dual of $L^{q,1}[0,\infty)$, its unit ball is weak* compact. Given $g = (g_k) \in (\sum_{k=1}^{\infty} \oplus L^{p,\infty}[0,k])_{\ell^\infty}$, let

$$Qg = (w^*) \lim_{k\to\mathcal{U}} g_k .$$

It is easy to see that Q is bounded as a map into $L^{p,\infty}[0,\infty)$, and that $Q\circ S$ is the identity on $L^{p,\infty}[0,\infty)$. This proves the proposition. ∎

The proof of Theorem 1 now follows as in the discussion in §2, using Theorems 13, 14, and Proposition 15 above.

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A Carlson type inequality with blocks and interpolation

by

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Abstract. An inequality, which generalizes and unifies some recently proved Carlson type inequalities, is proved. The inequality contains a certain number of "blocks" and it is shown that these blocks are, in a sense, optimal and cannot be removed or essentially changed. The proof is based on a special equivalent representation of a concave function (see [6, pp. 320–325]). Our Carlson type inequality is used to characterize Peetre's interpolation functor $\langle\ \rangle_\varphi$ (see [26]) and its Gagliardo closure on couples of functional Banach lattices in terms of the Calderón–Lozanovskiĭ construction.

Our interest in this functor is inspired by the fact that if $\varphi = t^\theta$ $(0<\theta<1)$, then, on couples of Banach lattices and their retracts, it coincides with the complex method (see [20], [27]) and, thus, it may be regarded as a "real version" of the complex method.

0. Introduction. In this paper we consider sequences $a_n, n = 1,2,\ldots$, of *nonnegative* numbers. In 1934 Carlson [8] proved the somewhat curious inequality

$$\sum a_n \le C\Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^2 a_n^2\Big)^{1/4} \tag{0.1}$$

and showed that $C = \pi^{1/2}$ is the best possible constant. Carlson also noted that (0.1) does *not* follow from Hölder's inequality in the following way:

$$\sum a_n \le \Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^{2h} a_n^2\Big)^{1/4}\Big(\sum n^{-h}\Big)^{1/2} = C(h)\Big(\sum a_n^2\Big)^{1/4}\Big(\sum n^{2h} a_n^2\Big)^{1/4}$$

because $C(h)\to\infty$ as $h\to 1+$. However, in 1936 Hardy [12] presented two elementary proofs of (0.1). In particular, he observed that (0.1) in fact follows even from Schwarz' inequality $\sum x_n y_n \le (\sum x_n^2)^{1/2}(\sum y_n^2)^{1/2}$ applied

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to the sequences $x_n = a_n(\alpha+\beta n^2)^{1/2}$ and $y_n = (\alpha+\beta n^2)^{-1/2}$ and he got the best constant $C = \pi^{1/2}$ by making a suitable choice of α and β. For some other generalizations and complements of (0.1) see e.g. [2, pp. 175–176]. Here we only remark that (0.1) and its generalizations have sense e.g. in some moment problems (see [14]) and in interpolation theory (see [11]).

In the sequel we assume that φ is a concave and positive function on $(0,\infty)$ such that $\varphi(t) \to 0$ as $t \to 0^+$ and $\varphi(t)/t \to 0$ as $t \to \infty$. Moreover, $\varphi(s,t)$ denotes the 1-homogeneous function of two variables defined by $\varphi(s,t) = s\varphi(t/s)$ if $s,t > 0$ and $\varphi(s,t) = 0$ if $s = 0$ or $t = 0$.

In 1977 Gustavsson and Peetre [11] (in connection with some interpolation problems for Orlicz spaces) considered the Carlson type inequality

$$\sum a_n \le C\varphi(\|\{a_n/\varphi(2^n)\}\|_{l_p}, \|\{2^n a_n/\varphi(2^n)\}\|_{l_q}, \tag{0.2}$$

and proved it for $\min(p,q) > 1$ and φ belonging to the class P^{+-} (see Section 1).

The first part of this paper is devoted to the question of the possibility and the form of a generalization of (0.2) for $\varphi \notin P^{+-}$. Our result (see Theorem 4) shows that if $\varphi \notin P^{+-}$, then (0.2) is not true and, thus, we must change the form of (0.2).

Let $\{2^n\}_{n\in\mathbb{Z}} = \bigcup_{n=1}^{\infty} \Omega_n$ with Ω_n pairwise disjoint. We consider the following Carlson type inequality:

$$\|\{a_n\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\Omega_n} \frac{a_k}{\varphi(2^k)}\Big\}_n\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\Omega_n} \frac{2^k a_k}{\varphi(2^k)}\Big\}_n\Big\|_{l_q}\Big), \tag{0.3}$$

with a constant C not depending on $\{a_n\}$. In our Theorem 3 we give a necessary and sufficient condition on Ω_n to ensure that (0.3) holds. We also present a construction for Ω_n (see Theorem 1). Moreover, we prove (see Theorem 2) that (0.3) can be written in the following "symmetric" form:

$$\|\{\varphi(a_n,b_n)\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} b_k\Big\}\Big\|_{l_q}\Big), \tag{0.4}$$

where $T_n, n = 1,2,\ldots$, are special blocks in $\mathbb{R}^2_+$ (see Figure 3). We note that blocks first appeared in Carlson's inequality (*not* in explicit form) in a work of Nilsson [22, Lemma 2.2]. In fact, this work was one starting point and motivation for us.

In the second part of this paper we show how our general Carlson type inequality "works" in our proof of the characterization of Peetre's interpolation functor $\langle\,\rangle_\varphi$ (see [26]) and its Gagliardo closure on couples of functional Banach lattices ([1]). Before the most general result of this kind was obtained

([1]) We want to mention that the first interpolation functor of this type was introduced (and even computed for the couple (L_{p_0}, L_{p_1})) in 1968 by Gagliardo [9].

by Nilsson [22]. However, he has some restrictions on φ and the couples of Banach lattices. Moreover, our proof is more "elementary" and we do not use e.g. such properties as K-monotonicity and K-divisibility. The main ideas of our proof are taken from a work of Kruglyak [15] (cf. [6, pp. 560–564]) and also our generalized Carlson inequality with blocks is used in a crucial way. For the reader's convenience we write all definitions and proofs in detail.

Finally, we remark that the interpolation properties of the Calderón–Lozanovskiĭ construction were investigated in a lot of works; in our opinion, the most important achievements were obtained, in chronological order, in [7], [29], [17], [19], [23], [11], [20], [3], [28], [10], [24], [15] and [22] (see also the books [25] and [21]).

1. On an equivalent representation of a concave function. Let P denote the set of all positive concave functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ ($\mathbb{R}_+ = (0,\infty)$). It is easy to see that $\varphi(t)$ is nondecreasing and that $\varphi(t)/t$ is nonincreasing. For $\varphi \in P$ we consider

$$s_\varphi(t) = \sup_{s>0} \varphi(st)/\varphi(s), \quad 0 < t < \infty.$$

We also define the following subsets of P:

$$P_0 = \{\varphi \in P : \lim_{t\to 0+} \varphi(t) = \lim_{t\to\infty} \varphi(t)/t = 0\},$$

$$P_1 = \{\varphi \in P : \lim_{t\to\infty} \varphi(t) = \lim_{t\to 0+} \varphi(t)/t = \infty\},$$

$$P^+ = \{\varphi \in P : \lim_{t\to 0+} s_\varphi(t) = 0\},$$

$$P^- = \{\varphi \in P : \lim_{t\to\infty} s_\varphi(t)/t = 0\}, \quad P^{+-} = P^+ \cap P^-.$$

In particular, we note that

$$P^{+-} \subset P_0 \cap P_1. \tag{1.1}$$

For a given $\varphi \in P_0$ Brudnyĭ–Kruglyak [5] (see also [6, pp. 320–325]) constructed a tricky increasing sequence such that

$$\varphi(t) \approx \sum \varphi(t_{2n+1}) \min(1, t/t_{2n+1})$$

and

$$\varphi(t) \approx \max_n(\varphi(t_{2n+1}) \min(1, t/t_{2n+1})),$$

where the equivalence constants are independent of φ and t. Since the geometric properties of this sequence will be used several times in this paper we describe briefly the construction of $\{t_n\}$ and notice some inequalities.

