[8] N. J. Kalton, *Endomorphisms of symmetric function spaces*, Indiana Univ. Math. J. 34 (2) (1985), 225–247.
[9] A. Kamińska, *Some remarks on Orlicz-Lorentz spaces*, Math. Nachr. 147 (1990), 29–38.
[10] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.
[11] G. Lorentz and T. Shimogaki, *Interpolation theorems for the pairs of spaces* (L^p, L^∞) *and* (L^1, L^q), Trans. Amer. Math. Soc. 139 (1971), 207–221.
[12] L. Maligranda, *A generalization of the Shimogaki theorem*, Studia Math. 71 (1981), 69–83.
[13] —, *Indices and interpolation*, Dissertationes Math. 234 (1985).
[14] M. Mastyło, *Interpolation of linear operators in Calderón–Lozanovskii spaces*, Comment. Math. 26 (2) (1986), 247–256.
[15] S. J. Montgomery-Smith, *Comparison of Orlicz–Lorentz spaces*, Studia Math. 103 (1992), 161–189.
[16] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207–226.
[17] Y. Raynaud, *On Lorentz–Sharpley spaces*, in: Proc. Workshop "Interpolation Spaces and Related Topics", Haifa, June 1990, IMCP, Vol. 5 (1992), 207–228.
[18] E. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. 96 (1990), 145–158.
[19] R. Sharpley, *Spaces* $\Lambda_\alpha(X)$ *and interpolation*, J. Funct. Anal. 11 (1972), 479–513.
[20] T. Shimogaki, *An interpolation theorem on Banach function spaces*, Studia Math. 31 (1968), 233–240.
[21] A. Torchinsky, *Interpolation of operators and Orlicz classes*, ibid. 59 (1976), 177–207.
[22] M. Zippin, *Interpolation of operators of weak type between rearrangement invariant function spaces*, J. Funct. Anal. 7 (1971), 267–284.

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Isomorphism of certain weak L^p spaces

by

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Abstract. It is shown that the weak L^p spaces $\ell^{p,\infty}$, $L^{p,\infty}[0,1]$, and $L^{p,\infty}[0,\infty)$ are isomorphic as Banach spaces.

Introduction. The Lorentz spaces play an important role in interpolation theory. They also form a class of Banach spaces generalizing the classical L^p spaces. In this paper, we continue the comparison of the Banach space structures of various weak L^p spaces begun in [2] and [3]. In [2], mimicking the construction of the Rademacher functions, it was shown that ℓ^2 can be embedded complementably into $\ell^{p,\infty}$. In [3], we showed that $\ell^{p,\infty}$ can in turn be embedded complementably (even as a sublattice) into $L^{p,\infty}[0,1]$. Here, we complete and extend these results by showing that, in fact, the three weak L^p spaces $\ell^{p,\infty}$, $L^{p,\infty}[0,1]$, and $L^{p,\infty}[0,\infty)$ are isomorphic as Banach spaces. This question was also mentioned in [1].

We start by recalling some standard definitions. Let (Ω,Σ,μ) be an arbitrary measure space. For $1<p<\infty$, the *weak* L^p *space* $L^{p,\infty}(\Omega,\Sigma,\mu)$ is the space of all Σ-measurable functions f such that $\{\omega : |f(\omega)|>0\}$ is σ-finite and

$$\|f\| \equiv \sup_B \frac{\int_B |f|\,d\mu}{\mu(B)^{1/q}} < \infty,$$

where $q=p/(p-1)$, and the supremum is taken over all measurable sets B with $0<\mu(B)<\infty$. If (Ω,Σ,μ) is a real interval I endowed with Lebesgue measure, we write $L^{p,\infty}(I)$; while $\ell^{p,\infty}$ and $\ell^{p,\infty}(m)$ will stand for the weak L^p spaces on $\mathbb{N}$ and $\{1,\ldots,m\}$ respectively, both with the counting measure.

