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DEPARTMENT OF MATHEMATICS
YAROSLAVL STATE UNIVERSITY
SOVETSKAYA 14
150 000 YAROSLAVL, RUSSIA

DEPARTMENT OF MATHEMATICS
LULEÅ UNIVERSITY
S-951 87 LULEÅ, SWEDEN

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Characterizing translation invariant projections on Sobolev spaces on tori by the coset ring and Paley projections

by

M. WOJCIECHOWSKI (Warszawa)

Abstract. We characterize those anisotropic Sobolev spaces on tori in the L^1 and uniform norms for which the idempotent multipliers have a description in terms of the coset ring of the dual group. These results are deduced from more general theorems concerning invariant projections on vector-valued function spaces on tori. This paper is a continuation of the author's earlier paper [W].

Introduction. In the present paper we study the translation invariant projections on the anisotropic Sobolev spaces $L^1_S(\mathbb{T}^d)$ and $C_S(\mathbb{T}^d)$ on the d-dimensional torus. Here S, called a *smoothness*, is a finite set of points of $\mathbb{R}^d$ with nonnegative integer coordinates containing the origin corresponding in an obvious way to a finite set of partial derivatives. The space $L^p_S(\mathbb{T}^d)$ is the completion of the trigonometric polynomials on the d-dimensional torus with respect to the norm

$$\|f\|_{S,p} = \Big(\int_{\mathbb{T}^d} \Big(\sum_{\alpha\in S} |D^\alpha f(x)|^2\Big)^{p/2} dx\Big)^{1/p}$$

where the integral is taken against the normalized Haar measure on $\mathbb{T}^d$, and the space $C_S(\mathbb{T}^d)$ is the completion of the trigonometric polynomials with respect to the norm

$$\|f\|_{S,\infty} = \sup_{x\in\mathbb{T}^d} \Big(\sum_{\alpha\in S} |D^\alpha f(x)|^2\Big)^{1/2}.$$

It is known (cf. [W]) that for some class of smoothnesses including the classical isotropic case the family of the supports of the multipliers of translation invariant projections on $L^1_S(\mathbb{T}^d)$ coincides with the coset ring of $\mathbb{Z}^d$ (denoted by $\mathrm{coset}(\mathbb{Z}^d)$), i.e. with the boolean ring generated by the cosets of all

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subgroups of $\mathbb{Z}^d$. On the other hand, such a description of translation invariant projections in terms of the coset ring does not extend to all smoothnesses (cf. [P-W1]).

Our main result, Theorem 5, says that in fact a dichotomy holds: either the translation invariant projections on $L^1_S(\mathbb{T}^d)$ are characterized by coset$(\mathbb{Z}^d)$ or there exists a Paley projection on $L^1_S(\mathbb{T}^d)$, i.e. a projection onto an infinite-dimensional hilbertian subspace of $L^1_S(\mathbb{T}^d)$.

We also study the translation invariant projections on the spaces $C_S(\mathbb{T}^d)$ and on polydisc algebras and we prove that they are always determined by the coset ring. This implies that every translation invariant projection on $C_S(\mathbb{T}^d)$ uniquely extends to a translation invariant projection on $C(\mathbb{T}^d)$ (the space of continuous functions on the torus). A similar fact for $L^p_S(\mathbb{T}^d)$ for $1 < p < \infty$ is proven in [P-W2].

Our results for Sobolev spaces are derived from similar results concerning translation invariant projections on certain translation invariant subspaces of the spaces $L^1(\mathbb{T}^d, E)$ and $C(\mathbb{T}^d, E)$ of E-valued functions on $\mathbb{T}^d$. Here E is a finite-dimensional complex Hilbert space.

The present paper is a continuation of [W] where a description of translation invariant projections on $L^1_S(\mathbb{T}^d)$ is given in a particular case (for S elliptic). The proofs in this paper are modifications of those of [W].

The paper consists of 4 sections. Section 1 contains preliminaries. We recall several notions from [W] and introduce some new properties of translation invariant subspaces of vector-valued function spaces. The cases of L^1 and uniform norm are treated in Sections 2 and 3 respectively. Section 4 is devoted to applications to Sobolev spaces.

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1. Preliminaries. Let us recall several definitions from [W]. $\mathbb{T}^d$ stands for the d-dimensional torus group, and $\mathbb{Z}^d$ for its dual group, i.e. the lattice in $\mathbb{R}^d$ consisting of points with integer coordinates. By $\mathbb{Z}^d_+$ we denote the set of points of $\mathbb{Z}^d$ with nonnegative coordinates. A linear manifold in $\mathbb{R}^d$ is called a *hyperplane*. We call a $(d-1)$-dimensional hyperplane in $\mathbb{R}^d$ *rational* if it is perpendicular to some nonzero vector with integer coordinates. For every hyperplane $H \subset \mathbb{R}^d$ passing through the origin the intersection $\mathbb{Z}^d \cap H$ is a subgroup of $\mathbb{Z}^d$. This subgroup is isomorphic to $\mathbb{Z}^{d'}$ for some $d' = 0, 1, \ldots, d-1$ and we will regard it as $\mathbb{Z}^{d'}$.