Let $q > 1$ be fixed. For $s \in (0,\infty)$ we denote by χ_s the closed interval defined in the following way:

$$t \in \chi_s \quad \text{iff} \quad \varphi(t) \le q\varphi(s)\min(1, t/s).$$

Geometrically χ_s means the set of all t such that the graph of $\varphi(t)$ is below that of $q\varphi(s)\min(1,t/s)$.

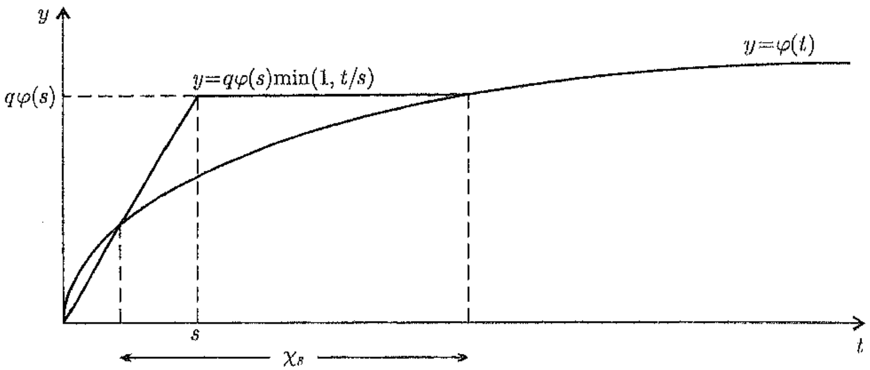

Fig. 1

We note that the right endpoint of χ_s can be $+\infty$ whenever φ is bounded and the left endpoint may be 0 whenever $\varphi(t)/t$ is bounded. The points t_{2n+1} in the Brudnyĭ–Kruglyak construction are chosen such that the intervals $\chi_{t_{2n+1}}$ cover $(0,\infty)$ and their interiors are disjoint. The right endpoint of $\chi_{t_{2n+1}}$ (equal to the left endpoint of $\chi_{t_{2n+3}}$ whenever t_{2n+3} exists) is, by definition, denoted as t_{2n+2}. Similarly, the left endpoint of $\chi_{t_{2n+1}}$ is denoted by t_{2n}.

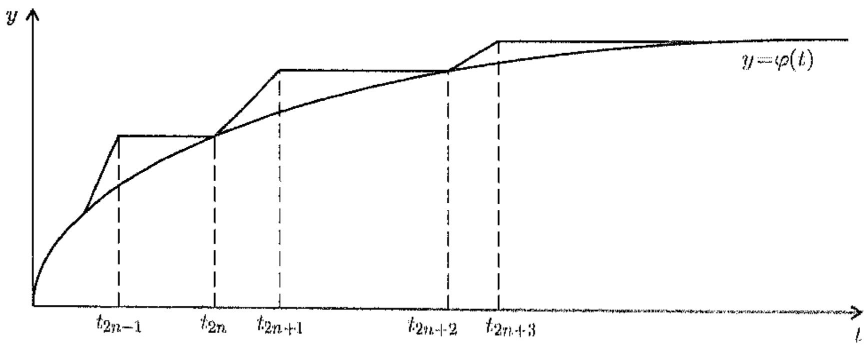

Fig. 2

From this construction we have the inequality

$$\varphi(t) \le q\min(1,t/t_{2n+1})\varphi(t_{2n+1}) \qquad \text{for } t \in \chi_{t_{2n+1}} = [t_{2n}, t_{2n+2}],$$

and also

$$\varphi(t_{2n+1})/t_{2n+1} = q^{-1}\varphi(t_{2n})/t_{2n}, \quad \varphi(t_{2n+2}) = q\varphi(t_{2n+1}), \quad t_{2n+2} \ge q^2 t_{2n}.$$

By making some straightforward calculations we obtain

$$\sum_n \varphi(t_{2n+1})\min(1,t/t_{2n+1}) \le \frac{q+1}{q-1}\varphi(t), \qquad 0 < t < \infty.$$

We note that in some questions of optimality the best choice of q is $q = 1+\sqrt{2}$ (cf. the proof of Theorem 1). The properties of the sequence $\{t_n\}$ are closely connected with the properties of φ. In particular, we will need the following lemma:

LEMMA 1. *Suppose that $\varphi \in P_0$ and let $\{t_n\}$ be the sequence from the above construction. Then $\varphi \in P^{+-}$ iff $\sup_n t_{2n+2}/t_{2n} < \infty$.*

Proof. Let $\varphi \in P^-$ and $q > 1$ be fixed. Then there exists $u > 1$ such that

$$\frac{\varphi(su)}{\varphi(s)} < \frac{1}{q}u \quad \text{for any } s > 0,$$

which we can rewrite as

$$\frac{\varphi(su)}{su} < \frac{1}{q}\frac{\varphi(s)}{s} \quad \text{for any } s > 0.$$

Put $s = t_{2n}$. Since

$$\frac{\varphi(t_{2n+1})}{t_{2n+1}} = \frac{1}{q}\frac{\varphi(t_{2n})}{t_{2n}}$$

it follows that $t_{2n+1}/t_{2n} < u$ (because $\varphi(t)/t$ is nonincreasing).

Similarly, if $\varphi \in P^+$, then $t_{2n+2}/t_{2n+1} < 1/v$, where $0 < v < 1$ is taken from the inequality $\varphi(sv) > \varphi(s)/q$ and where $s = t_{2n+2}$. Therefore if $\varphi \in P^{+-}$, then $\sup_n t_{2n+2}/t_{2n} \le u/v < \infty$.

Conversely, assume that $\sup_n t_{2n+2}/t_{2n} \le K < \infty$. For any $s \in (0,\infty)$ there exists n_0 such that $s \in [t_{2n_0}, t_{2n_0+2})$. Then $t_{2n_0+4} \le K^2 s$ and, according to the construction of $\{t_n\}$, we find that

$$\frac{\varphi(K^2 s)}{K^2 s} \le \frac{\varphi(t_{2n_0+4})}{t_{2n_0+4}} \le \frac{\varphi(t_{2n_0+3})}{t_{2n_0+3}} = \frac{1}{q}\frac{\varphi(t_{2n_0+2})}{t_{2n_0+2}} < \frac{1}{q}\frac{\varphi(s)}{s}.$$

Therefore $\varphi(K^{2n}s) \le q^{-n}K^{2n}\varphi(s)$ and we conclude that $s_\varphi(t)/t \to 0$ as $t \to \infty$, which means that $\varphi \in P^-$. The proof that $\varphi \in P^+$ is quite similar so we omit the details.