For a real-valued function defined on (Ω,Σ,μ), let f^* denote the decreasing rearrangement of $|f|$ [5]; similarly for (a_n^*), where (a_n) is a real sequence. It is well known that the expression

$$|||f||| \equiv \sup_{t>0} t^{1/p} f^*(t)$$

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satisfies $|||f||| \le \|f\| \le q|||f|||$. Let $L^{q,1}(\Omega,\Sigma,\mu)$ denote the space of all measurable functions f such that $\|f\|_{q,1} = \int_0^\infty t^{-1/p} f^*(t)\,dt < \infty$. Then $L^{p,\infty}(\Omega,\Sigma,\mu)$ is naturally isomorphic to the dual of $L^{q,1}(\Omega,\Sigma,\mu)$, where the isomorphism constant depends only on p. Finally, we note that any weak L^p space satisfies an upper p-estimate with constant 1 [5].

2. Main theorem. Fix $1 < p < \infty$ and $q = p/(p-1)$; our goal is to prove the following

THEOREM 1. *The Banach spaces $\ell^{p,\infty}, L^{p,\infty}[0,1]$, and $L^{p,\infty}[0,\infty)$ are isomorphic.*

Since the proof of the theorem goes through several intermediate embeddings and is rather circuitous, we first give an outline of the procedure. We start by recalling the following well known variant of Pełczyński's "decomposition method" [4]. The *square* of a Banach space E is the Banach space $E \oplus E$.

THEOREM 2. *Let E, F be Banach spaces which are isomorphic to their squares. Suppose that each is isomorphic to a complemented subspace of the other. Then they are isomorphic.*

Proof. Using the symbol "$\sim$" for "is isomorphic to", we find Banach spaces G and H such that $E \sim F \oplus G$ and $F \sim E \oplus H$. Then

$$E \oplus F \sim E \oplus (E \oplus H) \sim E \oplus H \sim F, \quad \text{and}$$
$$E \oplus F \sim (G \oplus F) \oplus F \sim G \oplus F \sim E\,. \ \blacksquare$$

It has already been mentioned that $\ell^{p,\infty}$ embeds complementably into $L^{p,\infty}[0,1]$ [3], and it is clear that $L^{p,\infty}[0,1]$ embeds complementably into $L^{p,\infty}[0,\infty)$. Since these spaces are obviously isomorphic to their squares, the proof of Theorem 1 will be complete if we can show that $L^{p,\infty}[0,\infty)$ embeds complementably into $\ell^{p,\infty}$. This we do in a number of steps.

For any $k \in \mathbb{N}$, let $X_k = (\sum_{n=0}^\infty \oplus \ell^{p,\infty}(k\cdot 2^n))_{\ell^\infty}$. Then, using "$\overset{c}{\hookrightarrow}$" to denote "embeds complementably into", we will show that

$$\begin{aligned} L^{p,\infty}[0,\infty) &\overset{c}{\hookrightarrow} \Big(\sum_{k=1}^\infty \oplus L^{p,\infty}[0,k]\Big)_{\ell^\infty} \\ &\overset{c}{\hookrightarrow} \Big(\sum_{k=1}^\infty \oplus X_k\Big)_{\ell^\infty} \\ &\overset{c}{\hookrightarrow} X_1 \overset{c}{\hookrightarrow} \ell^{p,\infty}\,. \end{aligned}$$

The third embedding in this chain is obvious, and the first is also straightforward. The second embedding is accomplished by showing that $L^{p,\infty}[0,k] \overset{c}{\hookrightarrow} X_k$ with uniform constants. This relies on the techniques of [2]. For the last link in the chain, we show that X_1 is isomorphic to a weak* closed subspace of $\ell^{p,\infty}$ generated by long blocks with constant coefficients. The complementation is then effected by a conditional expectation operator [5].

3. Finding $L^{p,\infty}[0,k]$ in X_k

LEMMA 3. *Let $k \in \mathbb{N}$ and let $f \in L^{p,\infty}[0,\infty)$. Then*

$$\sup_n 2^{n/q} \Big|\Big|\Big| \Big(\int_{(j-1)/2^n}^{j/2^n} f \Big)_{j=1}^{k\cdot 2^n} \Big|\Big|\Big| \le \|f\|\,.$$

Proof. Fix n. Since the expressions involved are rearrangement invariant, we may assume that $|\int_{(j-1)/2^n}^{j/2^n} f|$ decreases with j. Let $a_j = j^{1/p}|\int_{(j-1)/2^n}^{j/2^n} f|$. Then for $1 \le j \le k\cdot 2^n$,

$$\int_0^{j/2^n} |f| \ge \sum_{i=1}^{j} \Big| \int_{(i-1)/2^n}^{i/2^n} f \Big| \ge j^{1/q} a_j\,.$$

