a *Lipschitz operator* if there is a constant $C \geq 1$ such that $T(0)=0$ and $\|Tf - Tg\| \leq C\|f-g\|$; we define $\|T\|_{A_0\to B_0} = \inf C$.

Next we shall introduce another class of spaces and for that we need the following

LEMMA 1. *If X is a r.i. quasi-Banach space, $f \in X + L^\infty$ and $m(E) < \infty$ then $f\chi_E \in X$.*

Proof. Let $f = g + h$ with $g \in X$ and $h \in L^\infty$. Since $g\chi_E \in X$ and $|h\chi_E| \leq \|h\|_\infty \chi_E \in X$ the result follows immediately. ∎

We say that the couple (Y,Z) belongs to $\mathcal{M}(A_0,B_0)$ if there exists a constant $C \geq 1$ such that if $f \in Y$, $g \in B_0 + L^\infty$ and

$$(\mathcal{M}) \qquad \|f^*\chi_{[0,t]}\|_{A_0} \geq \|g^*\chi_{[0,t]}\|_{B_0}, \qquad \forall t > 0,$$

then $g \in Z$ and $\|g\|_Z \leq C\|f\|_Y$.

It is clear that

$$\mathcal{QLI}(A_0,B_0;L^\infty) \cup \mathcal{LPI}(A_0,B_0;L^\infty) \subseteq \mathcal{LI}(A_0,B_0;L^\infty).$$

Under some more restrictive assumptions the four classes of maps introduced above coincide.

PROPOSITION 1. *Let (A_0,B_0) be a couple of r.i. quasi-Banach spaces such that $\phi_{B_0}(t) \leq C\phi_{A_0}(t)$ for all $t > 0$. Then:*

(1.1) $\mathcal{M}(A_0,B_0) \subseteq \mathcal{QLI}(A_0,B_0;L^\infty)$.

(1.2) *If $I = [0,1]$ and A_0 is σ-order continuous then*

$$\mathcal{M}(A_0,B_0) \subseteq \mathcal{LPI}(A_0,B_0;L^\infty).$$

Proof. (1.1) Let (Y,Z) be a couple in $\mathcal{M}(A_0,B_0)$ and let $T: A_0+L^\infty \to B_0+L^\infty$ be a quasilinear map, bounded from A_0 into B_0 and from L^∞ into L^∞. Suppose that $\|T\|_{A_0\to B_0} \leq 1$ and $\|T\|_{L^\infty\to L^\infty} \leq 1$. We have to show that T is bounded from Y into Z. Let $f \in Y$. We only need to prove that

$$(*) \qquad \|(Tf)^*\chi_{[0,t]}\|_{B_0} \leq \|(Cf)^*\chi_{[0,t]}\|_{A_0}$$

for all $t > 0$.

We know that

$$\|(Tf)^*\chi_{[0,t]}\|_{B_0} = \sup \|(Tf)\chi_E\|_{B_0}$$

where E runs over the borelians in I with $m(E) \leq t$. Set $s = f^*(t)$ and define

$$f_{(s)} = \begin{cases} s & \text{if } f > s, \\ -s & \text{if } f < -s, \\ f & \text{otherwise,} \end{cases}$$

and $f^{(s)} = f - f_{(s)}$. Since $f_{(s)} \in L^\infty$ and $f^{(s)} \in A_0$ we have

$$\begin{aligned}\|(Tf_{(s)})\chi_E\|_{B_0} &\leq \|Tf_{(s)}\|_\infty \|\chi_E\|_{B_0} \leq \|f_{(s)}\|_\infty \phi_{B_0}(t) \leq Cs\phi_{A_0}(t) \\ &\leq C\|f^*(t)\chi_{[0,t]}\|_{A_0} \leq C\|f^*\chi_{[0,t]}\|_{A_0}.\end{aligned}$$

Now

$$\|(Tf^{(s)})\chi_E\|_{B_0} \leq \|Tf^{(s)}\|_{B_0} \leq C\|f^{(s)}\|_{A_0} \leq C\|f^*\chi_{[0,t]}\|_{A_0},$$

and hence we easily obtain the inequality $(*)$.

(1.2) In order to show that the couple (Y,Z) belongs to $\mathcal{LPI}(A_0,B_0;L^\infty)$ we will follow the ideas of [12, Lemma 4.4]. For the sake of completeness, we include the proof here. Suppose that T is a Lipschitz operator mapping $A_0 + L^\infty$ into $B_0 + L^\infty$ with $\|T\|_{A_0\to B_0} \leq 1$ and $\|T\|_{L^\infty\to L^\infty} \leq 1$.

If $0 < t \leq 1$ and $f \in Y$, we set $f^*(t) = s$. Consider $f_{(s)}$ and $[(Tf)^*]_{(s)}$ defined as before. Since $f_{(s)} \in L^\infty$ we have $\|T(f_{(s)})\|_\infty \leq s$, and so

$$|Tf - (Tf)_{(s)}| \leq |Tf - T(f_{(s)})|.$$

If $0 < x \leq t$ then

$$|(Tf)^*(x)| \leq |(Tf)^*(x) - [(Tf)^*]_{(s)}(x)| + [(Tf)^*]_{(s)}(x)$$

and thus

$$\begin{aligned}\|(Tf)^*\chi_{[0,t]}\|_{B_0} &\leq C(\||(Tf)^* - [(Tf)^*]_{(s)}|\chi_{[0,t]}\|_{B_0} + f^*(t)\|\chi_{[0,t]}\|_{B_0}) \\ &\leq C(\||Tf - (Tf)_{(s)}|\chi_{[0,t]}\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0}) \\ &\leq C(\||Tf - T(f_{(s)})|\chi_{[0,t]}\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0}) \\ &\leq C(\|Tf - T(f_{(s)})\|_{B_0} + \|f^*\chi_{[0,t]}\|_{A_0}) \\ &\leq C(\|f - f_{(s)}\|_{A_0} + \|f^*\chi_{[0,t]}\|_{A_0}) \\ &\leq C\|f^*\chi_{[0,t]}\|_{A_0}.\end{aligned}$$

This implies that T maps Y into Z. Next by using the fact that A_0 is σ-order continuous we can follow the proof of [11, Theorem 4.5], and conclude the proof of this part. (The constants C appearing above may change from line to line.) ∎

In order to obtain other implications we need to restrict our attention to Banach spaces.