2. A Carlson type inequality.

2. A Carlson type inequality. We begin by giving the following remarks to the inequality (0.2). Gustavsson–Peetre [11] have proved that (0.2) holds for $1 < \min(p,q) \le \infty$ if $\varphi \in P^{+-}$. On the other hand, the simple example $\varphi(t) = \min(1,t)$ shows that (0.2) does not hold in general for all $\varphi \in P_0$. For the limiting case $p = q = 1$ we have, for every $\varphi \in P$, the Jensen inequality

$$\sum \varphi(a_n, b_n) \le \varphi\Big(\sum a_n, \sum b_n\Big). \tag{2.1}$$

Let $\{t_n\}$ denote the sequence of numbers defined in the Brudnyĭ–Kruglyak construction (see Section 1) and put $\chi_n = [t_{2n}, t_{2n+2})$. Our generalization of Carlson's inequality reads:

THEOREM 1. *Assume that* $1 < p, q \le \infty$ *and* $\varphi \in P_0$. *Then, for any positive sequence* $\{a_n\}$,

$$\|\{a_n\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\chi_n} a_k/\varphi(2^k)\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\chi_n} 2^k a_k/\varphi(2^k)\Big\}\Big\|_{l_q}\Big), \tag{2.2}$$

with the constant C *not depending on* $\{a_n\}$ $(C \le (1+\sqrt{2})^2)$.

Proof. It is sufficient to prove (2.2) with $p = q = \infty$. We consider

$$A = \sup_n \sum_{k:2^k\in\chi_n} \frac{a_k}{\varphi(2^k)},$$

$$B = \sup_n \sum_{k:2^k\in\chi_n} \frac{2^k a_k}{\varphi(2^k)}, \quad M = B/A.$$

We denote by n_0 the index for which $M \in \chi_{n_0}$ and make the following decomposition:

$$\sum a_n = \sum_{n<n_0} \sum_{k:2^k\in\chi_n} a_k + \sum_{k:2^k\in\chi_n} a_k + \sum_{n>n_0} \sum_{k:2^k\in\chi_n} a_k.$$

If $q > 1$, then, according to the properties of the Brudnyĭ–Kruglyak construction, we have

$$\begin{aligned}\sum_{n<n_0} \sum_{k:2^k\in\chi_n} a_k &\le \sum_{n<n_0} \sum_{k:2^k\in\chi_n} \frac{a_k}{\varphi(2^k)} \max_{k:2^k\in\chi_n} \varphi(2^k)\\ &\le \sum_{n<n_0} \sum_{k:2^k\in\chi_n} \frac{a_k}{\varphi(2^k)} q\varphi(t_{2n+1})\\ &\le Aq(\varphi(M)/q + \varphi(M)/q^2 + \ldots)\\ &= A\varphi(M)q/(q-1) = \varphi(A,B)q/(q-1)\end{aligned}$$

and

$$\begin{aligned}\sum_{n>n_0} \sum_{k:2^k\in\chi_n} a_k &\le \sum_{n>n_0} \sum_{k:2^k\in\chi_n} \frac{2^k a_k}{\varphi(2^k)} \max_{k:2^k\in\chi_n} \frac{\varphi(2^k)}{2^k}\\ &\le \sum_{n>n_0} \sum_{k:2^k\in\chi_n} \frac{2^k a_k}{\varphi(2^k)} q\frac{\varphi(t_{2n+1})}{t_{2n+1}}\\ &\le Bq(\varphi(M)/(Mq) + \varphi(M)/(Mq^2) + \ldots)\end{aligned}$$

$$\begin{aligned}&= B\varphi(M)q/(M(q-1))\\ &= \varphi(A,B)q/(q-1).\end{aligned}$$

The third component will be estimated in the following way: If $\varphi(M) \ge \varphi(t_{2n_0+1})$, then

$$\begin{aligned}\sum_{k\in\chi_{n_0}} a_k &\le \sum_{k\in\chi_{n_0}} \frac{a_k}{\varphi(2^k)} \max_{k\in\chi_{n_0}} \varphi(2^k) \le \sum_{k\in\chi_{n_0}} \frac{a_k}{\varphi(2^k)} q\varphi(t_{2n_0+1})\\ &\le Aq\varphi(M) = q\varphi(A,B),\end{aligned}$$

and if $\varphi(M) < \varphi(t_{2n_0+1})$, then

$$\begin{aligned}\sum_{k\in\chi_{n_0}} a_k &\le \sum_{k\in\chi_{n_0}} \frac{2^k a_k}{\varphi(2^k)} \max_{k\in\chi_{n_0}} \frac{\varphi(2^k)}{2^k} \le \sum_{k\in\chi_{n_0}} \frac{2^k a_k}{\varphi(2^k)} q\frac{\varphi(t_{2n_0+1})}{t_{2n_0+1}}\\ &\le Bq\varphi(M)/M = q\varphi(A,B).\end{aligned}$$

By combining these inequalities we obtain

$$\begin{aligned}\sum a_n &\le \Big(\frac{q}{q-1} + \frac{q}{q-1} + q\Big)\varphi(A,B)\\ &= \frac{q(q+1)}{q-1}\varphi(A,B).\end{aligned}$$

The infimum over $q > 1$ is attained at $q = 1 + \sqrt{2}$ and we get (2.2) with $C = (1+\sqrt{2})^2$. The proof is complete.

Remark. From the proof of Theorem 1 we can also obtain the inequality

$$\|\{a_n\}\|_{l_1} \le \frac{q+1}{q-1}\varphi\Big(\Big\|\Big\{\frac{a_n}{\varphi(t_{2n+1})}\Big\}\Big\|_{l_p}, \Big\|\Big\{\frac{t_{2n+1}a_n}{\varphi(t_{2n+1})}\Big\}\Big\|_{l_q}\Big). \tag{2.3}$$

Next we will prove that the inequality (2.1) can be formulated in another useful symmetric form. We consider the following subsets of $\mathbb{R}_+^2$: T_n, $n = 1, 2, \ldots$, are the sectors in $\mathbb{R}_+^2$ between the lines $y = t_{2n}x$ and $y = t_{2n+2}x$, respectively (t_n are defined in the Brudnyĭ–Kruglyak construction).

THEOREM 2. *Let* $\varphi \in P_0$ *and* $1 < p, q \le \infty$. *Then the inequality* (2.2) *is equivalent to*

$$\sum_{n=1}^{\infty} \varphi(a_n, b_n) \le C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} b_k\Big\}\Big\|_{l_q}\Big). \tag{2.4}$$

Remark. Our proof shows that the constant C can be estimated by $2(1+\sqrt{2})^2$.

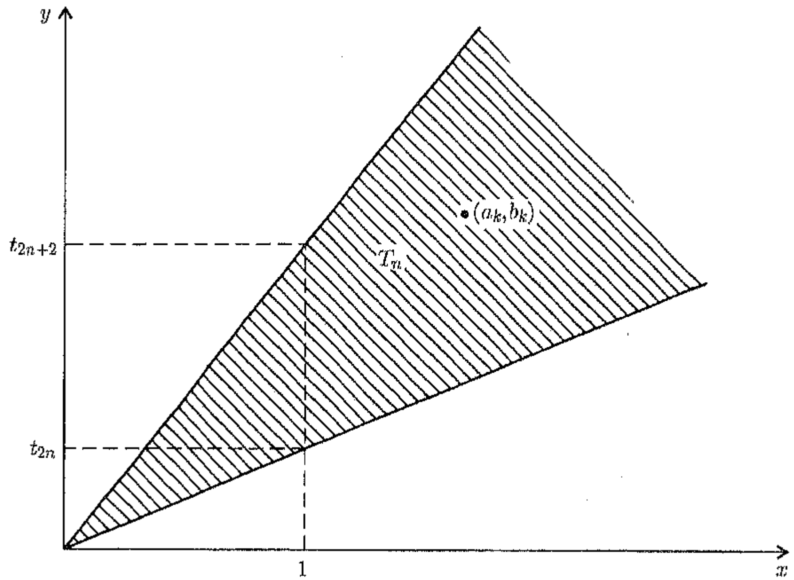

Fig. 3

Proof. (2.2)⇒(2.4). For $n = 1, 2, \ldots$ we define m_n to be such that $t_{2n} \le 2^{m_n} < t_{2n+2}$. We put $b_i = a_n$ if $i = m_n$ and $b_i = 0$ elsewhere. Then (2.2) implies that

$$\sum_n a_n = \sum_i b_i \le (1+\sqrt{2})^2 \varphi\bigg(\bigg\|\bigg\{\frac{a_n}{\varphi(2^{m_n})}\bigg\}\bigg\|_{l_p}, \bigg\|\bigg\{\frac{2^{m_n} a_n}{\varphi(2^{m_n})}\bigg\}\bigg\|_{l_q}\bigg).$$