Hence

$$\|f\| \ge (j/2^n)^{-1/q} \int_0^{j/2^n} |f| \ge 2^{n/q} a_j\,.$$

Taking the supremum over j finishes the proof. $\blacksquare$

We introduce some more notation. An element $x \in X_k$ will be written as $x = (x_n)_{n=0}^\infty$, where each $x_n \in \ell^{p,\infty}(k\cdot 2^n)$. Each x_n is in turn a finite real sequence $(x_n(j))_{j=1}^{k\cdot 2^n}$. Define

$$Y_k = \{x \in X_k : x_n(j) = 2^{-1/q}(x_{n+1}(2j-1) + x_{n+1}(2j)),\ 1 \le j \le k\cdot 2^n,\ n \ge 0\}\,.$$

We will say that a linear operator T mapping between Banach spaces is a K-*isomorphism* (into) if $K^{-1}\|x\| \le \|Tx\| \le K\|x\|$.

PROPOSITION 4. *Define $T_k : L^{p,\infty}[0,k] \to X_k$ by $T_k f = x$, with $x_n(j) = 2^{n/q} \int_{(j-1)/2^n}^{j/2^n} f$ for $1 \le j \le k\cdot 2^n, n \ge 0$. Then T_k is a q-isomorphism of $L^{p,\infty}[0,k]$ onto Y_k.*

Proof. Fix $f \in L^{p,\infty}[0,k]$ and let $x = T_k f$. By Lemma 3, $\sup_n |||x_n||| \le \|f\|$. Thus

$$\|T_k f\| = \sup_n \|x_n\| \le q \sup_n |||x_n||| \le q\|f\|\,.$$

On the other hand, letting E_n denote the conditional expectation operator [5] with respect to the partition $([(j-1)/2^n, j/2^n])_{j=1}^{k\cdot 2^n}$, it is easy to see that

$|||x_n||| = |||E_n f|||$. Now since $E_n f \to f$ in the weak* topology, we have

$$\begin{aligned}\|f\| &\le \limsup \|E_n f\| \le q \limsup |||E_n f|||\\ &= q \limsup |||x_n||| \le q \limsup \|x_n\| \le q\|T_k f\|.\end{aligned}$$

This proves that T_k is a q-isomorphism. Clearly, T_k maps into Y_k. Conversely, for $x \in Y_k$, let $f_n = 2^{n/p}\sum_{j=1}^{k\cdot 2^n} x_n(j)\chi_{n,j}$, where $\chi_{n,j}$ is the characteristic function of the interval $[(j-1)/2^n, j/2^n]$. Then $|||f_n||| = |||x_n|||$. In particular, (f_n) is a bounded sequence in $L^{p,\infty}[0,k]$. Now if f is a weak* cluster point of the sequence (f_n), then, using the fact that $x \in Y_k$, it is easy to see that $T_k f = x$. Hence T_k maps onto Y_k, as required. ∎

We proceed to show that Y_k is complemented in X_k. Fix $k \in \mathbb{N}$ and let

$$Z_k = \Big\{x \in X_k : \text{there exists } i \text{ such that } \sum_{j=1}^{k\cdot 2^n} x_n(j) = 0 \text{ for all } n \ge i\Big\}.$$

It is clear that Z_k is a linear subspace of X_k. Also define $u \in X_k$ so that $u_n(j) = (k\cdot 2^n)^{-1/p}, 1 \le j \le k\cdot 2^n, n \ge 0$.

LEMMA 5. *Define $\phi : \operatorname{span}\{Z_k, \{u\}\} \to \mathbb{R}$ by*

$$\phi(z + au) = a \quad \text{for all } z \in Z_k \text{ and } a \in \mathbb{R}.$$

Then $\|\phi\| \le 1$ with respect to the norm on X_k.