PROPOSITION 2. *Let A_0 be a r.i. Banach space and let B_0 be a r.i. quasi-Banach space. Suppose that $\phi_{A_0} \leq C\phi_{B_0}$ and $1/\phi_{B_0} \in B_0$. Then*

$$\mathcal{LI}(A_0,B_0;L^\infty) \subseteq \mathcal{M}(A_0,B_0).$$

Proof. Let $f \in Y$, $g \in B_0 + L^\infty$ such that

$$\|f^*\chi_{[0,t]}\|_{A_0} \geq \|g^*\chi_{[0,t]}\|_{B_0}, \qquad \forall t > 0.$$

For every $k \in \mathbb{Z}$ we define $E_k = [2^k, 2^{k+1})$. We have

$$g^*(t) = \sum_{k=-\infty}^{\infty} g^*(t)\chi_{E_k}(t) \leq \sum_{k=-\infty}^{\infty} g^*(2^k)\chi_{E_k}(t)$$

$$\leq \sum_{k=-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \|f^*\chi_{[0,2^k]}\|_{A_0}\chi_{E_k}(t).$$

For every $k \in \mathbb{Z}$ we can choose a function $h_k \in A_0'$ with $\|h_k\|_{A_0'} \leq 1$ such that

$$\|f^*\chi_{[0,2^k]}\|_{A_0} \leq 2 \int_I f^*\chi_{[0,2^k]}h_k.$$

Then

$$g^*(t) \leq 2 \sum_{k=-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \Big(\int_0^{2^k} f^*h_k\Big)\chi_{E_k}(t).$$

For any locally integrable function φ on I we define the "linear" operator T by

$$T\varphi(t) = 2 \sum_{k=-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)} \Big(\int_0^{2^k} \varphi h_k\Big)\chi_{E_k}(t).$$

It is clear that if $\varphi \in L^\infty$ and $t \in E_k$ we have

$$|T\varphi(t)| \leq \frac{2}{\phi_{B_0}(2^k)} \|\varphi\|_\infty \|\chi_{[0,2^k]}\|_{A_0} \leq C\|\varphi\|_\infty.$$

On the other hand, if $\varphi \in A_0$ then

$$|T\varphi(t)| \leq 2\|\varphi\|_{A_0} \sum_{k=-\infty}^{\infty} \frac{1}{\phi_{B_0}(2^k)}\chi_{E_k}(t).$$

For $t \in E_k$ the triangle inequality of the quasinorm implies that

$$\phi_{B_0}(t) \leq \phi_{B_0}(2^{k+1}) \leq C\phi_{B_0}(2^k),$$

hence

$$|T\varphi(t)| \leq C\|\varphi\|_{A_0} \sum_{k=-\infty}^{\infty} \frac{\chi_{E_k}(t)}{\phi_{B_0}(t)} = C\|\varphi\|_{A_0} \frac{1}{\phi_{B_0}(t)}.$$

Since $1/\phi_{B_0} \in B_0$ we see that T is also bounded from A_0 into B_0. Eventually we conclude that $g \in B_0$ and $\|g\|_{B_0} \leq C\|f\|_{A_0}$ because $g^* \leq Tf^* \in B_0$. ∎

The above results imply the following

THEOREM 1. *Let A_0 be a r.i. Banach space and let B_0 be a r.i. quasi-Banach space. Suppose that $C^{-1}\phi_{B_0} \leq \phi_{A_0} \leq C\phi_{B_0}$ for some constant C and $1/\phi_{B_0} \in B_0$. Then:*

(i) $\mathcal{LI}(A_0, B_0; L^\infty) = \mathcal{M}(A_0, B_0)$.

(ii) *A couple of r.i. quasi-Banach spaces (Y, Z) belongs to any of these classes if and only if the quasilinear operator*

$$Q\varphi(t) = \frac{1}{\phi_{B_0}(t)} \|\varphi\chi_{[0,t]}\|_{A_0}$$

is bounded from Y into Z for nonincreasing functions.

(iii) *The couples (A_0, L^∞) and (B_0, L^∞) are relative Calderón couples.*

Proof. (i) follows from Propositions 1 and 2. For (ii), observe that Q is bounded from A_0 into B_0 and from L^∞ into L^∞ and so if $(Y, Z) \in \mathcal{LI}(A_0, B_0; L^\infty)$ $(= \mathcal{QLI}(A_0, B_0; L^\infty))$, then Q is bounded from Y into Z. On the other hand, note that condition $(\mathcal{M})$ implies that $g^*(t) \leq Qf^*(t)$ and, therefore, the boundedness of Q for nonincreasing functions implies that $(Y, Z) \in \mathcal{M}(A_0, B_0; L^\infty)$.