By using this inequality and (2.1) we find that

$$\text{(2.5)}\qquad \|\{\varphi(a_n, b_n)\}\|_{l_1} = \sum_n \sum_{k:(a_k,b_k)\in T_n} \varphi(a_k, b_k)$$
$$\le \sum_n \varphi\Big(\sum_{k:(a_k,b_k)\in T_n} a_k, \sum_{k:(a_k,b_k)\in T_n} b_k\Big) := \sum_n A_n$$
$$\le (1+\sqrt{2})^2 \varphi\bigg(\bigg\|\bigg\{\frac{A_n}{\varphi(2^{m_n})}\bigg\}\bigg\|_{l_p}, \bigg\|\bigg\{\frac{2^{m_n} A_n}{\varphi(2^{m_n})}\bigg\}\bigg\|_{l_q}\bigg).$$

If

$$s_n = \sum_{k:(a_k,b_k)\in T_n} b_k \Big/ \sum_{k:(a_k,b_k)\in T_n} a_k,$$

then from the definition of T_n it follows that $t_{2n} \le s_n < t_{2n+2}$. Therefore we can choose m_n so that $2^{m_n} \in \chi_n$ and $s_n \le 2^{m_n+1}$. Hence $\varphi(s_n) \le$

$\varphi(2^{m_n+1}) \le 2\varphi(2^{m_n})$ and it follows that

$$\frac{A_n}{\varphi(2^{m_n})} = \sum_{k:(a_k,b_k)\in T_n} a_k \frac{\varphi(s_n)}{\varphi(2^{m_n})} \le 2 \sum_{k:(a_k,b_k)\in T_n} a_k,$$

and

$$\frac{2^{m_n} A_n}{\varphi(2^{m_n})} = \sum_{k:(a_k,b_k)\in T_n} a_k \frac{2^{m_n}\varphi(s_n)}{\varphi(2^{m_n})} \le 2 \sum_{k:(a_k,b_k)\in T_n} a_k s_n = 2 \sum_{k:(a_k,b_k)\in T_n} b_k.$$

By inserting these inequalities into (2.5) we find that (2.4) holds.

(2.4)⇒(2.2). Let c_k, $k = 1, 2, \ldots$, be arbitrary positive numbers. By taking $a_k = c_k/\varphi(2^k)$ and $b_k = 2^k c_k/\varphi(2^k)$ we find that $b_k/a_k = 2^k$ and therefore $(a_k, b_k) \in T_n$ iff $2^k \in \chi_n$. Hence, according to (2.4), we have

$$\|\{c_k\}\|_{l_1} = \|\{\varphi(a_k, b_k)\}\|_{l_1}$$
$$\le C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in T_n} a_k\Big\}\Big\|_{l_q}\Big)$$
$$\le C\varphi\bigg(\bigg\|\bigg\{\sum_{k:2^k\in\chi_n} \frac{c_k}{\varphi(2^k)}\bigg\}\bigg\|_{l_p}, \bigg\|\bigg\{\sum_{k:2^k\in\chi_n} \frac{2^k c_k}{\varphi(2^k)}\bigg\}\bigg\|_{l_q}\bigg),$$

i.e. (2.2) holds and the proof is complete.

3. An optimality result concerning the system of blocks in Theorem 1. The main purpose in this section is to explain how the block decomposition in Theorem 1 is in a sense "optimal". First we note that if each $\chi_n = [t_{2n}, t_{2n+2})$ is decomposed into no more than M subsets $\chi_n^1, \ldots, \chi_n^M$ (M is *independent* of n), then, by using the inequality

$$\Big(\sum_{k:2^k\in\chi_n} b_k\Big)^p \le M^p \max_{i=1,\ldots,M} \Big(\sum_{k:2^k\in\chi_n^i} b_k\Big)^p \le M^p \sum_{i=1}^M \Big(\sum_{k:2^k\in\chi_n^i} b_k\Big)^p$$

and Theorem 1 we find that, for $\varphi \in P_0$,

$$\|\{a_n\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\chi_n} a_k/\varphi(2^k)\Big\}_n\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\chi_n} 2^k a_k/\varphi(2^k)\Big\}_n\Big\|_{l_q}\Big)$$
$$\le CM\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\chi_n^i} a_k/\varphi(2^k)\Big\}_{i,n}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\chi_n^i} 2^k a_k/\varphi(2^k)\Big\}_{i,n}\Big\|_{l_q}\Big).$$

This means that our Carlson type inequality (2.1) also holds with the system of blocks χ_n^i, $n = 1, 2, \ldots$, $i = 1, \ldots, M$, but with a constant of order M. Therefore it is tempting to suggest that there are only such possibilities to obtain a Carlson type inequality with blocks for a general $\varphi \in P_0$. We give a more precise meaning to this suggestion in our next theorem.

THEOREM 3. *Let $\varphi \in P_0$, $1 < r = \min(p,q) \le \infty$ and $\chi_n = [t_{2n}, t_{2n+2}]$, where t_n are the numbers defined in the Brudnyĭ–Kruglyak construction. Moreover, assume that $\{2^n\}_{n\in\mathbb{Z}} = \bigcup_{n=1}^{\infty} \Omega_n$ with Ω_n pairwise disjoint. Then the following conditions are equivalent:*

(i) *The inequality*

$$\|\{a_n\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\Omega_n}\frac{a_k}{\varphi(2^k)}\Big\}_n\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\Omega_n}\frac{2^k a_k}{\varphi(2^k)}\Big\}_n\Big\|_{l_q}\Big) \tag{3.1}$$

holds for any nonnegative sequence $\{a_n\}_{n\in\mathbb{Z}}$ with a constant C independent of $\{a_n\}$.

(ii) *There exists a finite positive number M such that, for any $k \in \mathbb{Z}$,*

$$\operatorname{card}(\{n : \Omega_n \cap \chi_k \ne \emptyset\}) \le M. \tag{3.2}$$

Proof. (i)⇒(ii). Assume the contrary, i.e. for any $M > 0$ we can find $k_0 = k(M)$ such that the number of n for which $\Omega_n \cap \chi_{k_0} \ne \emptyset$ is greater than M. Choose distinct points $2^{n_1}, \ldots, 2^{n_M}$, each in one of these intersections. Then, according to (3.1), we have

$$\sum_{k=1}^{M} a_{n_k} \le C\varphi\Big(\Big\|\Big\{\frac{a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_p}, \Big\|\Big\{\frac{2^{n_k}a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_q}\Big). \tag{3.3}$$

(Note that now each block contains only one element.)