Proof. For all $n \ge 0$, the functional x'_n on X_k given by $x'_n(x) = \sum_{j=1}^{k\cdot 2^n} x_n(j)$ has norm $\le (k\cdot 2^n)^{1/q}$. Now $z \in Z_k$ implies $x'_n(z) = 0$ for all large n. Thus

$$\begin{aligned}&\|z + au\| \le 1\\ \Rightarrow\ &|x'_n(z+au)| \le (k\cdot 2^n)^{1/q}\\ \Rightarrow\ &|a| x'_n(u) \le (k\cdot 2^n)^{1/q}\\ \Rightarrow\ &|a| \le 1. \ ∎\end{aligned}$$

Let Φ be a norm preserving extension of ϕ to all of X_k. For an element $x \in X_k$, the *support* of x, $\operatorname{supp} x$, is $\{(n,j) : x_n(j) \ne 0\}$. As it is clear that any finitely supported element of X_k is in Z_k, we must have $\Phi = 0$ on $(\sum_{n=0}^{\infty} \oplus \ell^{p,\infty}(k\cdot 2^n))_{c_0}$.

If A is a subset of $\Gamma = \{(n,j) : 1 \le j \le k\cdot 2^n, n \ge 0\}$, the operator on X_k given by multiplication by the characteristic function of A, which we denote by χ_A, is a norm one projection on X_k. For any $(n,j) \in \Gamma$, let $A_{n,j} = \operatorname{supp} T_k\chi_{[(j-1)/2^n, j/2^n]}$. Then let $\Phi_{n,j} = \chi'_{A_{n,j}}\Phi$.

LEMMA 6. *For every $n \ge 0$, $(\Phi_{n,j})_{j=1}^{k\cdot 2^n}$ is a sequence of pairwise disjoint (in the lattice sense) functionals on X_k. Moreover, $\Phi_{n+1,2j-1} + \Phi_{n+1,2j} = \Phi_{n,j}$ for all $n \ge 0$, $1 \le j \le k\cdot 2^n$.*

Proof. Note that $A_{n+1,2j-1} \cup A_{n+1,2j} = A_{n,j}$. Then both assertions follow easily from the fact that $A_{n,j} \cap A_{n,j'}$ is finite for $1 \le j \ne j' \le k\cdot 2^n$, and Φ is 0 on finitely supported elements. ∎

Given $n \ge 0$, $1 \le j \le k\cdot 2^n$, define $S_{n,j} : X_k \to X_k$ by $S_{n,j}x = z$, where $z_m = 0$ for $0 \le m < n$, and $z_m(i) = x_m(i + (j-1)2^{m-n} \bmod (k\cdot 2^m))$ for $m \ge n$, $1 \le i \le k\cdot 2^m$. It is clear that $S_{n,j}$ has norm 1, and that $x - S_{n,j}x \in Z_k$.

LEMMA 7. *For $n \ge 0$, $1 \le j \le k\cdot 2^n$, $\|\Phi_{n,j}\| = \|\Phi_{n,1}\|$.*

Proof. For any $x \in X_k$,

$$\Phi_{n,j}(x) = \Phi(\chi_{A_{n,j}}x) = \Phi(S_{n,j}\chi_{A_{n,j}}x)$$

since $\Phi = 0$ on Z_k. But the support of $S_{n,j}\chi_{A_{n,j}}x$ is contained in $A_{n,1}$. Therefore,

$$\Phi_{n,j}(x) = \Phi_{n,1}(S_{n,j}\chi_{A_{n,j}}x) \le \|\Phi_{n,1}\|\|x\|$$

since both $S_{n,j}$ and $\chi_{A_{n,j}}$ are norm 1 operators. Thus $\|\Phi_{n,j}\| \le \|\Phi_{n,1}\|$. The reverse inequality can be obtained similarly. ∎

LEMMA 8. *For all $n \ge 0$, $\|\Phi_{n,1}\| \le (k\cdot 2^n)^{-1/q}\|\Phi\|$.*

Proof. Since $\ell^{p,\infty}$ satisfies an upper p-estimate with constant 1, so does X_k. Hence X'_k satisfies a lower q-estimate with constant 1. By Lemma 6, $\Phi = \sum_{j=1}^{k\cdot 2^n} \Phi_{n,j}$ and the summands are pairwise disjoint. Then by Lemma 7,

$$\|\Phi\| \ge \Big(\sum_{j=1}^{k\cdot 2^n} \|\Phi_{n,j}\|^q\Big)^{1/q} = (k\cdot 2^n)^{1/q}\|\Phi_{n,1}\|. \ ∎$$