A set $A \subset \mathbb{Z}^d$ is called *essentially periodic* with *essential period* $\varrho \in \mathbb{Z}^d$, $\varrho^{(j)} \neq 0$ for $j = 1, \ldots, d$, and *exceptional family* $H_1, \ldots, H_k$ of $(d-1)$-dimensional hyperplanes if there exists $B \subset \mathbb{Z}^d$ such that the symmetric difference

$$A \div B \subset \sum_{j=1}^{k} H_j$$

where B is a periodic set of period $\varrho = (\varrho^{(j)})_{j=1}^d$, i.e.

$$B + (0, \ldots, 0, \varrho^{(j)}, 0, \ldots, 0) = B \quad \text{for } j = 1, \ldots, d.$$

Let E be an arbitrary finite-dimensional complex Hilbert space with norm $|\cdot|_E$. Then $L^1(\mathbb{T}^d, E)$ denotes the space of equivalence classes of E-valued functions on $\mathbb{T}^d$ absolutely summable with respect to the Haar measure with the norm

$$\|f\| = \int_{\mathbb{T}^d} |f(x)|_E \, dx .$$

By $G(E, 1)$ we denote the Grassmannian of one-dimensional subspaces of E and by $d(\cdot, \cdot)$ the usual metric on $G(E, 1)$, i.e. $d(X, Y) =$ the Hausdorff distance of the sets $X \cap B_E(0, 1)$ and $Y \cap B_E(0, 1)$.

A *one-dimensional bundle* (or briefly a *bundle*) is a function $\psi : \mathbb{Z}^d \to G(E, 1)$. By $L^1_\psi(\mathbb{T}^d)$ (or L^1_ψ for short) we denote the closed linear subspace of the space $L^1(\mathbb{T}^d, E)$ generated by the set $\{xe^{2\pi i \langle \gamma, t \rangle} : \gamma \in \mathbb{Z}^d,\ x \in \psi(\gamma)\}$. Replacing the L^1 norm by the sup norm we define similarly the space $C(\mathbb{T}^d, E)$ and its subspace $C_\psi(\mathbb{T}^d)$.

Given any translation invariant operator $P : L^1_\psi(\mathbb{T}^d) \to L^1_\psi(\mathbb{T}^d)$ the corresponding *multiplier* $\widehat{P}$ is a function from $\mathbb{Z}^d$ into the complex numbers such that $P(x_\gamma e^{2\pi i \langle \gamma, \cdot \rangle}) = \widehat{P}(\gamma) x_\gamma e^{2\pi i \langle \gamma, \cdot \rangle}$ for every $\gamma \in \mathbb{Z}^d$ and $x_\gamma \in \psi(\gamma)$ (cf. [W]). If P is a translation invariant projection then $\widehat{P} : \mathbb{Z}^d \to \{0, 1\}$. Hence every translation invariant projection P on $L^1_\psi(\mathbb{T}^d)$ corresponds to some subset of $\mathbb{Z}^d$, namely to the *support* of $\widehat{P}$ $(= \{\gamma \in \mathbb{Z}^d : \widehat{P}(\gamma) \neq 0\})$.

For any hyperplane $H \subset \mathbb{R}^d$ passing through the origin, $\psi_{|H}$ is the restriction of a bundle ψ to $H \cap \mathbb{Z}^d$. A set $F \subset \mathbb{Z}^d$ is called ε-*stable* for a bundle ψ if $d(\psi(\gamma_1), \psi(\gamma_2)) < \varepsilon$ for any $\gamma_1, \gamma_2 \in F$. Recall (cf. [W]) that a bundle ψ is called *stable* if for every $m > 0$ and $\varepsilon > 0$ there exists $M > 0$ such that $|\gamma| > M$ implies that the ball $B(\gamma, m)$ is ε-stable. A bundle ψ is called *asymptotically symmetric* if for every $\varepsilon > 0$ there exists $M > 0$ such that if $|\gamma| > M$ then the set $\{\gamma, -\gamma\}$ is ε-stable.

Now we introduce certain properties of a bundle which we use in this paper. For any bundle ψ on $\mathbb{Z}^d$ and finite family $\mathcal{H}$ of $(d-1)$-dimensional rational hyperplanes we will say "ψ is stab$(\mathcal{H})$" provided for every $\varepsilon > 0$ and $m > 0$ there exists $M = M(\varepsilon, m) > 0$ such that $\min_{H \in \mathcal{H}} \operatorname{dist}(\gamma, H) > M$ and $|\gamma| > M$ implies that $B(\gamma, m)$ is ε-stable. Similarly "ψ is sym$(\mathcal{H})$" provided for every $\varepsilon > 0$ there exists $M = M(\varepsilon) > 0$ such that $\min_{H \in \mathcal{H}} \operatorname{dist}(\gamma, H) > M$ and $|\gamma| > M$ implies that $\{\gamma, -\gamma\}$ is ε-stable.

Now we define inductively two classes of bundles: $\mathbf{S}$ and $\mathbf{SS}$.