For (iii) we recall (see [6]) that the couples (A_0, L^∞), (B_0, L^∞) are termed *relative Calderón couples* if a couple $(Y, Z) \in \mathcal{LI}(A_0, B_0; L^\infty)$ if and only if for some constant $C > 0$ and for each $f \in Y$, whenever the inequality $K(t, g; B_0, L^\infty) \leq K(t, f; A_0, L^\infty)$ holds for all $t > 0$ and some $g \in B_0 + L^\infty$ then $g \in Z$ and $\|g\|_Z \leq C\|f\|_Y$ (K is the K-functional introduced by Peetre). It is easy to prove that the quasilinear operator Q satisfies

$$C^{-1}Q\varphi(t) \leq \frac{1}{\phi_{B_0}(t)} K(\phi_{B_0}(t), \varphi; A_0, L^\infty) \leq CQ\varphi(t)$$

for all $t > 0$ and for all nonnegative nonincreasing functions $\varphi \in A_0 + L^\infty$. Therefore (iii) holds trivially. ∎

Remarks. (i) Under the same hypotheses as in Proposition 2, but supposing only that A_0 is quasi-Banach, we can prove in a simpler way that $\mathcal{QLI}(A_0, B_0; L^\infty) \subseteq \mathcal{M}(A_0, B_0)$. The operator we have to use instead of T is Q.

(ii) Proposition 2 is not true when A_0 is quasi-Banach. For instance, let $I = [0, 1]$, $A_0 = B_0 = L^{p,\infty}$, $0 < p \leq 1$. In this case it is easy to see that $(L^p, L^p) \notin \mathcal{M}(L^{p,\infty}, L^{p,\infty}; L^\infty) = \mathcal{QLI}(L^{p,\infty}, L^{p,\infty}; L^\infty)$, but by using a result by Kalton [8, Theorem 1.1], we can deduce that $(L^p, L^p) \in \mathcal{LI}(L^{p,\infty}, L^{p,\infty}; L^\infty)$ (this result was quoted to the authors by Oscar Blasco).

2. Applications. Not all r.i. Banach spaces B_0 satisfy the condition $1/\phi_{B_0} \in B_0$. In order to study this property we introduce the Lorentz spaces following [2], [12], [19], [22].

In what follows we assume that X is a r.i. Banach space. We use the following notation:

- $\Lambda(X)$ is the space of all measurable functions with

$$\|f\|_{\Lambda(X)} = \int_I f^*(t)\,d\phi_X(t) < \infty\,.$$

Since ϕ_X is concave, the expression $\|f\|_{\Lambda(X)}$ is a norm and moreover $\Lambda(X)$ is a r.i. Banach space.

- $M(X)$ is the space of all measurable functions f for which f^{**} exists and

$$\|f\|_{M(X)} = \sup_{t\in I} \phi_X(t) f^{**}(t) < \infty\,.$$

Recall that f^{**}, the *Hardy transform* of f^*, is defined by

$$H(f^*)(t) = f^{**}(t) = \frac{1}{t}\int_0^t f^*\,.$$

$M(X)$ is again a r.i. Banach space.

- $M^*(X)$ is the space of all measurable functions for which

$$\|f\|_{M^*(X)} = \sup_{t\in I} \phi_X(t) f^*(t) < \infty\,.$$

The function $\|\cdot\|_{M^*(X)}$ is a quasinorm on $M^*(X)$.

It is clear that for these spaces we have:

(i) $X \subseteq M^*(X)$, $\Lambda(X) \hookrightarrow X \hookrightarrow M(X)$, $M(X) \subseteq M^*(X)$,
(ii) $\phi_{\Lambda(X)} = \phi_{M^*(X)} = \phi_X = \phi_{M(X)}$,
(iii) $1/\phi_X \in M^*(X)$,
(iv) $1/\phi_X \in X \Leftrightarrow X = M^*(X)$.

Lemma 2. *Let X be a r.i. Banach space. The following conditions are equivalent*:

(a) *The space $M^*(X)$ is convexifiable* (*i.e. there is a norm on $M^*(X)$ equivalent to $\|\cdot\|_{M^*(X)}$*).
(b) $M(X) = M^*(X)$.
(c) $1/\phi_X \in M(X)$.
(d) *There exists a constant $C > 0$ such that*

$$\frac{\phi_X(t)}{t}\int_0^t \frac{ds}{\phi_X(s)} \le C, \qquad \forall t \in I\,.$$

(e) $\|f\|_{M(X)} \sim \|f\|_{M^*(X)}$, $\forall f \in M^*(X)$.

Proof. We only sketch (a)$\Rightarrow$(b). We may assume there is a r.i. norm $|||\cdot|||$ on $M^*(X)$ equivalent to $\|\cdot\|_{M^*(X)}$. If $f \in M^*(X)$ and $t > 0$ then

$$f^{**}(t)\phi_X(t) \le C|||f^{**}(t)\chi_{[0,t]}||| \le C|||f^*\chi_{[0,t]}|||$$

and therefore $\|f\|_{M(X)} \le C\|f\|_{M^*(X)}$. ∎

From the preceding comments it is clear that $B_0 = M^*(A_0)$ is the only space (after renorming) satisfying the conditions of Theorem 1.