LEMMA 9. *Given real numbers $b_1, \ldots, b_i$ and a sequence (c_j) in the unit ball of $\ell^{p,\infty}$,*

$$\Big\|\Big(\sum_{j=1}^{i} b_j c_{li+j}\Big)_{l=0}^{\infty}\Big\|_{\ell^{p,\infty}} \le q^2 \sum_{j=1}^{i} b_j^*(j^{1/q} - (j-1)^{1/q}).$$

Proof. Let

$$K = \Big\|\Big(\sum_{j=1}^{i} b_j c_{li+j}\Big)_{l=0}^{\infty}\Big\|_{\ell^{p,\infty}}.$$

By rearranging (c_j), we may assume without loss of generality that $|\sum_{j=1}^{i} b_j c_{li+j}|$ is a decreasing function of l. Recall that $|||x||| \ge \|x\|/q$ for all $x \in \ell^{p,\infty}$. Thus, given $\varepsilon > 0$, there exists r such that

$$(r+1)^{1/p}\Big|\sum_{j=1}^{i} b_j c_{ri+j}\Big| > (K - \varepsilon)/q.$$

Then, since $c_j^* \le j^{-1/p}$ for all $j \ge 1$,

$$\frac{(K-\varepsilon)(r+1)^{1/q}}{q} \le \sum_{l=0}^{r}\Big|\sum_{j=1}^{i} b_j c_{li+j}\Big| \le \sum_{j=1}^{i}\sum_{l=0}^{r}|b_j c_{li+j}|$$
$$\le \sum_{j=1}^{i} b_j^* \sum_{l=0}^{r}((j-1)(r+1)+l+1)^{-1/p}$$
$$\le q(r+1)^{1/q}\sum_{j=1}^{i} b_j^*(j^{1/q}-(j-1)^{1/q}).$$

Multiplying by $q(r+1)^{-1/q}$ finishes the proof. ∎

LEMMA 10. *For any real sequence (a_j),*

$$\Big\|\sum_{j=1}^{k\cdot 2^n} a_j\Phi_{n,j}\Big\| \le qk^{-1/q}\Big\|\sum_{j=1}^{k\cdot 2^n} a_j\chi_{[(j-1)/2^n,\,j/2^n]}\Big\|_{q,1},$$

where $\|\cdot\|_{q,1}$ denotes the norm on $L^{q,1}[0,k]$.

Proof. Fix $x \in X_k$ with norm ≤ 1; then

$$\Big\langle x, \sum_{j=1}^{k\cdot 2^n} a_j\Phi_{n,j}\Big\rangle = \sum_{j=1}^{k\cdot 2^n} a_j\langle \chi_{A_{n,j}}x, \Phi\rangle$$
$$= \sum_{j=1}^{k\cdot 2^n} a_j\langle S_{n,j}\chi_{A_{n,j}}x, \Phi\rangle = \langle y, \Phi_{n,1}\rangle,$$

where $y = \sum_{j=1}^{k\cdot 2^n} a_j S_{n,j}\chi_{A_{n,j}}x$. By Lemma 9,

$$\|y_m\| \le q^2\sum_{j=1}^{k\cdot 2^n} a_j^*(j^{1/q}-(j-1)^{1/q})$$

for all $m \ge 0$. Hence

$$\Big\langle x, \sum_{j=1}^{k\cdot 2^n} a_j\Phi_{n,j}\Big\rangle \le \|y\|\,\|\Phi_{n,1}\| \le q^2\|\Phi\|\sum_{j=1}^{k\cdot 2^n} a_j^*\frac{j^{1/q}-(j-1)^{1/q}}{(k\cdot 2^n)^{1/q}}$$
$$\le qk^{-1/q}\Big\|\sum_{j=1}^{k\cdot 2^n} a_j\chi_{[(j-1)/2^n,\,j/2^n]}\Big\|_{q,1}.$$

Here the second inequality follows from Lemma 8, while the third inequality is true because $\|\Phi\| \le 1$. ∎

PROPOSITION 11. *Define $P_k : X_k \to X_k$ by $P_k x = y$, where*

$$y_n(j) = (k\cdot 2^n)^{1/q}\langle x, \Phi_{n,j}\rangle$$

for all $n \ge 0$, $1 \le j \le k\cdot 2^n$. Then P_k is a projection from X_k onto Y_k of norm $\le q^2$.