We divide the interval $\chi_{k_0} = [t_{2k_0}, t_{2k_0+2})$ into $[t_{2k_0}, t_{2k_0+1})$ and $[t_{2k_0+1}, t_{2k_0+2})$. Then at least half of the points in $\{2^{n_k}\}$ is in one of these subintervals. First, we assume that they are in $[t_{2k_0}, t_{2k_0+1})$. We put $a_{n_k} = 0$ if 2^{n_k} is in the second interval and obtain

$$t_{2k_0} \le \Big\|\Big\{\frac{2^{n_k}a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r} \Big/ \Big\|\Big\{\frac{a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r} \le t_{2k_0+1}.$$

On the interval $[t_{2k_0}, t_{2k_0+1})$ we have $\varphi(t) \le qt\varphi(t_{2k_0+1})/t_{2k_0+1}$ for some $q > 1$ from the construction of $\{t_k\}$ (see Section 1). Therefore, by using (3.3) and making some simple calculations, we find that

$$\sum_{k=1}^{M} a_{n_k} \le Cq\Big\|\Big\{\frac{2^{n_k}a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r}\frac{\varphi(t_{2k_0+1})}{t_{2k_0+1}} \le Cq\|\{a_{n_k}\}_{k=1}^{M}\|_{l_r}.$$

Here we can choose at least $M/2$ of the numbers a_{n_k} strictly positive and since $r > 1$ we get a contradiction as $M \to \infty$.

Assume now that at least half of the points in $\{2^{n_k}\}$ is in $[t_{2k_0+1}, t_{2k_0+2})$. We let $a_{n_k} = 0$ if 2^{n_k} is in the other interval and find that

$$t_{2k_0+1} \le \Big\|\Big\{\frac{2^{n_k}a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r} \Big/ \Big\|\Big\{\frac{a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r} \le t_{2k_0+2}.$$

This yields that $\varphi(t) \le qt\varphi(t_{2k_0+1})$ on $[t_{2k_0+1}, t_{2k_0+2})$. Hence, according to (3.3), we have

$$\sum_{k=1}^{M} a_{n_k} \le Cq\varphi(t_{2k_0+1})\Big\|\Big\{\frac{a_{n_k}}{\varphi(2^{n_k})}\Big\}_{k=1}^{M}\Big\|_{l_r} \le Cq\|\{a_{n_k}\}_{k=1}^{M}\|_{l_r},$$

which leads to a contradiction as before.

(ii)⇒(i). Assume that the uniform bound $M < \infty$ from (3.2) exists and put $\chi_n^i = \Omega_n \cap \chi_i$. For fixed i no more than M of the χ_n^i are nonempty. Therefore we can use the inequality before Theorem 3 to obtain

$$\|\{a_n\}\|_{l_1} \le CM\varphi\Big(\Big\|\Big\{\sum_{k:2^k\in\chi_n^i} a_k/\varphi(2^k)\Big\}_{i,n}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:2^k\in\chi_n^i} 2^k a_k/\varphi(2^k)\Big\}_{i,n}\Big\|_{l_q}\Big).$$

Moreover, we note that $\Omega_n = \bigcup_{i\in\mathbb{Z}} \chi_n^i$, $\chi_n^i \cap \chi_n^j = \emptyset$ if $i \ne j$, and when passing from the blocks χ_n^i to the bigger blocks Ω_n the expression in the right-hand side can only increase; this yields (3.1) and completes the proof of Theorem 3.

4. A precise version of the Gustavsson–Peetre inequality.

The Gustavsson–Peetre inequality is connected with the following subsets of $\mathbb{R}_+^2$: S_n, $n = 1, 2, \ldots$, are the sectors in $\mathbb{R}_+^2$ between the lines $y = 2^n x$ and $y = 2^{n+1}x$, respectively. The following equivalence theorem is to be compared with our Theorem 2:

THEOREM 4. *Let $\varphi \in P_0$ and let $1 < \min(p,q) \le \infty$. The following statements are equivalent:*

(i) $$\varphi \in P^{+-},$$

(ii) $$\sum a_n \le C\varphi(\|\{a_n/\varphi(2^n)\}\|_{l_p}, \|\{2^n a_n/\varphi(2^n)\}\|_{l_q}),$$

(iii) $$\|\{\varphi(a_n, b_n)\}\|_{l_1} \le C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} b_k\Big\}\Big\|_{l_q}\Big).$$

Proof. (i)⇒(ii). See Gustavsson–Peetre [11, Prop. 3.1], where instead of p, q we must take $\max(p,q)$, and also the book [21].

(ii)⇒(i). First we note that (ii) can be interpreted as (3.1) with blocks $\Omega_n = \{2^n\}$ consisting of only one element. Assume that $\varphi \notin P^{+-}$. Then, according to Lemma 1, for any $k > 0$ there exists $n = n(k)$ such that $t_{2n+2}/t_{2n} > 2^k$, i.e. χ_n contains at least $k-1$ blocks Ω_n. Since k was arbitrary we get a contradiction with the optimality theorem (Theorem 3).

(ii)⇒(iii). We note that

$$\|\{\varphi(a_n, b_n)\}\|_{l_1} = \sum_n \sum_{k:(a_k,b_k)\in S_n} \varphi(a_k, b_k)$$

$$\leq \sum_n \sum_{k:(a_k,b_k)\in S_n} \varphi(a_k, 2^{n+1}a_k)$$
$$= 2\sum_n \sum_{k:(a_k,b_k)\in S_n} a_k\varphi(2^n).$$

Therefore (ii) implies that

$$\|\{\varphi(a_n,b_n)\}\|_{l_1} \leq C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{2^n \sum_{k:(a_k,b_k)\in S_n} a_k\Big\}\Big\|_{l_q}\Big)$$
$$\leq C\varphi\Big(\Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} a_k\Big\}\Big\|_{l_p}, \Big\|\Big\{\sum_{k:(a_k,b_k)\in S_n} b_k\Big\}\Big\|_{l_q}\Big).$$

(iii)$\Rightarrow$(ii). The proof is similar to the proof of the implication (2.4)$\Rightarrow$(2.2) in Theorem 2 so we omit the details.

5. A characterization of Peetre's interpolation functor on couples of Banach lattices. Let us recall some notations and definitions from interpolation theory (see [4], [6]). We say that $\vec{A} = (A_0, A_1)$ is a *Banach couple* if the Banach spaces A_0 and A_1 are both continuously embedded in some Hausdorff topological vector space. Let

$$\Delta(\vec{A}) = A_0 \cap A_1 \quad \text{and} \quad \Sigma(\vec{A}) = A_0 + A_1$$

with

$$\|a\|_{\Delta(\vec{A})} = \max_{i=0,1} \|a\|_{A_i} \quad \text{and} \quad \|a\|_{\Sigma(\vec{A})} = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}),$$

respectively. A Banach space A is called *intermediate* between A_0 and A_1 if $\Delta(\vec{A}) \subset A \subset \Sigma(\vec{A})$ (here and in the sequel $\subset$ denotes continuous imbedding). If this is the case, then we denote by A^0 the closure of $\Delta(\vec{A})$ in A.

The *Gagliardo closure* with respect to $\Sigma(\vec{A})$ is denoted by A^c and defined in the following way: $a \in A^c$ iff there exists a sequence $\{a_n\} \in A$ such that, for some $\lambda < \infty$, $\|a_n\|_A \leq \lambda$ and $\|a_n - a\|_{\Sigma(\vec{A})} \to 0$ as $n \to \infty$. A^c is a Banach space with the norm $\|a\|_{A^c} = \inf \lambda$. The Banach couple $\vec{A} = (A_0, A_1)$ is called *regular* if $A_i^0 = A_i$, $i = 0, 1$. If $\vec{A} = (A_0, A_1)$ and $\vec{B} = (B_0, B_1)$ are two Banach couples, then we say that T is a continuous linear operator from $\vec{A}$ to $\vec{B}$ ($T \in L(\vec{A}, \vec{B})$) if it is a linear operator from $\Sigma(\vec{A})$ to $\Sigma(\vec{B})$ and if $T|A_i : A_i \to B_i$ ($i = 0, 1$) are continuous. We then set

$$\|T\|_{L(\vec{A},\vec{B})} := \max_{i=0,1}(\|T\|_{A_i\to B_i}).$$

Let $\vec{E} = (E_0, E_1)$ be a fixed Banach couple and let E be an intermediate space between E_0 and E_1. For an arbitrary Banach couple $\vec{A} = (A_0, A_1)$, the *orbit* of E in $\Sigma(\vec{A})$ is the space of all $a \in \Sigma(\vec{A})$ which admit the representation

$$a = \sum_{n=1}^{\infty} T_n f_n,$$

where the series converges in $\Sigma(\vec{A})$, with $f_n \in E$, $T_n \in L(\vec{E}, \vec{A})$, and $\sum_{n=1}^{\infty} \|T_n\|_{L(\vec{E},\vec{A})} \|f_n\|_E < \infty$.