When $A_0 = \Lambda(X)$ (and so, $B_0 = M^*(X)$), the corresponding quasilinear operator is

$$Q_{\Lambda(X)}\varphi(t) = \frac{1}{\phi_X(t)}\int_0^t \varphi(x)\,d\phi_X(x) = H(\varphi\circ\phi_X^{-1})\circ\phi_X(t)\,. \tag{2.1}$$

Note that this operator is actually "linear" and then Theorem 1 is more or less immediate since the linearization made in Proposition 2 is not needed. This case was already studied by Maligranda [12]. In the general case of $(X, M(X))$ the quasilinear operator is

$$Q_X\varphi(t) = \frac{1}{\phi_X(t)}\|\varphi\chi_{[0,t]}\|_X\,.$$

Next we are going to apply the theory to the class of classical Lorentz spaces $\Lambda(w,q)$ with nonmonotone weights. Let w be an a.e. positive weight defined on $I = [0,\infty)$ such that $\int_0^t w < \infty$, $\forall t < \infty$, and $\int_0^\infty w = \infty$. We recall that the classical Lorentz space $\Lambda(w,q)$, $0 < q \le \infty$, is the class of all real-valued measurable functions on I such that

$$\|f\|_{\Lambda(w,q)} = \begin{cases} \left(\int_I f^*(t)^q w(t)\,dt\right)^{1/q} < \infty & \text{if } 0 < q < \infty,\\ \sup_{t>0} f^*(t)w(t) < \infty & \text{if } q = \infty.\end{cases}$$

For $q = \infty$ we will only consider nondecreasing weights w. Ariño and Muckenhoupt [1] showed that given $0 < q < \infty$, there exists a constant $C > 0$ such that the Hardy operator satisfies

$$\|Hf\|_{\Lambda(w,q)} \le C\|f\|_{\Lambda(w,q)}$$

for all nonnegative and nonincreasing functions f on $\mathbb{R}$ if and only if the weight w satisfies

$$\int_t^\infty \frac{w(x)}{x^q}\,dx \le \frac{B}{t^q}\int_0^t w(x)\,dx \tag{AM_q}$$

for some constant $B > 0$ and for all $t > 0$. Moreover, for $1 \le q < \infty$, condition (AM_q) implies that $\Lambda(w,q)$ is a Banach space. Sawyer [18] proved that the converse is true for $1 < q < \infty$. Raynaud gave also another equivalent condition by using quasiconcavity conditions for the function $W(t) = \int_0^t w(x)\,dx$ (see [17]).

In the case $q = \infty$ and w nondecreasing, the same arguments as for Lemma 2 show that $\Lambda(w,\infty)$ is a Banach space if and only if the weight w satisfies

$$(A_1) \qquad \frac{w(t)}{t}\int_0^t \frac{dx}{w(x)} \le C$$

for some constant $C > 0$ and for all $t > 0$.

If we suppose that the weights satisfy the conditions (AM_q) or (A_1) then

$$\|f^{**}\|_{\Lambda(w,q)} \sim \|f\|_{\Lambda(w,q)},$$

and conversely.

In the next statements, when we say that $\Lambda(w,q)$ is a Banach space we will mean that conditions (AM_q) or (A_1) are satisfied.

PROPOSITION 3. *Let X be a r.i. Banach space and suppose that $\Lambda(w,q)$ is a Banach space. Then:*

(i) *For $1 \le q < \infty$, $\Lambda(w,q) \in \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ if and only if*

$$(2.2) \qquad \int_t^\infty \frac{w(x)}{\phi_X(x)^q}\,dx \le \frac{B}{\phi_X(t)^q}\int_0^t w(x)\,dx, \qquad \forall t > 0.$$

(ii) *$\Lambda(w,\infty) \in \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ if and only if*

$$(2.3) \qquad \int_0^t \frac{d\phi_X(x)}{w(x)} \le C\frac{\phi_X(t)}{w(t)}, \qquad \forall t > 0.$$

(iii) *If $1 \le q < \infty$, then $\Lambda(w,q) \in \mathcal{LI}(X, M^*(X); L^\infty) \Leftrightarrow$ condition (2.2) holds.*

(iv) *$\Lambda(w,\infty) \in \mathcal{LI}(X, M^*(X); L^\infty)$ if and only if*

$$(2.4) \qquad \frac{w(t)}{\phi_X(t)}\left\|\frac{\chi_{[0,t]}}{w}\right\|_X \le C, \qquad \forall t > 0.$$

Proof. First of all we remark that condition (2.2) implies that w satisfies (AM_q).

(i) We only have to prove that the operator $Q_{\Lambda(X)}$ defined by

$$Q_{\Lambda(X)}f = H(f\circ\phi_X^{-1})\circ\phi_X$$

is bounded on $\Lambda(w,q)$ for nonnegative and nonincreasing functions. We have

$$\begin{aligned}\|Q_{\Lambda(X)}f\|^q_{\Lambda(w,q)} &= \int_0^\infty H(f\circ\phi_X^{-1})(\phi_X(t))^q w(t)\,dt\\ &= \int_0^\infty H(f\circ\phi_X^{-1})(y)^q w(\phi_X^{-1}(y))(\phi_X^{-1})'(y)\,dy\\ &\le C\int_0^\infty f(\phi_X^{-1}(y))^q w(\phi_X^{-1}(y))(\phi_X^{-1})'(y)\,dy\\ &= C\int_0^\infty f(x)^q w(x)\,dx = C\|f\|^q_{\Lambda(w,q)}\end{aligned}$$

where, by using condition (AM_q), the inequality is satisfied if and only if the weight

$$v(y) = w(\phi_X^{-1}(y))(\phi_X^{-1})'(y)$$

satisfies

$$\int_t^\infty \frac{v(y)}{y^q}\,dx \le \frac{B}{t^q}\int_0^t v(y)\,dy, \qquad \forall t > 0,$$

for some constant $B > 0$, and this inequality is equivalent to (2.2).