Proof. Since $(\ell^{q,1})' = \ell^{p,\infty}$ isomorphically, there is a constant K (actually q suffices) such that

$$\|(a_j)\| = K\sup_{(b_j)\in U}\sum a_j b_j$$

for all $(a_j) \in \ell^{p,\infty}$, where U denotes the unit ball of $\ell^{q,1}$. Thus, given $x \in X_k$ and $n \ge 0$,

$$(k\cdot 2^n)^{1/q}\|(\langle x, \Phi_{n,j}\rangle)_{j=1}^{k\cdot 2^n}\| \le q(k\cdot 2^n)^{1/q}\sup_{(b_j)\in U}\sum b_j\langle x, \Phi_{n,j}\rangle$$
$$\le q(k\cdot 2^n)^{1/q}\sup_{(b_j)\in U}\|x\|\Big\|\sum b_j\Phi_{n,j}\Big\|$$
$$\le q^2 2^{n/q}\|x\|\sup_{(b_j)\in U}\Big\|\sum_{j=1}^{k\cdot 2^n} b_j\chi_{[(j-1)/2^n,\,j/2^n]}\Big\|_{q,1} \quad \text{by Lemma 10}$$
$$= q^2\|x\|.$$

Hence $\|P_k\| \le q^2$. Using Lemma 6, it is easy to see that P_k maps into Y_k. Conversely, let $y \in Y_k$. Then for $n \ge 0$, $1 \le j \le k\cdot 2^n$,

$$y_n(j)(k\cdot 2^n)^{1/p}\chi_{A_{n,j}}u - \chi_{A_{n,j}}y \in Z_k.$$

Note also that $S_{n,j}\chi_{A_{n,j}}u - \chi_{A_{n,1}}u \in Z_k$. Hence for $1 \le j \le k\cdot 2^n$,

$$\Phi(\chi_{A_{n,j}}u) = \Phi(S_{n,j}\chi_{A_{n,j}}u) = \Phi(\chi_{A_{n,1}}u).$$

Thus

$$1 = \Phi(u) = \sum_{j=1}^{k\cdot 2^n}\Phi(\chi_{A_{n,j}}u) = k\cdot 2^n\,\Phi(\chi_{A_{n,1}}u).$$

Therefore,

$$\Phi_{n,j}(y) = \Phi(\chi_{A_{n,j}}y) = y_n(j)(k\cdot 2^n)^{1/p}\Phi(\chi_{A_{n,j}}u)$$
$$= y_n(j)(k\cdot 2^n)^{1/p}\Phi(\chi_{A_{n,1}}u) = \frac{y_n(j)}{(k\cdot 2^n)^{1/q}}.$$

Thus $P_k y = y$ for all $y \in Y_k$, as required. ∎

Propositions 4 and 11 combine to give

PROPOSITION 12. *For all $k \in \mathbb{N}$, $L^{p,\infty}[0,k]$ is q-isomorphic to a subspace of X_k which is complemented in X_k by a projection of norm $\le q^2$.*

The following theorem, the main goal of this section, now follows easily.

THEOREM 13. *The space* $(\sum_{k=1}^{\infty} \oplus L^{p,\infty}[0,k])_{\ell^\infty}$ *is isomorphic to a complemented subspace of* $(\sum_{k=1}^{\infty} \oplus X_k)_{\ell^\infty}$.

4. Embedding X_1 into $\ell^{p,\infty}$ complementably. Choose a strictly increasing sequence of integers $(m_n)_{n=0}^{\infty}$ so that $m_0 = 1$ and

$$m_n \ge \sum_{j=1}^{n-1} 2^j m_j \quad \text{for all } n \ge 1\,.$$

Then choose a pairwise disjoint sequence of subsets of $\mathbb{N}$, $(B_{n,j})_{j=1}^{2^n}{}_{n=0}^{\infty}$, so that $|B_{n,j}| = m_n$, where $|B|$ is the cardinality of the set B. Finally, let $w_{n,j} = \chi_{B_{n,j}}$ for $1 \le j \le 2^n$, $n \ge 0$. If W is the weak* closed subspace of $\ell^{p,\infty}$ generated by $(w_{n,j})$, i.e.,

$$W = \{(a_i) \in \ell^{p,\infty} : (a_i) \text{ is constant on each } B_{n,j}\}\,,$$

then we will show that X_1 is isomorphic to W. Since W is complemented in $\ell^{p,\infty}$ by the conditional expectation operator with respect to the σ-algebra generated by $(B_{n,j})$, we will have proved

THEOREM 14. X_1 *embeds complementably into* $\ell^{p,\infty}$.