DEFINITION. If ψ is a bundle on $\mathbb{Z}$ then we say $\psi \in \mathbf{S}$ (resp. $\psi \in \mathbf{SS}$) if ψ is stable (resp. ψ is stable and asymptotically symmetric). For $d > 1$ a bundle ψ on $\mathbb{Z}^d$ belongs to $\mathbf{S}$ (resp. $\mathbf{SS}$) if there exists a finite family $\mathcal{H}$ of $(d-1)$-dimensional rational pairwise nonparallel hyperplanes such that ψ is $\operatorname{stab}(\mathcal{H})$ (resp. ψ is $\operatorname{stab}(\mathcal{H})$ and ψ is $\operatorname{sym}(\mathcal{H})$) and for every $H \in \mathcal{H}$ the bundle $\psi_{|H} \in \mathbf{S}$ ($\psi_{|H} \in \mathbf{SS}$).

($\mathbf{S}$ stands for *almost stable* and $\mathbf{SS}$ stands for *almost stable and weakly symmetric.*)

We will also use yet another property of a bundle:

DEFINITION. A bundle $\psi : \mathbb{Z}^d \to G(E, 1)$ is called *shiftable* if for every $\gamma \in \mathbb{Z}^d$ there exists $x_\gamma \in \psi(\gamma)$ such that for every $\beta \in \mathbb{Z}^d$ the operator given by

$$T_\beta\Big(\sum a_\gamma x_\gamma e^{2\pi i\langle\gamma,\cdot\rangle}\Big) = \sum a_{\gamma+\beta} x_\gamma e^{2\pi i\langle\gamma,\cdot\rangle}$$

is bounded simultaneously on $L^1_\psi(\mathbb{T}^d)$ and on $C_\psi(\mathbb{T}^d)$.

Note that if a bundle ψ on $\mathbb{Z}^d$ is shiftable then $\psi_{|H}$ is also shiftable for every hyperplane $H \subset \mathbb{Z}^d$ passing through the origin.

The Sobolev space $L^1_S(\mathbb{T}^d)$ can be identified with the space $L^1_\psi(\mathbb{T}^d)$ for an appropriate bundle ψ. For every partial derivative $D = \partial_1^{\alpha_1} \ldots \partial_d^{\alpha_d}$ we denote by $\widehat{D}$ the *symbol* of D, i.e. the polynomial on $\mathbb{R}^d$ given by $\widehat{D}(\xi) = (i\xi^{(1)})^{\alpha_1} \ldots (i\xi^{(d)})^{\alpha_d}$. For any smoothness S the *fundamental polynomial* Q_S is defined as

$$Q_S(\xi) = \sum_{D\in S} |\widehat{D}(\xi)|^2 .$$

With a d-dimensional smoothness S we associate the bundle $\psi_S : \mathbb{Z}^d \to G(E, 1)$ defined as follows. For $\gamma \in \mathbb{Z}^d$ we put $\psi_S(\gamma) = \operatorname{span}\{x_\gamma\}$ where $x_\gamma = (\widehat{D}(\gamma)/Q_S(\gamma)^{1/2})_{D\in S} \in E$. Here E is a complex Hilbert space of dimension $\operatorname{card} S$. Then the Sobolev space $L^1_S(\mathbb{T}^d)$ (resp. $C_S(\mathbb{T}^d)$) is invariantly and isometrically isomorphic to $L^1_{\psi_S}(\mathbb{T}^d)$ (resp. $C_{\psi_S}(\mathbb{T}^d)$) (for details cf. [W]).

$A(\mathbb{D}^d)$ stands for the polydisc algebra, i.e. the subspace of $C(\mathbb{T}^d)$ consisting of functions whose Fourier transforms are supported by $\mathbb{Z}^d_+$.

2. The main result. The main technical result of the present paper is the following

THEOREM 1. *If $\psi \in \mathbf{SS}$ is a shiftable bundle on $\mathbb{Z}^d$ and $P : L^1_\psi \to L^1_\psi$ is a translation invariant projection then* $\operatorname{supp} \widehat{P} \in \operatorname{coset}(\mathbb{Z}^d)$.

In order to prove Theorem 1 we need some lemmas. First observe that all technical lemmas of Section 2 of [W] are true if we replace the stable bundle ψ by a bundle $\psi \in \mathbf{S}$ and an arbitrary unbounded sequence $(\alpha'_n)_{n=1}^\infty \subset \mathbb{Z}^d$ by one for which the sequence $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha'_n, H)$ is unbounded. Hence we have the following three lemmas:

LEMMA 1. *If $F \subset \mathbb{Z}^d$ is an n-element set which is $1/(3n)$-stable for the bundle ψ then there exists a translation invariant isomorphism $H : L^1_{|F} \to L_{\psi|F}$ with $\|H\| \cdot \|H^{-1}\| \le 2$.* ■

LEMMA 2. *Let $\psi \in \mathbf{S}$ and let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection. Then each sequence $(\alpha'_n)_{n=1}^\infty \subset \mathbb{Z}^d$ for which the sequence $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha'_n, H)$ is unbounded contains an unbounded subsequence $(\alpha_n)_{n=1}^\infty$ such that $\lim_{n\to\infty} \widehat{P}(\gamma+\alpha_n)$ exists for each $\gamma \in \mathbb{Z}^d$ and the formula*

$$\widehat{R}(\gamma) = \lim_{n\to\infty} \widehat{P}(\gamma+\alpha_n) \quad \text{for } \gamma \in \mathbb{Z}^d$$

determines a translation invariant projection $R : L^1(\mathbb{T}^d) \to L^1(\mathbb{T}^d)$. ■