(ii) This proof is simpler. Suppose that (2.3) holds. If $f \in \Lambda(w,\infty)$ we have

$$\begin{aligned}\sup_{t>0}\frac{w(t)}{\phi_X(t)}\int_0^t f^*(x)\,d\phi_X(x) &\le \|f\|_{\Lambda(w,\infty)}\sup_{t>0}\frac{w(t)}{\phi_X(t)}\int_0^t\frac{d\phi_X(x)}{w(x)}\\ &\le C\|f\|_{\Lambda(w,\infty)}.\end{aligned}$$

For the converse implication, consider for each $t > 0$ the function $f^*(x) = w(x)^{-1}\chi_{[0,t]}(x) \in \Lambda(w,\infty)$. Then the inequality

$$\frac{w(t)}{\phi_X(t)}\int_0^t f^*(x)\,d\phi_X(x) \le C\sup_{t>0} f^*(t)w(t), \qquad \forall t > 0,$$

implies (2.3).

(iii) We know that $\Lambda(w,q) \in \mathcal{LI}(X, M^*(X); L^\infty) \Leftrightarrow$ the operator

$$Q_X(f)(t) = \frac{1}{\phi_X(t)}\|f\chi_{[0,t]}\|_X$$

is bounded in $\Lambda(w,q)$ for nonincreasing and nonnegative functions. Since $Q_X(f) \le Q_{\Lambda(X)}(f)$, we see that condition (2.2) implies the interpolation property. On the other hand, since

$$Q_X(\chi_{[0,s]})(t) = \chi_{[0,s]}(t) + \frac{\phi_X(s)}{\phi_X(t)}\chi_{[s,\infty)}$$

we deduce that if Q_X is bounded on $\Lambda(w,q)$, then

$$\int_0^s w(x)\,dx + \phi_X(s)^q \int_s^\infty \frac{w(x)}{\phi_X(x)^q}\,dx \le C \int_0^s w(x)\,dx$$

and hence (2.2) holds.

(iv) Suppose $\Lambda(w,\infty) \in \mathcal{LI}(X, M^*(X); L^\infty)$. By observing that $1/w \in \Lambda(w,\infty)$ and that

$$Q_X\left(\frac{1}{w}\right)(t) = \frac{1}{\phi_X(t)}\left\|\frac{\chi_{[0,t]}}{w}\right\|_X$$

we get condition (2.4).

For the converse, we know that $w(x)f^*(x) \le \|f\|_{\Lambda(w,\infty)}$, $\forall x > 0$ and $\forall f \in \Lambda(w,\infty)$. Furthermore,

$$\begin{aligned} Q_X(f^*)(t) &= \frac{1}{\phi_X(t)}\|f^*\chi_{[0,t]}\|_X \\ &\le \frac{C}{\phi_X(t)}\|f\|_{\Lambda(w,\infty)}\left\|\frac{\chi_{[0,t]}}{w}\right\|_X \le \frac{C}{w(t)}\|f\|_{\Lambda(w,\infty)} \end{aligned}$$

and since w is nondecreasing we conclude that Q_X is bounded on $\Lambda(w,\infty)$. ∎

Remarks. (i) Assertions (i) and (ii) of Proposition 3 should be compared with those appearing in [19]. In his paper Sharpley deals with the case of interpolation between $(\Lambda(X_1), M(X_1))$ and $(\Lambda(X_2), M(X_2))$ and characterizes when $\Lambda^\alpha(Y)$ or $M(Y)$ are interpolated spaces. Observe that $\Lambda^\alpha(Y)$ and $M(Y)$ are particular cases of $\Lambda(w,q)$.

(ii) We observe from Propositions 4 and 6 below that the spaces $\Lambda(w,q)$, $1 \le q < \infty$, belong to $\mathcal{LI}(X, M^*(X); L^\infty)$ and $\mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$ simultaneously. Furthermore, if $X = L^p$, $1 < p < q$, the weight w satisfies condition $(AM_{q/p})$ and then the space $\Lambda(w,q)$ is p-convex. The converse is also true, i.e., for $q > p \ge 1$, if $\Lambda(w,q)$ is p-convex then $(\Lambda(w,q), \Lambda(w,q)) \in \mathcal{LI}(L^p, L^{p,\infty}; L^\infty)$. This result should be compared with those appearing in [17].

(iii) If $\Lambda(X) \ne X = M^*(X)$ (e.g., $X = L^{p,\infty}$, $p > 1$) then it is easy to prove $(X,X) \in \mathcal{LI}(X, M^*(X); L^\infty)$ but $(X,X) \notin \mathcal{LI}(\Lambda(X), M^*(X); L^\infty)$.

(iv) $(\Lambda(w,\infty), \Lambda(w,\infty)) \in \mathcal{LI}(L^{p,1}, L^{p,\infty}; L^\infty) \Leftrightarrow \Lambda(w,\infty)$ is p-convex.

When $X = L^p$, $1 < p < \infty$, it would not be difficult to prove that for Y p-convex, $(Y,Y) \in \mathcal{LI}(L^p, L^{p,\infty}; L^\infty) \Leftrightarrow (Y,Y) \in \mathcal{LI}(L^{p,1}, L^{p,\infty}; L^\infty)$ (see [10] for definition of p-convex spaces), but actually we can prove a stronger result:

PROPOSITION 4. *Let $1 \le p < \infty$ and let Y be a r.i. space. The following assertions are equivalent:*

(i) $(Y,Y) \in \mathcal{LI}(L^{p,1}, L^{p,\infty}; L^\infty)$.

(ii) $(Y,Y) \in \mathcal{LI}(L^{p,r}, L^{p,\infty}; L^\infty)$ *for some* $1 < r < \infty$.

(iii) *The upper Boyd index* $\alpha(Y) < 1/p$. (See [4] for the definition of Boyd indices.)