This space, which we shall denote by $\mathrm{Orb} = \mathrm{Orb}_E^{\vec{E}}(\vec{A})$, was introduced by Aronszajn and Gagliardo [1]. It is a Banach space with the norm

$$\|a\|_{\mathrm{Orb}} = \inf \sum_{n=1}^{\infty} \|T_n\|_{L(\vec{E},\vec{A})} \|f_n\|_E,$$

where the infimum is taken over all admissible representations.

It is easy to see that $\mathrm{Orb}_E^{\vec{E}}(\vec{A})$ is an exact interpolation functor, which means that if $T \in L(\vec{A}, \vec{B})$, then

$$T : \mathrm{Orb}_E^{\vec{E}}(\vec{A}) \to \mathrm{Orb}_E^{\vec{E}}(B) \quad \text{and} \quad \|T\|_{\mathrm{Orb}\to\mathrm{Orb}} \leq \|T\|_{L(\vec{A},\vec{B})}.$$

Moreover, if the exact interpolation functor F is such that $E \overset{1}{\subset} F(\vec{E})$, then $\mathrm{Orb}_E^{\vec{E}}(\vec{A}) \overset{1}{\subset} F(\vec{A})$ for every Banach couple $\vec{A}$.

Our main interest in this construction is when

$$\vec{E} = \vec{c}_0 = (c_0, c_0(\{2^{-n}\})), \qquad E = c_0(\{1/\varphi(2^n)\}),$$

where $\varphi \in P_0$ and

$$\|\{a_n\}_{n=-\infty}^{\infty}\|_{c_0} = \max_{n\in\mathbb{Z}} |a_n|,$$
$$\|\{a_n\}_{n=-\infty}^{\infty}\|_{c_0(\{2^{-n}\})} = \|\{a_n 2^{-n}\}_{n=-\infty}^{\infty}\|_{c_0}.$$

This orbit construction was introduced in 1971 by Peetre [26] in another equivalent form (see [13]) and we denote it by G_φ^0. Thus Peetre's interpolation functor is defined by the formula

$$G_\varphi^0 = \mathrm{Orb}_{c_0(\{1/\varphi(2^n)\})}^{\vec{c}_0}.$$

If we everywhere replace c_0 by l_∞, then we obtain the corresponding functor

$$G_\varphi^\infty = \mathrm{Orb}_{l_\infty(\{1/\varphi(2^n)\})}^{\vec{l}_\infty}.$$

In the sequel we say that $\vec{X} = (X_0, X_1)$ is a *Banach couple of lattices* if it is a Banach couple of functional lattices defined on a measure space (Ω, μ). Let $\varphi \in P$, put

$$\varphi(s,t) = s\varphi(t/s) \quad \text{and} \quad \varphi(s,t) = 0 \text{ if } s = 0 \text{ or } t = 0,$$

and consider (see [7], [18]) the *Calderón–Lozanovskiĭ space* $\varphi(\vec{X})$ equipped with the norm

$$\|f\|_{\varphi(\vec{X})} = \inf \max(\|f_0\|_{X_0}, \|f_1\|_{X_1}),$$

where inf is taken over all representations of $|f|$ in the form $|f| = \varphi(|f_0|, |f_1|)$, $f_i \in X_i$, $i = 0, 1$.

For example, for the couples $\vec{c}_0, \vec{l}_\infty$ we have *with equality of norms*

$$\varphi(\vec{c}_0) = c_0(\{1/\varphi(2^n)\}) \quad \text{and} \quad \varphi(\vec{l}_\infty) = l_\infty(\{1/\varphi(2^n)\}).$$

Our main interpolation result reads:

THEOREM 5. *If $\varphi \in P_0$, then, for any couple of Banach lattices $\vec{X}$,*

(a) $G^0_\varphi(\vec{X}) = \varphi(\vec{X})^0$, *and*

(b) $[G^0_\varphi(\vec{X})]^c = [\varphi(\vec{X})]^c$.

The constants in the equivalence of norms of the spaces in both (a) *and* (b) *do not depend on $\vec{X}$ and φ.*

Remark. For the special case when $\varphi \in P_0 \cap P_1$ and $\vec{X}$ is a regular couple of lattices this theorem was proved by Nilsson [22].

Our idea of proof is taken from [15] (see also [6]) and for the reader's convenience we begin by stating the following lemmas, which are of independent interest:

LEMMA 2. *If $\varphi \in P_0$, then, for any couple of Banach lattices $\vec{X}$,*

$$G^0_\varphi(\vec{X}) \overset{(1+\sqrt{2})^2}{\subset} \varphi(\vec{X}).$$

Remark. Another proof (without estimate of the imbedding constant) was given by Ovchinnikov [25].

LEMMA 3. *If $\varphi \in P_0$, then, for any couple of Banach lattices $\vec{X}$,*

$$\varphi(\vec{X}) \overset{2}{\subset} G^\infty_\varphi(\vec{X}).$$

LEMMA 4. *If $\varphi \in P_0$, then, for any Banach couple $\vec{X}$,*

(i) $G^\infty_\varphi(\vec{X})^0 \overset{5}{\subset} G^0_\varphi(\vec{X}) \overset{1}{\subset} G^\infty_\varphi(\vec{X})^0$, *and*

(ii) $G^\infty_\varphi(\vec{X}) \overset{1}{\subset} [G^0_\varphi(\vec{X})]^c$.

Remark. For $\varphi \in P_0 \cap P_1$, (i) was proved by Janson [13] (see also [16]).

In the proof of Lemma 2 we use our block version of Carlson's inequality in a crucial way. We postpone the proofs of the lemmas until the next section.

Proof of Theorem 5. (a) By using Lemmas 2–4 and $G^0_\varphi(\vec{X}) \equiv G^0_\varphi(\vec{X})^0$ (this equality with equal norms follows from Lemma 3(i)) we find

that

$$\varphi(\vec{X})^0 \overset{2}{\subset} G^\infty_\varphi(\vec{X})^0 \overset{5}{\subset} G^0_\varphi(\vec{X}) \equiv G^0_\varphi(\vec{X})^0 \overset{(1+\sqrt{2})^2}{\subset} \varphi(\vec{X})^0,$$

and, in particular, $G^0_\varphi(\vec{X}) = \varphi(\vec{X})^0$.

(b) According to Lemmas 2, 3 and 4(ii) we have

$$G^0_\varphi(\vec{X}) \overset{(1+\sqrt{2})^2}{\subset} \varphi(\vec{X}) \overset{2}{\subset} G^\infty_\varphi(\vec{X}) \overset{1}{\subset} [G^0_\varphi(\vec{X})]^c,$$

which yields that $[G^0_\varphi(\vec{X})]^c = [\varphi(\vec{X})]^c$.