LEMMA 3. *Let $Q : L^1_\psi \to L^1_\psi$ be a translation invariant projection ($\psi \in \mathbf{S}$ on $\mathbb{Z}^d$). Assume that Q satisfies either*

(i) *there exist $M_0 > 0$, a sequence $(x_k)_{k=1}^\infty \subset \mathbb{R}^d$ with $|x_k| = 1$ for $k = 1, 2, \ldots$ and a sequence of balls $(B(\alpha_k, r_k))_{k=1}^\infty$ with $(\alpha_k)_{k=1}^\infty \subset \mathbb{Z}^d$ for which the sequence $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha_k, H)$ is unbounded and $\lim r_k = \infty$ such that for $k = 1, 2, \ldots$*

$$\alpha_k \in \operatorname{supp} \widehat{Q} \cap B(\alpha_k, r_k) \subset \{z : \langle z - \alpha_k, x_k\rangle \le M_0\},$$

or

(ii) *there exists a sequence of balls $(B(a_k, s_k))_{k=1}^\infty$ with $\lim s_k = \infty$ and sequences $(a_k)_{k=1}^\infty \subset \mathbb{R}^d$ and $(\alpha_k)_{k=1}^\infty \subset \mathbb{Z}^d$ for which the sequence $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha_k, H)$ is unbounded and $|\alpha_k - a_k| = s_k$ for $k = 1, 2, \ldots$ such that for $k = 1, 2, \ldots$*

$$\widehat{Q}(\alpha_k) = 1 \quad \text{and} \quad \operatorname{supp} \widehat{Q} \cap B(a_k, s_k) = \emptyset .$$

Then there exist $M > 0$, $x \in \mathbb{Z}^d$ and a subsequence $(\beta_k)_{k=1}^\infty$ of $(\alpha_k)_{k=1}^\infty$ such that for $k = 1, 2, \ldots$

$$\beta_k \in \operatorname{supp} \widehat{Q} \cap B(\beta_k, k) \subset \{z : |\langle z - \beta_k, x\rangle| \le M\}. \text{ ■}$$

Next we prove the following

LEMMA 4. *Suppose that the assertion of Theorem 1 is valid for all $d' < d$ and all shiftable bundles $\phi \in \mathbf{SS}$ on $\mathbb{Z}^{d'}$. Let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection. Then $H \cap \operatorname{supp} \widehat{P} \in \operatorname{coset}(\mathbb{Z}^d)$ for every $(d-1)$-dimensional rational hyperplane $H \subset \mathbb{R}^d$.*

Proof. Define $R : L^1_{\psi|H+\gamma} \to L^1_{\psi|H+\gamma}$ by $R = T_\gamma \circ P \circ T_{-\gamma}$ where $\gamma \in \mathbb{Z}^d$ is chosen so that $0 \in H + \gamma$. Certainly R is a translation invariant projection and $\psi_{|H+\gamma} \in \mathbf{SS}$ and it is a shiftable bundle on $\mathbb{Z}^{d-1}$. Hence, from our assumption it follows that $\operatorname{supp}\widehat{R} \in \operatorname{coset}(\mathbb{Z}^d \cap H)$. To complete the proof observe that $H \cap \operatorname{supp}\widehat{P} = \operatorname{supp}\widehat{R} + \gamma$. ∎

We will denote by $\Im$ the family of all components of $\mathbb{R}^d - \bigcup \mathcal{H}$ which are not contained in any strip determined by two parallel rational hyperplanes.

The crucial lemma in the proof of Theorem 1 is the following

LEMMA 5. *For every $K \in \Im$ and every translation invariant projection $P : L^1_\psi \to L^1_\psi$ there exists a set $A \in \operatorname{coset}(\mathbb{Z}^d)$ such that $\operatorname{supp}\widehat{P} \cap K = A \cap K$. In particular, if $d = 1$ then $\operatorname{supp}\widehat{P} \in \operatorname{coset}(\mathbb{Z})$.*

Proof. The proof of Lemma 5 is similar to that of Theorem 1 of [W] and therefore we restrict ourselves to point out the modifications only. We use induction on the dimension. Assume the validity of the assertion of Lemma 5 for all integers d' with $0 \le d' \le d-1$ and for all $\psi \in \mathbf{SS}$ on $\mathbb{Z}^{d'}$.

First observe that the inductive hypothesis implies

($*'$) for every shiftable bundle $\psi \in \mathbf{SS}$ on $\mathbb{Z}^d$, for every translation invariant projection $P : L^1_\psi \to L^1_\psi$ and for every $(d-1)$-dimensional hyperplane H of $\mathbb{R}^d$ there exists a set $A \in \operatorname{coset}(\mathbb{Z}^d)$ such that $K \cap \operatorname{supp}\widehat{P} \cap H = K \cap A$.

To prove ($*'$) we simply apply Lemma 4.

Let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection for some $\psi \in \mathbf{SS}$. Assume to the contrary

(A′) $\operatorname{supp}\widehat{P} \cap K \ne A \cap K$ for every $A \in \operatorname{coset}(\mathbb{Z}^d)$.