Proof. We only have to prove (ii)⇒(i). We restrict ourselves to the case $r = p$ because the other cases are similar. By Proposition 3, we know that the operator

$$Q_p f(t) = \frac{1}{t^{1/p}}\Big(\int_0^t f^p\Big)^{1/p}$$

is bounded in Y for nonincreasing nonnegative functions, i.e. $\|Q_p f\|_Y \le C\|f\|_Y$ for some constant $C > 0$. Let $f = f^* \in Y$. It is quite easy to compute that

$$Q_p^{(n+1)} f(t) = \left(\int_0^1 f^p(tx)\frac{[\log(1/x)]^n}{n!}\,dx\right)^{1/p}$$

for any natural number $n \in \mathbb{N}$. If $\varepsilon C < 1$ we define

$$Sf(t) = \Big(\sum_{n=0}^\infty [\varepsilon^n Q_p^{(n+1)} f(t)]^p\Big)^{1/p}.$$

Since $S_N f(t) = (\sum_{n=0}^N [\varepsilon^n Q_p^{(n+1)} f(t)]^p)^{1/p} \in Y$, $S_N f(t) \uparrow Sf(t)$ as $N \to \infty$, and

$$\|S_N f\|_Y \le \Big\|\sum_{n=0}^N \varepsilon^n Q_p^{(n+1)} f\Big\|_Y \le \Big(\sum_{n=0}^\infty \varepsilon^n C^{n+1}\Big)\|f\|_Y$$

we find that $Sf \in Y$ and $\|Sf\|_Y \le C'\|f\|_Y$. But

$$\begin{aligned} Sf(t) &= \left(\sum_{n=0}^\infty \int_0^1 f(tx)^p \frac{[\varepsilon\log(1/x)]^n}{n!}\,dx\right)^{1/p} \\ &= \left(\int_0^1 f(tx)^p \sum_{n=0}^\infty \frac{[\varepsilon\log(1/x)]^n}{n!}\,dx\right)^{1/p} \\ &= \left(\int_0^1 f(tx)^p \frac{dx}{x^\varepsilon}\right)^{1/p} = \frac{1}{t^{(1-\varepsilon)/p}}\Big(\int_0^t f(x)^p\,d(x^{1-\varepsilon})\Big)^{1/p} \\ &= \frac{1}{t^{(1-\varepsilon)/p}}\|f\chi_{[0,t]}\|_{L^{p/(1-\varepsilon),p}}. \end{aligned}$$

Hence $(Y,Y) \in \mathcal{LI}(L^{p/(1-\varepsilon),p}, L^{p/(1-\varepsilon),\infty}; L^\infty)$ and so (i) is true. ∎

In the last part of this paper we consider a similar situation to the previous one in the framework of Lorentz–Orlicz spaces. Different versions of this class of spaces appear in [13], [21]; they have also been studied in [9], [14], [15] and [17]. Here we follow [13].

In the sequel φ will denote an *Orlicz function*, i.e. a convex, nondecreasing function on $[0,\infty)$ such that $\varphi(0)=0$ and $\lim_{t\to\infty}\varphi(t)=\infty$. We also suppose that φ satisfies the Δ_2 condition: there exists a constant $C>0$ such that $\varphi(2t)\le C\varphi(t)$ for all $t>0$, or equivalently, there exists $1\le q<\infty$ so that

$$\varphi(at)\le a^q\varphi(t),\qquad \forall a\ge 1,\ \forall t>0\,.$$

The weight w is supposed to satisfy the same conditions appearing in the definition of Lorentz spaces, namely, w is an a.e. positive weight defined on $[0,\infty)$ such that $\int_0^t w<\infty, \forall t<\infty$, and $\int_0^\infty w=\infty$.

The space $\Lambda(w,\varphi)$ is the class of real-valued measurable functions on I so that $\int_I \varphi(f^*(t))w(t)\,dt<\infty$.

The next results extend Ariño–Muckenhoupt's inequality.

LEMMA 3. *Let φ be an Orlicz function. Suppose that there exists a constant $B>0$ such that*

$$(A_\varphi)\qquad \int_t^\infty \varphi\Big(\frac{at}{x}\Big)w(x)\,dx\le B\varphi(a)\int_0^t w(x)\,dx\,,\qquad \forall t>0,\ \forall a>0\,.$$

Then, for some $\alpha<1$ and $D>0$, we have

$$\int_t^\infty \psi\Big(\frac{at}{x}\Big)w(x)\,dx\le D\psi(a)\int_0^t w(x)\,dx\,,\qquad \forall t>0,\ \forall a>0\,,$$

where $\psi(t)=\varphi(t^\alpha)$.

Proof. We can adapt the arguments of [1] to our more general situation. Only a few changes are necessary. Using the notation of [1] the number $\alpha<1$ is chosen in such a way that

$$2S^{1/q}=2\Big(\frac{2B+1}{2B+2}\Big)^{1/q}<2^\alpha<2\,. \blacksquare$$

As a consequence of this lemma and with the same argument as in [1], we obtain

PROPOSITION 5. *Let φ be an Orlicz function. A weight w satisfies condition (A_φ) if and only if there exists a constant $B'>0$ such that for every nonnegative nonincreasing function f on $(0,\infty)$ we have*

$$\int_0^\infty \varphi(H(f))w\le B'\int_0^\infty \varphi(f)w$$

where $H(f)$ is the Hardy transform of f.

Remark. If w satisfies (A_φ) then the expression

$$\|f\|_{\Lambda(w,\varphi)}=\inf\Big\{\varrho:\ \int_0^\infty \varphi(f^*/\varrho)w\le 1\Big\}$$

defines a quasinorm on $\Lambda(w,\varphi)$ which is equivalent to the norm

$$|||f|||=\|Hf^*\|_{\Lambda(w,\varphi)}$$

and therefore $\Lambda(w,\varphi)$ is a Banach space.