To show the isomorphism between X_1 and W, define

$$Rx = \bigvee_{n=0}^{\infty} \bigvee_{j=1}^{2^n} \frac{x_n(j)}{m_n^{1/p}} w_{n,j}$$

for all $x \in X_1$, where the suprema refer to the pointwise order on the vector lattice of all real sequences. We first show that R maps into $\ell^{p,\infty}$. Fix $x \in X_1$. Let A be the set of all ordered pairs (n,j) such that $1 \le j \le 2^n$ and

$$|m_n^{-1/p} x_n(j)| < |m_l^{-1/p} x_l(i)| \tag{1}$$

for some $l > n$, $1 \le i \le 2^l$. Since $\sup_j |m_n^{-1/p} x_n(j)| \le m_n^{-1/p} \|x\| \to 0$ as $n \to \infty$, for every $(n,j) \in A$, there exists $(l,i) \in A^c$ satisfying equation (1), with $l > n$. Let $\Psi : A \to A^c$ be a choice function such that $\Psi(n,j) = (l,i)$ satisfies (1) with respect to (n,j), and $l > n$. Let

$$C_{l,i} = \bigcup_{(n,j) \in \Psi^{-1}\{(l,i)\}} B_{n,j}$$

for all $(l,i) \in A^c$. Then

$$\sum_{(n,j)\in\Psi^{-1}\{(l,i)\}} \left|\frac{x_n(j)}{m_n^{1/p}}\right| w_{n,j} < \left|\frac{x_l(i)}{m_l^{1/p}}\right| \sum_{(n,j)\in\Psi^{-1}\{(l,i)\}} w_{n,j} = \left|\frac{x_l(i)}{m_l^{1/p}}\right| \chi_{C_{l,i}}\,.$$

Note that

$$|C_{l,i}| \le \sum_{n=0}^{l-1} \sum_{j=1}^{2^n} |B_{n,j}| = \sum_{n=0}^{l-1} \sum_{j=1}^{2^n} m_n \le m_l$$

by the choice of (m_n). It follows that the sequence $\bigvee_{(n,j)\in A} m_n^{-1/p} |x_n(j)| w_{n,j}$ can be rearranged so that it is $\le \bigvee_{(n,j)\notin A} m_n^{-1/p} |x_n(j)| w_{n,j}$ in the pointwise order. Since the norm of $\ell^{p,\infty}$ is rearrangement invariant, we have

$$\Big\| \bigvee_{(n,j)\in A} m_n^{-1/p} x_n(j) w_{n,j} \Big\| \le \Big\| \bigvee_{(n,j)\notin A} m_n^{-1/p} x_n(j) w_{n,j} \Big\|\,. \tag{2}$$

On the other hand, for all $n \ge 0$, let $(x_n^*(j))_{j=1}^{2^n}$ denote the decreasing rearrangement of $(|x_n(j)|)_{j=1}^{2^n}$. Then

$$\begin{aligned}
\Big\| \bigvee_{(n,j)\notin A} m_n^{-1/p} x_n(j) w_{n,j} \Big\| &\le q \Big|\Big|\Big| \bigvee_{(n,j)\notin A} m_n^{-1/p} x_n(j) w_{n,j} \Big|\Big|\Big| \\
&\le q \sup_{n,j} \frac{x_n^*(j)(jm_n + \sum_{i=1}^{n-1} 2^i m_i)^{1/p}}{m_n^{1/p}} \\
&\le q \sup_{n,j} \frac{x_n^*(j)\,((j+1)m_n)^{1/p}}{m_n^{1/p}} \\
&\le q\, 2^{1/p} \sup_{n,j} j^{1/p} x_n^*(j) = q 2^{1/p} \sup_n |||x_n||| \\
&\le q\, 2^{1/p} \sup_n \|x_n\| = q 2^{1/p} \|x\|\,.
\end{aligned}$$

Together with equation (2), this shows that R is bounded as a map into $\ell^{p,\infty}$. On the other hand, for all $x \in X_1$,

$$\|Rx\| \ge \sup_n \Big\| \sum_{j=1}^{2^n} \frac{x_n(j)}{m_n^{1/p}} w_{n,j} \Big\| = \|x\|\,.$$

Therefore, R is an embedding, as claimed.