The proof consists of 4 steps. The first is the proof of the implication (A′)⇒(B′) where (B′) is the following modification of (B) from [W]:

(B′) there exists a translation invariant projection $Q : L^1_\psi \to L^1_\psi$ such that $\operatorname{supp}\widehat{Q} \cap K \ne A \cap K$ for every $A \in \operatorname{coset}(\mathbb{Z}^d)$, and for some sequence of balls $(B_n)_{n=1}^\infty \subset K$ with unbounded sequence of radii, $\operatorname{supp}\widehat{Q} \cap B_n = \emptyset$ for $n = 1, 2, \ldots$

In this step we repeat the construction of step 1 from [W] beginning, instead of an arbitrary unbounded sequence $(\alpha'_n)_{n=1}^\infty \subset \mathbb{Z}^d$, from a sequence such that $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha'_n, H)$ is unbounded.

It follows from ($*'$) that one can assume without loss of generality that

($**'$) $\operatorname{supp}\widehat{Q} \cap K$ is not contained in a union of finitely many $(d-1)$-dimensional hyperplanes.

The second step is the proof of the implication (B′)⇒(C′) provided Q satisfies ($**'$). Here (C′) is the following modification of (C) from [W]:

(C′) for $n = 1, 2, \ldots$ there exists a ball $C_n = B(a_n, n)$ with $a_n \in \mathbb{R}^d$ and a point $\alpha_n \in \mathbb{Z}^d$ with $|\alpha_n - a_n| = n$ such that $\operatorname{supp}\widehat{Q} \cap C_n = \emptyset$, $\widehat{Q}(\alpha_n) = 1$ and $\min_{H\in\mathcal{H}} \operatorname{dist}(\alpha_n, H) \to \infty$. Moreover, $|\langle y_k, \alpha_n - a_i\rangle| \ge n$ for $i, k < n$ where $(y_k)_{k=1}^\infty$ is any enumeration of the set $\{\gamma/|\gamma| : \gamma \in \mathbb{Z}^d,\ \gamma \ne 0\}$.

To prove this implication it is enough to repeat the proof of step 2 from [W] letting now $\mathcal{A}$ be the family of those components of the set $K - \bigcup \mathcal{R}_n$ (the symbol $\mathcal{R}_n$ is defined in [W]) which are not contained in any strip determined by two parallel rational hyperplanes. For this "new" family the property (12) of [W] is true (the proof is the same).

The other two steps: the proofs of the implications (C′)⇒(D) and (D)⇒*contradiction*, are the same as in [W]. ∎

The Cartesian product of a $(d-1)$-dimensional ball contained in a $(d-1)$-dimensional hyperplane and the line (half-line) perpendicular to this hyperplane is called a *cylinder* (*half-cylinder*) and the diameter of the defining ball is called the *width* of the cylinder. If the line is rational (contains a nonzero vector with integer coordinates) this cylinder is called *rational*.

The elements K_1, K_2 of $\Im$ are said to be *opposite* provided there exists a rational cylinder with arbitrarily large width which contains two half-cylinders, one contained in K_1 and the other in K_2.

LEMMA 6. *Given $K', K'' \in \Im$ there exists a chain $\{K_1, \ldots, K_n\} \subset \Im$ such that $K_1 = K'$, $K_n = K''$ and K_i is opposite to K_{i+1} for $i = 1, \ldots, n-1$.*

Proof. Every $K \in \Im$ is defined by a system of inequalities:

$$\langle a_H, x\rangle > b_H \quad \text{for } H \in \mathcal{H} \tag{1}$$

where a_H is a suitable vector perpendicular to the hyperplane H and $b_H \in \mathbb{R}$.

Let us call $P \in \Im$ *antipodal* to K if it is defined by the system

$$\langle a_H, x\rangle < b_H \quad \text{for } H \in \mathcal{H}.$$

First we will prove that any two antipodal elements of $\Im$ are opposite. To do this, take any $r > 0$ and choose $\gamma \in \mathbb{Z}^d$ so that $\langle a_H, \gamma\rangle > 0$ for all $H \in \mathcal{H}$ (this is possible because the set K is nonempty). Then for every $b \in \mathbb{R}^d$ there exists $M(b)$ such that

$$\langle a_H, t\gamma + b\rangle > b_H \tag{2}$$

for $H \in \mathcal{H}$ and $t > M(b)$ and

$$\langle a_H, t\gamma + b\rangle < b_H \tag{3}$$

for $H \in \mathcal{H}$ and $t < -M(b)$. Since $B(0, r)$ is compact, we can find $M > 0$ such that (2) and (3) hold for all $b \in B(0, r)$ and $t > M$ or $t < -M$ respectively. Hence there is a rational cylinder of width $2r$ containing two half-cylinders, one of them contained in K and the other in the element of $\Im$ antipodal to K.