By introducing the Simonenko indices (see [13]) we can give necessary or sufficient conditions for a weight to satisfy condition (A_φ). Given an Orlicz convex function φ and a number $T>0$ we define

$$p_T=\inf_{t\ge T}\frac{t\varphi'(t)}{\varphi(t)}\,,\qquad q_T=\sup_{t\ge T}\frac{t\varphi'(t)}{\varphi(t)}$$

where $\varphi'(t)$ is the right derivative of the Orlicz function φ. We also introduce $p_0=\inf p_T$ and $q_0=\sup q_T$. It is clear that $1\le p_0\le p_T\le q_T\le q_0<\infty$ and

$$\alpha^{q_T}\varphi(t)\le\varphi(\alpha t)\le\alpha^{p_T}\varphi(t)$$

whenever $\alpha\le 1$ and $T\le\alpha t$.

PROPOSITION 6. *Let φ be an Orlicz function and let w be a weight. Then:*

(i) *If a weight w satisfies condition (AM_{p_0}) then it also satisfies (A_φ). Furthermore, the Hardy transform is bounded on $\Lambda(w,\varphi)$ and $\Lambda(w,\varphi)$ is a Banach space.*

(ii) *If the Hardy transform is bounded on $\Lambda(w,\varphi)$ then w satisfies condition (AM_{q_0}).*

Proof. (i) First of all suppose that w satisfies (AM_{p_0}). Then

$$\int_t^\infty \varphi\Big(\frac{at}{x}\Big)w(x)\,dx\le\varphi(a)\int_t^\infty\Big(\frac{t}{x}\Big)^{p_0}w(x)\,dx$$

and hence w satisfies (A_φ) and so the Hardy transform is bounded on $\Lambda(w,\varphi)$. The other statements are clear.

(ii) Now we assume that there exists a constant $C\ge 1$ such that

$$\|Hf\|_{\Lambda(w,\varphi)}\le C\|f\|_{\Lambda(w,\varphi)}$$

for all nonincreasing functions $f\in\Lambda(w,\varphi)$. In particular, given $t>0$ if $s=\|\chi_{[0,t]}\|^{-1}_{\Lambda(w,\varphi)}$, we obtain

$$\|H(\chi_{[0,t]})\|_{\Lambda(w,\varphi)}\le\frac{C}{s}\,.$$

Therefore

$$\int_0^t \varphi\left(\frac{s}{C}\right) w(x)\,dx + \int_t^\infty \varphi\left(\frac{st}{Cx}\right) w(x)\,dx \le 1\,.$$

Since $\varphi(s)/C^{q_0} \le \varphi(s/C)$ and

$$\varphi\left(\frac{st}{Cx}\right) \ge \varphi\left(\frac{s}{C}\right)\left(\frac{t}{x}\right)^{q_0} \ge \varphi(s)\left(\frac{t}{Cx}\right)^{q_0}$$

we have

$$\int_0^t w(x)\,dx + \int_t^\infty \left(\frac{t}{x}\right)^{q_0} w(x)\,dx \le C^{q_0} \int_0^t w(x)\,dx\,,$$

and this completes the proof. ∎

For the space $\Lambda(w,\varphi)$ on the unit interval $I=[0,1]$ (or more generally $I=[0,l]$ for $l<\infty$) we can give a more precise result.

Proposition 7. *Suppose $I=[0,l]$ ($l<\infty$) and let φ be an Orlicz function. Let*

$$p = \liminf_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)} \quad \textit{and} \quad q = \limsup_{t\to\infty} \frac{t\varphi'(t)}{\varphi(t)}\,.$$

Then:

(i) *If w satisfies condition (AM_p) then the Hardy transform is bounded on $\Lambda(w,\varphi)$ and $\Lambda(w,\varphi)$ is a Banach space.*

(ii) *If the Hardy transform is bounded on $\Lambda(w,\varphi)$ then w satisfies condition $(AM_{q+\varepsilon})$ for all $\varepsilon>0$.*

Proof. (i) If w satisfies (AM_p) then by Lemma 2.1 of [1], w also satisfies $(AM_{p-\varepsilon})$ for some $\varepsilon>0$. Then there exists $T>0$ such that $p-\varepsilon<p_T$. Define now the function $\overline{\varphi}_T$ by

$$\overline{\varphi}_T(t) = \begin{cases} \varphi(t) & \text{if } t\ge T,\\ \varphi(T)(t/T)^{p_T} & \text{otherwise.}\end{cases}$$

Note that $\overline{\varphi}_T$ is an Orlicz function for which

$$p_T = \inf_{t>0} \frac{t\overline{\varphi}'(t)}{\overline{\varphi}(t)}\,.$$

Hence by the preceding proposition the Hardy transform is bounded on $\Lambda(w,\overline{\varphi}_T)$. Thus it is also bounded on $\Lambda(w,\varphi)$, as $m(I)<\infty$ and the Orlicz functions are equivalent at infinity.

(ii) Let ε be a positive number. There exists $T>0$ such that

$$\frac{t\varphi'(t)}{\varphi(t)} \le q_T < q+\varepsilon\,, \quad \forall t\ge T\,.$$

If we define the Orlicz function $\overline{\varphi}_T$ as before it is clear that $\sup_{t>0} t\overline{\varphi}'(t)/\overline{\varphi}(t) \le q_T$. Since $\overline{\varphi}_T$ is equivalent to φ at infinity and the Hardy transform is bounded on $\Lambda(w,\overline{\varphi}_T)$, by using again Proposition 6, we deduce that w satisfies (AM_{q_T}) and so $(AM_{q+\varepsilon})$. This completes the proof. ∎

Now we can state the corresponding interpolation results whose proofs follow the same lines as in Proposition 3.