5. Proof of the main theorem

PROPOSITION 15. $L^{p,\infty}[0,\infty)$ *embeds complementably into* $(\sum_{k=1}^{\infty} \oplus L^{p,\infty}[0,k])_{\ell^\infty}$.

Proof. This is rather straightforward. Define $S : L^{p,\infty}[0,\infty) \to (\sum_{k=1}^{\infty} \oplus L^{p,\infty}[0,k])_{\ell^\infty}$ by

$$Sf = (f\chi_{[0,1]}, f\chi_{[0,2]}, \ldots)$$

for all $f \in L^{p,\infty}[0,\infty)$. Clearly, S is an isometric embedding. Now choose a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and regard $L^{p,\infty}[0,k]$ as the subspace of $L^{p,\infty}[0,\infty)$ consisting of all functions supported on $[0,k]$. Since $L^{p,\infty}[0,\infty)$ is the dual of $L^{q,1}[0,\infty)$, its unit ball is weak* compact. Given $g = (g_k) \in (\sum_{k=1}^{\infty} \oplus L^{p,\infty}[0,k])_{\ell^\infty}$, let

$$Qg = (w^*) \lim_{k\to\mathcal{U}} g_k .$$

It is easy to see that Q is bounded as a map into $L^{p,\infty}[0,\infty)$, and that $Q\circ S$ is the identity on $L^{p,\infty}[0,\infty)$. This proves the proposition. ∎

The proof of Theorem 1 now follows as in the discussion in §2, using Theorems 13, 14, and Proposition 15 above.

References

[1] N. L. Carothers and S. J. Dilworth, *Subspaces of $L_{p,q}$*, Proc. Amer. Math. Soc. 104 (1988), 537–545.

[2] D. H. Leung, *$\ell^{p,\infty}$ has a complemented subspace isomorphic to ℓ^2*, Rocky Mountain J. Math. 22 (1992), 943–952.

[3] —, *Embedding $\ell^{p,\infty}$ into $L^{p,\infty}[0,1]$ complementably*, Bull. London Math. Soc. 23 (1991), 583–586.

[4] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer, Berlin 1977.

[5] —,—, *Classical Banach Spaces II. Function Spaces*, Springer, Berlin 1979.

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A Carlson type inequality with blocks and interpolation

by

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Abstract. An inequality, which generalizes and unifies some recently proved Carlson type inequalities, is proved. The inequality contains a certain number of "blocks" and it is shown that these blocks are, in a sense, optimal and cannot be removed or essentially changed. The proof is based on a special equivalent representation of a concave function (see [6, pp. 320–325]). Our Carlson type inequality is used to characterize Peetre's interpolation functor $\langle\ \rangle_\varphi$ (see [26]) and its Gagliardo closure on couples of functional Banach lattices in terms of the Calderón–Lozanovskiĭ construction.

Our interest in this functor is inspired by the fact that if $\varphi = t^\theta$ $(0<\theta<1)$, then, on couples of Banach lattices and their retracts, it coincides with the complex method (see [20], [27]) and, thus, it may be regarded as a "real version" of the complex method.

0. Introduction. In this paper we consider sequences $a_n, n = 1, 2, \ldots$, of *nonnegative* numbers. In 1934 Carlson [8] proved the somewhat curious inequality

$$\sum a_n \le C\Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^2 a_n^2\Big)^{1/4} \tag{0.1}$$

and showed that $C = \pi^{1/2}$ is the best possible constant. Carlson also noted that (0.1) does *not* follow from Hölder's inequality in the following way:

$$\sum a_n \le \Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^{2h} a_n^2\Big)^{1/4}\Big(\sum n^{-h}\Big)^{1/2} = C(h)\Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^{2h} a_n^2\Big)^{1/4}$$

because $C(h)\to\infty$ as $h\to 1+$. However, in 1936 Hardy [12] presented two elementary proofs of (0.1). In particular, he observed that (0.1) in fact follows even from Schwarz' inequality $\sum x_n y_n \le (\sum x_n^2)^{1/2}(\sum y_n^2)^{1/2}$ applied

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