Let us call $K_1, K_2 \in \Im$ *neighboring* if there exists a $(d-1)$-dimensional cone contained in $\overline{K}_1 \cap \overline{K}_2$. Note that this definition is equivalent to the definition of the relation "$\sim$" from step 2 of Section 2 of [W] (because hyperplanes from $\mathcal{H}$ are pairwise not parallel). Hence it follows from property (12) of [W] that any two elements of $\Im$ can be joined by a chain of neighboring elements of $\Im$. Hence to prove Lemma 6 it is enough to show that for any two neighboring elements of $\Im$, say K and K', there exists $P \in \Im$ which is opposite to both K and K'. The element of $\Im$ antipodal to K is the one we need. Indeed, we have shown above that K and P are opposite so to end the proof it is enough to check that K' and P are opposite. If K is defined by a system (1) then there exists $H_0 \in \mathcal{H}$ such that K' is defined by

$$(4)\qquad \begin{cases} \langle a_H, x\rangle > b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, x\rangle < b_{H_0}. \end{cases}$$

Hence we have

$$\begin{cases} \langle a_H, x\rangle > b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, x\rangle = b_{H_0} \end{cases}$$

for $x \in \overline{K'} \cap H_0$ and

$$\begin{cases} \langle a_H, x\rangle < b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, x\rangle = b_{H_0} \end{cases}$$

for $x \in \overline{P} \cap H_0$. Since the restriction of $\mathcal{H}$ to H_0 gives a $(d-1)$-dimensional case of Lemma 6, the inductive hypothesis implies that there exists a rational line $\alpha + t\beta$ such that $\alpha + n\beta \in \mathbb{Z}^d \cap H_0$ for all $n \in \mathbb{Z}$ and

$$(5)\qquad \begin{cases} \langle a_H, \alpha + t\beta\rangle > b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, \alpha + t\beta\rangle = b_{H_0} \end{cases}$$

for sufficiently large positive t and

$$\begin{cases} \langle a_H, \alpha + t\beta\rangle < b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, \alpha + t\beta\rangle = b_{H_0} \end{cases}$$

for sufficiently large negative t. From (5) we derive that

$$\langle a_H, \beta\rangle > 0 \quad \text{for } H \in \mathcal{H} - \{H_0\},$$
$$\langle a_{H_0}, \beta\rangle = 0 \quad \text{and} \quad \langle a_{H_0}, \alpha\rangle = b_{H_0}.$$

Hence there exists a continuous positive function $N : \mathbb{R}^d \to \mathbb{R}$ such that for every γ belonging to the half-space $\langle a_{H_0}, x\rangle < b_{H_0}$, if $t > N(\gamma)$ then

$$(6)\qquad \begin{cases} \langle a_H, \gamma + t\beta\rangle > b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, \gamma + t\beta\rangle < b_{H_0}, \end{cases}$$

and if $t > N(\gamma)$ then

$$(7)\qquad \begin{cases} \langle a_H, \gamma + t\beta\rangle > b_H & \text{for } H \in \mathcal{H} - \{H_0\},\\ \langle a_{H_0}, \gamma + t\beta\rangle = b_{H_0}. \end{cases}$$

By a compactness argument for every ball B contained in the half-space $\langle a_{H_0}, x\rangle < b_{H_0}$ we can find N such that (6) and (7) hold for every $\gamma \in B$ and $t > N$ or $t < -N$ respectively. This means that K' and P are opposite. ■

Proof of Theorem 1. We use induction on d. The case $d = 1$ follows immediately from Theorem 1 of [W] because a one-dimensional bundle ψ belongs to **SS** iff it is stable and asymptotically symmetric. Assume the validity of the inductive hypothesis for all integers d' with $1 \le d' \le d-1$ and for all shiftable bundles $\psi \in \mathbf{SS}$ on $\mathbb{Z}^{d'}$. Let $P : L^1_\psi \to L^1_\psi$ be a translation invariant projection for some shiftable bundle $\psi \in \mathbf{SS}$ on $\mathbb{Z}^d$. Assume to the contrary that

$$(\mathrm{N})\qquad \operatorname{supp} \widehat{P} \notin \operatorname{coset}(\mathbb{Z}^d).$$

Lemma 5 implies that for every $K \in \Im$ there exists a set $A_K \in \operatorname{coset}(\mathbb{Z}^d)$ such that $\operatorname{supp} \widehat{P} \cap K = A_K \cap K$. Since each A_K is essentially periodic (cf. Fact 1 from [W]), there exists a finite family $\mathcal{G}$ of $(d-1)$-dimensional hyperplanes such that for every $K \in \Im$ there exists a periodic set B_K satisfying $\operatorname{supp} \widehat{P} \cap (K - \bigcup \mathcal{G}) = B_K \cap (K - \bigcup \mathcal{G})$. From Lemma 4 it follows that $\operatorname{supp} \widehat{P} \cap G \in \operatorname{coset}(\mathbb{Z}^d)$ for every $G \in \mathcal{G}$. Hence (N) implies that there exist $K', K'' \in \Im$ such that $B_{K'} \ne B_{K''}$. Using now Lemma 6 we deduce that there exists a chain $K' = K_1, K_2, \ldots, K_k = K''$ with K_i opposite to K_{i+1} for $i = 1, \ldots, k-1$. Hence there are opposite elements $K, K' \in \Im$ with