Proposition 8. *Suppose that φ is an Orlicz function and w satisfies condition (A_φ). Let X be a r.i. space. Then the following assertions are equivalent:*

(i) *Any linear or quasilinear operator T which is bounded from $\Lambda(X)$ into $M^*(X)$ and from L^∞ into L^∞ satisfies*

$$\int_0^\infty \varphi((Tf)^*(x))w(x)\,dx \le C' \int_0^\infty \varphi(f^*(x))w(x)\,dx$$

for some constant $C'>0$ and for any function $f\in\Lambda(w,\varphi)$.

(ii) *The same as in* (i) *but with operators mapping X (instead of $\Lambda(X)$) into $M^*(X)$.*

(iii) *There exists a constant $D>0$ such that*

$$\int_t^\infty \varphi\left(\frac{\phi_X(t)}{\phi_X(x)}\right) w(x)\,dx \le D \int_0^t w(x)\,dx\,.$$

Moreover, if one of these conditions is satisfied for $X=L^p$, $1\le p<\infty$, then the space $\Lambda(w,\varphi)$ is p-convex.

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References

[1] M. A. Ariño and B. Muckenhoupt, *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. 320 (2) (1990), 727–735.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[3] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, 1976.

[4] D. W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math. 21 (1969), 1245–1254.

[5] A. P. Calderón, *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz*, Studia Math. 26 (1966), 273–299.

[6] M. Cwikel, *K-divisibility of the K-functional and Calderón couples*, Ark. Mat. 22 (1) (1984), 39–62.

[7] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. Stud. 116, North-Holland, Amsterdam 1985.

[8] N. J. Kalton, *Endomorphisms of symmetric function spaces*, Indiana Univ. Math. J. 34 (2) (1985), 225–247.
[9] A. Kamińska, *Some remarks on Orlicz–Lorentz spaces*, Math. Nachr. 147 (1990), 29–38.
[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.
[11] G. Lorentz and T. Shimogaki, *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. 139 (1971), 207–221.
[12] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. 71 (1981), 69–83.
[13] —, *Indices and interpolation*, Dissertationes Math. 234 (1985).
[14] M. Mastyło, *Interpolation of linear operators in Calderón–Lozanovskii spaces*, Comment. Math. 26 (2) (1986), 247–256.
[15] S. J. Montgomery-Smith, *Comparison of Orlicz–Lorentz spaces*, Studia Math. 103 (1992), 161–189.
[16] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
[17] Y. Raynaud, *On Lorentz–Sharpley spaces*, in: Proc. Workshop "Interpolation Spaces and Related Topics", Haifa, June 1990, IMCP, Vol. 5 (1992), 207–228.
[18] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.
[19] R. Sharpley, *Spaces* $\Lambda_\alpha(X)$ *and interpolation*, J. Funct. Anal. 11 (1972), 479–513.
[20] T. Shimogaki, *An interpolation theorem on Banach function spaces*, Studia Math. 31 (1968), 233–240.
[21] A. Torchinsky, *Interpolation of operators and Orlicz classes*, ibid. 59 (1976), 177–207.
[22] M. Zippin, *Interpolation of operators of weak type between rearrangement invariant function spaces*, J. Funct. Anal. 7 (1971), 267–284.

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Isomorphism of certain weak L^p spaces

by

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Abstract. It is shown that the weak L^p spaces $\ell^{p,\infty}$, $L^{p,\infty}[0,1]$, and $L^{p,\infty}[0,\infty)$ are isomorphic as Banach spaces.

Introduction. The Lorentz spaces play an important role in interpolation theory. They also form a class of Banach spaces generalizing the classical L^p spaces. In this paper, we continue the comparison of the Banach space structures of various weak L^p spaces begun in [2] and [3]. In [2], mimicking the construction of the Rademacher functions, it was shown that ℓ^2 can be embedded complementably into $\ell^{p,\infty}$. In [3], we showed that $\ell^{p,\infty}$ can in turn be embedded complementably (even as a sublattice) into $L^{p,\infty}[0,1]$. Here, we complete and extend these results by showing that, in fact, the three weak L^p spaces $\ell^{p,\infty}$, $L^{p,\infty}[0,1]$, and $L^{p,\infty}[0,\infty)$ are isomorphic as Banach spaces. This question was also mentioned in [1].

We start by recalling some standard definitions. Let (Ω,Σ,μ) be an arbitrary measure space. For $1 < p < \infty$, the *weak L^p space* $L^{p,\infty}(\Omega,\Sigma,\mu)$ is the space of all Σ-measurable functions f such that $\{\omega : |f(\omega)| > 0\}$ is σ-finite and

$$\|f\| \equiv \sup_B \frac{\int_B |f|\,d\mu}{\mu(B)^{1/q}} < \infty,$$

where $q = p/(p-1)$, and the supremum is taken over all measurable sets B with $0 < \mu(B) < \infty$. If (Ω,Σ,μ) is a real interval I endowed with Lebesgue measure, we write $L^{p,\infty}(I)$; while $\ell^{p,\infty}$ and $\ell^{p,\infty}(m)$ will stand for the weak L^p spaces on $\mathbb{N}$ and $\{1,\ldots,m\}$ respectively, both with the counting measure.

For a real-valued function defined on (Ω,Σ,μ), let f^* denote the decreasing rearrangement of $|f|$ [5]; similarly for (a_n^*), where (a_n) is a real sequence. It is well known that the expression

$$|||f||| \equiv \sup_{t>0} t^{1/p} f^*(t)$$

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