We close this section by stating the following interpolation result:

COROLLARY. *If $\varphi \in P_0$, then $\varphi(\cdot)^0$ and $\varphi(\cdot)^c$ are interpolation functors on the category of couples of Banach lattices with interpolation constants $\le 10(1+\sqrt{2})^2$ and $\le 2(1+\sqrt{2})^2$, respectively.*

6. Proofs of the lemmas

Proof of Lemma 2. *First step.* It is sufficient to prove that for any interval $(a, b) \subset \mathbb{R}_+$ and $T : \vec{c}_0 \to \vec{X}$ we have, for $\varphi\chi_{(a,b)} = \sum \varphi\chi_{(a,b)}(2^n)e_n$,

$$\|T(\varphi\chi_{(a,b)})\|_{\varphi(\vec{X})} \le (1+\sqrt{2})^2 \|T\|_{L(\vec{c}_0,\vec{X})}. \tag{6.1}$$

In fact, since $\varphi(\vec{c}_0) = c_0(\{1/\varphi(2^n)\})$ it follows from (6.1) that, for any $f \in \varphi(\vec{c}_0)$ with finite support, we have

$$\|T(f)\|_{\varphi(\vec{X})} \le (1+\sqrt{2})^2 \|T\|_{L(\vec{c}_0,\vec{X})} \|f\|_{\varphi(\vec{c}_0)}. \tag{6.2}$$

If $f \in \varphi(\vec{c}_0)$ does not have a finite support, then f is a sum $f = \sum_{n=1}^\infty f_n$ of functions with finite support and such that

$$\sum_{n=1}^\infty \|f_n\|_{\varphi(\vec{c}_0)} \le (1+\varepsilon)\|f\|_{\varphi(\vec{c}_0)}, \quad \varepsilon > 0.$$

The series $\sum T f_n$ is absolutely convergent to Tf in $\Sigma(\vec{X})$ and, according to (6.2), it is also absolutely convergent in $\varphi(\vec{X})$ so we conclude that it is absolutely convergent to Tf in $\varphi(\vec{X})$. Moreover,

$$\begin{aligned}\|Tf\|_{\varphi(\vec{X})} &\le (1+\sqrt{2})^2 \|T\|_{L(\vec{c}_0,\vec{X})} \Big(\sum_{n=1}^\infty \|f_n\|_{\varphi(\vec{c}_0)}\Big) \\ &\le (1+\varepsilon)(1+\sqrt{2})^2 \|T\|_{L(\vec{c}_0,\vec{X})} \|f\|_{\varphi(\vec{c}_0)}.\end{aligned}$$

Letting $\varepsilon \to 0^+$ we find that (6.2) holds for *all* $f \in \varphi(\vec{c}_0)$. Now, if $f \in G^0_\varphi(\vec{X})$, then $f=\sum_{n=1}^{\infty}T_nf_n$ and

$$\|f\|_{\varphi(\vec{X})}=\Big\|\sum_{n=1}^{\infty}T_nf_n\Big\|_{\varphi(\vec{X})}\le\sum_{n=1}^{\infty}\|T_nf_n\|_{\varphi(\vec{X})}$$
$$\le(1+\sqrt{2})^2\sum_{n=1}^{\infty}\|T_n\|_{L(\vec{c}_0,\vec{X})}\|f_n\|_{\varphi(\vec{c}_0)},$$

i.e., $\|f\|_{\varphi(\vec{X})}\le(1+\sqrt{2})^2\|f\|_{G^0_\varphi(\vec{X})}$.

Second step. Let $\widetilde{\varphi}=\varphi\chi_{(a,b)}$, $0<a<b<\infty$, and consider, for $\chi_n=[t_{2n},t_{2n+2})$ (see Section 1),

$$g_0=\max_n\frac{|T(\widetilde{\varphi}\chi_n)|}{\varphi(t_{2n+1})}\quad\text{and}\quad g_1=\max_n\frac{t_{2n+1}}{\varphi(t_{2n+1})}|T(\widetilde{\varphi}\chi_n)|,$$

where, by abuse of notation, we also write χ_n for the characteristic function of the interval χ_n. We will prove that

$$\|g_i\|_{X_i}\le q\|T\|_{L(\vec{c}_0,\vec{X})},\quad i=0,1. \tag{6.3}$$

Since $\widetilde{\varphi}$ has finite support and $t_n\to0$ as $n\to-\infty$ and $t_n\to+\infty$ as $n\to+\infty$ it follows that the maximum in both expressions defining g_0 and g_1 can be taken over only a finite number of indices, say N. According to the well-known inequality

$$\max_{n=1,\dots,N}|x_n|\le2^{-N}\sum_{\varepsilon_n=\pm1}\Big|\sum_{n=1}^{N}\varepsilon_nx_n\Big|$$

we have

$$g_0\le2^{-N}\sum_{\varepsilon_n=\pm1}\Big|\sum_{n=1}^{N}\frac{\varepsilon_n}{\varphi(t_{2n+1})}T(\widetilde{\varphi}\chi_n)\Big|$$
$$=2^{-N}\sum_{\varepsilon_n=\pm1}\Big|T\Big(\sum_{n=1}^{N}\frac{\varepsilon_n}{\varphi(t_{2n+1})}\widetilde{\varphi}\chi_n\Big)\Big|$$

and, thus,

$$\|g_0\|_{X_0}\le2^{-N}\sum_{\varepsilon_n=\pm1}\|T\|_{L(\vec{c}_0,\vec{X})}\Big\|\sum_{n=1}^{N}\frac{\varepsilon_n}{\varphi(t_{2n+1})}\widetilde{\varphi}\chi_n\Big\|_{c_0}.$$

Since the $\widetilde{\varphi}\chi_n$ for different n have disjoint supports and $|\varphi\chi_n|\le q\varphi(t_{2n+1})$ it follows that $\|g_0\|_{X_0}\le q\|T\|_{L(\vec{c}_0,\vec{X})}$. Similarly we find that $\|g_1\|_{X_1}\le q\|T\|_{L(\vec{c}_0,\vec{X})}$, and (6.3) is proved.

Third step. According to our generalized Carlson inequality (in the form (2.3)) we have

$$|T\widetilde{\varphi}|\le\sum_n|T(\widetilde{\varphi}\chi_n)|$$
$$=\sum_n\varphi\Big(\frac{|T(\widetilde{\varphi}\chi_n)|}{\varphi(t_{2n+1})},\frac{t_{2n+1}}{\varphi(t_{2n+1})}|T(\widetilde{\varphi}\chi_n)|\Big)\le\frac{q+1}{q-1}\varphi(g_0,g_1).$$

Therefore $T\widetilde{\varphi}\in\varphi(\vec{X})$ and, in view of (6.3), we conclude that

$$\|T\widetilde{\varphi}\|_{\varphi(\vec{X})}\le\frac{q(q+1)}{q-1}\|T\|_{L(\vec{c}_0,\vec{X})}.$$

By taking infimum over $q>1$ we obtain the estimate (6.1) and the proof is complete.

Remark. In the proof of the first step we did not at all use the structure of $\vec{X}$ and $\varphi(\vec{X})$.

Proof of Lemma 3. The proof of this lemma is standard and well-known (see e.g. [25, Lemma 8.2.1]) but for the reader's convenience we present the idea of the proof: Let $\|f\|_{\varphi(\vec{X})}<1$, i.e. that there exist $f_0\in X_0$ and $f_1\in X_1$ such that $|f|\le\varphi(|f_0|,|f_1|)$ and $\|f_i\|_{X_i}\le1$, $i=0,1$. Let $\Omega_n=\{\omega\in\Omega:2^n\le|f_1(\omega)|/|f_0(\omega)|<2^{n+1}\}$, consider the mapping $A:\vec{l}_\infty\to\vec{X}$ such that $Ae_n=f\chi_{\Omega_n}$ and make an extension to $\Sigma(\vec{l}_\infty)$ formally defined by $A(\sum\lambda_ne_n)=\sum\lambda_nAe_n$. Then $A\varphi=f$ (here it is important that $\varphi\in P_0$) and $\|A\|_{L(\vec{l}_\infty,\vec{X})}\le2$.