$$(8)\qquad B_K \ne B_{K'}.$$

Let us consider the operator $R : L^1_\psi \to L^1_\psi$ given by the formula

$$R = (P - T_K) \circ (P - T_K)$$

where T_K is the convolution with an idempotent measure satisfying $\operatorname{supp} \widehat{T}_K = A_K$. Certainly R is a translation invariant projection, $\operatorname{supp} \widehat{R} \cap K = \emptyset$ and $\operatorname{supp} \widehat{R} \cap K' = A_K \div A_{K'}$. The set $A_K \div A_{K'}$ is essentially periodic and, by (8), it has nonempty periodic part $(= B_K \div B_{K'})$. Hence there exists $r > 0$ such that for every rational cylinder C of width greater than r the set $\operatorname{supp} \widehat{R} \cap K' \cap C$ is infinite. Since K and K' are opposite we deduce that there exist a cylinder C and half-cylinders D and D' contained in C such that

$$\operatorname{supp} \widehat{R} \cap K' \cap D' \quad \text{is infinite}$$

and

$$\operatorname{supp} \widehat{R} \cap K' \cap D \quad \text{is empty.}$$

This implies that there exists a rational line L such that supp $\widehat{R} \cap L$ is infinite and for some half-line $M \subset L$ the intersection supp $\widehat{R} \cap M$ is empty. This means in particular that supp $\widehat{P} \cap L \notin \operatorname{coset}(\mathbb{Z})$. Hence because ψ is shiftable there exists a translation invariant projection R on L^1_ψ and a rational line L' with $0 \in L'$ such that supp $\widehat{R} \cap L' \notin \operatorname{coset}(\mathbb{Z})$. This contradicts Lemma 5 because the bundle $\psi_{|L'}$ is shiftable and belongs to **SS**, and $S = R_{|L^1_{\psi|L}}$ is a translation invariant operator acting on the space $L^1_{\psi|L}$. ∎

3. Translation invariant projections on $C_\psi(\mathbb{T}^d)$. The case of the space $C_\psi(\mathbb{T}^d)$ is simpler than that of $L^1_\psi(\mathbb{T}^d)$. It does not involve the symmetry of the bundle.

THEOREM 2. *If $\psi \in \mathbf{S}$ is a shiftable bundle and $P : C_\psi(\mathbb{T}^d) \to C_\psi(\mathbb{T}^d)$ is a translation invariant projection then* supp $\widehat{P} \in \operatorname{coset}(\mathbb{Z}^d)$.

Proof. Repeat the proof of Theorem 1 replacing the L^1 norm by the uniform norm except that the whole step 4 of Lemma 5 (where the weak symmetry was involved) must be replaced by the argument taken from Section 3 of [W] involving the Rudin–Shapiro construction (instead of Riesz products). ∎

Define now the bundle $\phi^d_+ : \mathbb{Z}^d \to G(\mathbb{C}^2, 1)$ by

$$\phi^d_+(\gamma) = \begin{cases} (1,0) & \text{for } \gamma \in \mathbb{Z}^d_+, \\ (0,1) & \text{otherwise.} \end{cases}$$

A bundle ψ will be called *ordered* if it is isomorphic to $\phi^d_{+|H}$ for some hyperplane $H \subset \mathbb{R}^d$ (not necessarily passing through the origin). More precisely, this means that there exists a hyperplane $H \subset \mathbb{R}^d$ passing through the origin, $\alpha \in \mathbb{Z}^d$ and an isomorphism $i : \mathbb{Z}^{d'} \to H \cap \mathbb{Z}^d$ such that $\psi(\gamma) = \phi^d_+(i(\gamma) + \alpha)$. Such bundles usually fail to be shiftable. Nevertheless the coset ring description of translation invariant projections holds for the space $C_{\phi^d_+}(\mathbb{T}^d)$.

THEOREM 3. *If ψ is an ordered bundle on $\mathbb{Z}^d$ and $P : C_\psi(\mathbb{T}^d) \to C_\psi(\mathbb{T}^d)$ is a translation invariant projection then* supp $\widehat{P} \in \operatorname{coset}(\mathbb{Z}^d)$.

Proof. Obviously $\psi \in \mathbf{S}$. The only place in the proof of Theorem 2 where the property that the bundle is shiftable is involved is checking that supp $\widehat{P} \cap H \in \operatorname{coset}(H \cap \mathbb{Z}^d)$ for every hyperplane $H \subset \mathbb{R}^d$ and every translation invariant projection $P : C_\psi \to C_\psi$. To prove this for an ordered bundle observe that $\psi_{|H}$ is isomorphic to some ordered bundle ψ' on $\mathbb{Z}^{d'}$ for some $d' < d$ and use the inductive hypothesis. Every one-dimensional ordered bundle is shiftable. ∎

4. Application to Sobolev spaces. We begin this section with an application to Sobolev spaces. First we recall the concept of odd and even smoothnesses (cf. [P-W1]).

DEFINITION. A smoothness S is called *odd* provided either there are $a, b \in S$ with $\sum_{j=1}^d a(j) \not\equiv \sum_{j=1}^d b(j) \bmod 2$ and a $(d-1)$-dimensional hyperplane $H \subset \mathbb{Z}^d$ given by the equation

$$H = \{x \in \mathbb{R}^d : \langle x, \beta\rangle = 1\} \quad \text{for some } \beta = (\beta(j)) \in \mathbb{R}^d$$

such that

(i) $\langle a, \beta\rangle = \langle b, \beta\rangle = 1$,
(ii) $\langle c, \beta\rangle \le 1$ for all $c \in S$,
(iii) $\beta(j) > 0$ for $j = 1, \ldots, d$,

or the same property holds for some lower dimensional smoothness which is the intersection of S with some coordinate plane. A smoothness is called *even* if it is not odd.

Applications of Theorem 1 base on the following

PROPOSITION 1. *If S is even then $\psi_S \in \mathbf{SS}$.*

Proof. We use induction on the dimension. It is clear that $\psi_S \in \mathbf{SS}$ for every one-dimensional smoothness S. Let us assume that we have already proved Proposition 1 for all d'-dimensional smoothnesses for $d' < d$. Let $\mathcal{H}$ be the family of all coordinate hyperplanes of the form $\{x \in \mathbb{R}^d : x(i) = 0\}$ for some $i \in \{1, \ldots, d\}$. Then ψ is stab$(\mathcal{H})$, by [P-W1, Proposition 1.1].

To prove that ψ is sym$(\mathcal{H})$ suppose that this is not true. Hence there exists a sequence (γ_n) satisfying $\lim_k \inf_j |\gamma_k(j)| = \infty$ such that $d(\psi(\gamma_n), \psi(-\gamma_n)) > C$ for $n = 1, 2, \ldots$ We have

$$d(\psi(\gamma_n), \psi(-\gamma_n)) = \min_{\varepsilon = \pm 1} \frac{(\sum_{D \in S} |\widehat{D}(\gamma_n) + \varepsilon \widehat{D}(-\gamma_n)|^2)^{1/2}}{Q_S(\gamma_n)^{1/2}}.$$

Hence we see that, after passing to a subsequence if necessary, there are $D_1, D_2 \in S$ such that

$$|\widehat{D}_1(\gamma_n) + \widehat{D}_1(-\gamma_n)| Q_S(\gamma_n)^{-1/2} > C'$$

and

$$|\widehat{D}_2(\gamma_n) - \widehat{D}_2(-\gamma_n)| Q_S(\gamma_n)^{-1/2} > C'$$

for some $C' > 0$ and $n = 1, 2, \ldots$ But this means that D_1 has even order while D_2 has odd order and $\inf_n |\widehat{D}_i(\gamma_n)| Q_S(\gamma_n)^{-1/2} > 0$ for $i = 1, 2$. Hence the smoothness S has property (O′) from Definition 1.2 of [P-W1] and therefore Proposition 1.2 of [P-W1] gives that the intersection of S with some coordinate plane has property (O). This means that S is odd. A contradiction.

To end the proof it is enough to observe that $H \cap S$ is a $(d-1)$-dimensional smoothness for every d-dimensional smoothness S and every $H \in \mathcal{H}$. Hence we can use the inductive hypothesis. ∎

Analogously we have

PROPOSITION 2. *$\psi_S \in \mathrm{S}$ for every smoothness S.*

The proof of this proposition is even simpler than the previous one because the part concerning property $\mathrm{sym}(\mathcal{H})$ can be omitted.

We also have the following obvious

FACT 1. *Every bundle corresponding to a smoothness is shiftable.*

Proof. This follows from the boundedness of the operator of multiplication by a character in Sobolev spaces on $\mathbb{T}^d$. ∎

Now, from Theorem 1, Proposition 1 and Fact 1 we have

THEOREM 4. *For every even smoothness S the translation invariant projections on $L^1_S(\mathbb{T}^d)$ are characterized by* $\mathrm{coset}(\mathbb{Z}^d)$. ∎

Theorem 4 together with the main result of [P-W1], namely the equivalence of oddness of a smoothness and the existence of a Paley projection on $L^1_S(\mathbb{T}^d)$, yield the following dichotomy.

THEOREM 5. *For every smoothness S either there exists a Paley projection on $L^1_S(\mathbb{T}^d)$ or the translation invariant projections on $L^1_S(\mathbb{T}^d)$ are characterized by* $\mathrm{coset}(\mathbb{Z}^d)$.

For Sobolev spaces with uniform norms we obtain from Theorem 2, Proposition 2 and Fact 1 the following

THEOREM 6. *For every smoothness S the translation invariant projections on $C_S(\mathbb{T}^d)$ are characterized by* $\mathrm{coset}(\mathbb{Z}^d)$. ∎

Finally, we use our method to prove the known characterization of the translation invariant projections on the polydisc algebra (cf. [K]).

THEOREM 7. *For any translation invariant projection P on $A(\mathbb{D}^d)$ there is an $A \in \mathrm{coset}(\mathbb{Z}^d)$ satisfying* $\mathrm{supp}\,\widehat{P} \cap \mathbb{Z}^d_+ = A \cap \mathbb{Z}^d_+$.

Proof. Define $Rf(t) = \Pi_1(f(t))$ where $\Pi_1 : \mathbb{C}^2 \to \mathbb{C}^2$ is the projection onto the first coordinate axis. Certainly $R : C_{\phi^d_+}(\mathbb{T}^d) \to A(\mathbb{D}^d)$ is a bounded projection. Because R and P commute, $P \circ R$ is a translation invariant projection on $C_{\phi^d_+}(\mathbb{T}^d)$, so we apply Theorem 3. ∎

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
ŚNIADECKICH 8
00-950 WARSZAWA, POLAND

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