Proof of Lemma 4. (i) The second imbedding is proved in the following way: If $\|f\|_{\varphi(\vec{c}_0)}\le1$, then the operator $A_f:\vec{l}_\infty\to\vec{c}_0$ defined by $A_f(e_n)=f(2^n)/\varphi(2^n)$ has the norm $\|A_f\|_{L(\vec{l}_\infty,\vec{c}_0)}\le1$. Therefore, if $\|x\|_{G^0_\varphi}<1$, then x has a representation $x=\sum_{n=1}^{\infty}T_nf_n$ with

$$\sum_{n=1}^{\infty}\|T_n\|_{L(\vec{c}_0,\vec{X})}\|f_n\|_{\varphi(\vec{c}_0)}<1,$$

and, thus, $x=\sum_{n=1}^{\infty}T_nA_{f_n}\varphi$ with

$$\sum_{n=1}^{\infty}\|T_nA_{f_n}\|_{L(\vec{l}_\infty,\vec{X})}\|\varphi\|_{l_\infty(\{1/\varphi(2^n)\})}<1,$$

and the second imbedding is proved.

In order to prove the first imbedding it is sufficient to show that

$$\|x\|_{G^0_\varphi(\vec{X})}\le5\|x\|_{G^\infty_\varphi(\vec{X})}\quad\text{for }x\in\Delta(\vec{X}). \tag{6.4}$$

Let $x\in\Delta(\vec{X})$ and $\|x\|_{G^\infty_\varphi(\vec{X})}<1$. Then $x=A\varphi$ for some $A:\vec{l}_\infty\to\vec{X}$ with $\|A\|_{L(\vec{l}_\infty,\vec{X})}<1$. We have

$$x=A(\varphi\chi_{(-\infty,-N)})+A(\varphi\chi_{[-N,N]})+A(\varphi\chi_{(N,+\infty)}).$$

First we note that $\|\varphi\chi_{[-N,N]}\|_{\varphi(\vec{c}_0)} \le 1$ and since the restriction of A to $\vec{c}_0$ is an operator with norm < 1 it follows that

$$\|A(\varphi\chi_{[-N,N]})\|_{G^0_\varphi(\vec{X})} < 1\,.$$

Therefore it is sufficient to prove that for every $\varepsilon > 0$ and for sufficiently large N, we have the estimates

$$\|A(\varphi\chi_{(-\infty,-N)})\|_{G^0_\varphi(\vec{X})} \le 2+\varepsilon \quad \text{and} \quad \|A(\varphi\chi_{(N,\infty)})\|_{G^0_\varphi(\vec{X})} \le 2+\varepsilon\,.$$

The proofs of these estimates are quite similar so we only prove the second one. Consider an operator $B : \vec{c}_0 \to \vec{X}$ such that $Be_n = 0$ if $n \ne N$ and for $n = N$ we have $B(\varphi(2^N)e_N) = A(\varphi\chi_{(N,\infty)})$. We have to prove that for large N

$$\|B\|_{L(\vec{c},\vec{X})} \le 2+\varepsilon\,,$$

i.e.

$$\begin{aligned} \|A(\varphi\chi_{(N,\infty)})\|_{X_0} &\le (2+\varepsilon)\varphi(2^N)\,,\\ \|A(\varphi\chi_{(N,\infty)})\|_{X_1} &\le (2+\varepsilon)2^{-N}\varphi(2^N)\,. \end{aligned} \tag{6.5}$$

Since φ is concave and $\|A\|_{L(\vec{l}_\infty,\vec{X})} < 1$ it follows that

$$\|A(\varphi\chi_{(N,\infty)})\|_{X_1} \le \|\varphi\chi_{(N,\infty)}\|_{l_\infty(\{2^{-N}\})} \le 2^{-N}\varphi(2^N)\,,$$

and the second inequality in (6.5) holds (even with constant 1). For the proof of the first, we consider the following cases:

1°. $\lim_{N\to\infty}\varphi(2^N) = +\infty$. We choose N so large that $\|A\varphi\|_{X_0} < \varphi(2^N)$ and find that

$$\|A(\varphi\chi_{(N,\infty)})\|_{X_0} \le \|A\varphi\|_{X_0} + \|A(\varphi\chi_{(-\infty,N)})\|_{X_0} < 2\varphi(2^N)\,.$$

2°. $\lim_{N\to\infty}\varphi(2^N) = C < \infty$. Now this yields that $\|\varphi\|_{l_\infty} = C$ and $\|\varphi\chi_{(-\infty,N)}\|_{l_\infty} \le C$. Therefore

$$\|A(\varphi\chi_{(N,\infty)})\|_{X_0} \le \|A\varphi\|_{X_0} + \|A(\varphi\chi_{(-\infty,N)})\|_{X_0} < 2C$$

and for large N we also have $2C < (2+\varepsilon)\varphi(2^N)$. Therefore (6.5) holds and, thus, (6.4) follows, which means that (i) is proved.

(ii) follows from the minimality of G^∞_φ in the couple $\vec{l}_\infty$ and from the fact that $G^0_\varphi(\vec{X})$ is an exact interpolation functor whose Gagliardo closure on the pair $\vec{l}_\infty$ contains φ (because for the imbedding operator $A : \vec{c}_0 \to \vec{l}_\infty$ we have $A(\varphi(\vec{c}_0)) = c_0(\{1/\varphi(2^n)\})$ with Gagliardo closure $l_\infty(\{1/\varphi(2^n)\})$).

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Characterizing translation invariant projections on Sobolev spaces on tori by the coset ring and Paley projections

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Abstract. We characterize those anisotropic Sobolev spaces on tori in the L^1 and uniform norms for which the idempotent multipliers have a description in terms of the coset ring of the dual group. These results are deduced from more general theorems concerning invariant projections on vector-valued function spaces on tori. This paper is a continuation of the author's earlier paper [W].

Introduction. In the present paper we study the translation invariant projections on the anisotropic Sobolev spaces $L^1_S(\mathbb{T}^d)$ and $C_S(\mathbb{T}^d)$ on the d-dimensional torus. Here S, called a *smoothness*, is a finite set of points of $\mathbb{R}^d$ with nonnegative integer coordinates containing the origin corresponding in an obvious way to a finite set of partial derivatives. The space $L^p_S(\mathbb{T}^d)$ is the completion of the trigonometric polynomials on the d-dimensional torus with respect to the norm

$$\|f\|_{S,p} = \Big(\int_{\mathbb{T}^d} \Big(\sum_{\alpha\in S} |D^\alpha f(x)|^2\Big)^{p/2} dx\Big)^{1/p}$$

where the integral is taken against the normalized Haar measure on $\mathbb{T}^d$, and the space $C_S(\mathbb{T}^d)$ is the completion of the trigonometric polynomials with respect to the norm

$$\|f\|_{S,\infty} = \sup_{x\in\mathbb{T}^d} \Big(\sum_{\alpha\in S} |D^\alpha f(x)|^2\Big)^{1/2}.$$

It is known (cf. [W]) that for some class of smoothnesses including the classical isotropic case the family of the supports of the multipliers of translation invariant projections on $L^1_S(\mathbb{T}^d)$ coincides with the coset ring of $\mathbb{Z}^d$ (denoted by $\operatorname{coset}(\mathbb{Z}^d)$), i.e. with the boolean ring generated by the cosets of all

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