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Weighted estimates for commutators of linear operators

by

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Abstract. We study boundedness properties of commutators of general linear operators with real-valued BMO functions on weighted L^p spaces. We then derive applications to particular important operators, such as Calderón–Zygmund type operators, pseudo-differential operators, multipliers, rough singular integrals and maximal type operators.

1. Introduction. The purpose of this paper is to study boundedness properties of commutators of real-valued BMO functions with general linear operators on weighted L^p spaces. Indeed, we will give a general result, Theorem 2.13, from which we will derive applications to particular important operators, such as Calderón–Zygmund type operators, pseudo-differential operators, multipliers, rough singular integrals, and maximal type operators.

More specifically, given a linear operator T acting on functions and given a function b, we define formally the commutator $[b,T]$ as

$$[b,T]f = bT(f) - T(bf).$$

The first results on this commutator were obtained by Coifman, Rochberg, and Weiss [8] in their study of certain factorization theorems for generalized Hardy spaces. They showed that if T is a classical singular integral operator with smooth kernel and $b \in \mathrm{BMO}$, then the commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$, for $1 < p < \infty$. They also showed that the condition $b \in \mathrm{BMO}$ is necessary when $T = R_j$, the jth Riesz transform in $\mathbb{R}^n$, for $j = 1, \ldots, n$.

The proof they gave of the sufficient condition is based on a delicate good-λ inequality involving several auxiliary operators. Some years later, J. O. Strömberg [20] provided a much simpler proof using the sharp maximal operator of C. Fefferman and E. M. Stein. Coifman, Rochberg, and Weiss outlined in the same paper a different approach, which is less direct but

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shows the close relationship with the existence of weighted inequalities for the operator T. Roughly speaking, their idea is that an appropriate weighted inequality for T provides an unweighted inequality for $[b,T]$ if $b \in \mathrm{BMO}$. In this paper, we exploit that idea further to obtain one and two weighted inequalities for $[b,T]$.

The organization of the paper is as follows. In Section 2, we state and prove the main result, with the aid of several auxiliary results. In Section 3, we show the broad use of these results by considering various applications of interest.

The notation we use is standard. We will write $f : A \to B$ to denote a function defined on A with values in B, with no assumption of continuity. Given Banach spaces A and B, $\mathcal{L}(A,B)$ will be the space of continuous linear operators $T : A \to B$ with operator norm $\|T\|$. Given p with $1 \le p \le \infty$, p' will satisfy $1/p + 1/p' = 1$.

2. Main result. A nonnegative, locally integrable function on $\mathbb{R}^n$ is called a *weight*. We will consider weights which satisfy the following conditions.

DEFINITION 2.1. Let $1 \le p < \infty$. A weight w satisfies the A_p *condition*, $w \in A_p$, if there is a constant $C > 0$ so that

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,dx\right)^{p-1} \le C, \quad \text{for } 1 < p < \infty,$$

or

$$\frac{1}{|Q|}\int_Q w(x)\,dx \le C \operatorname*{ess\,inf}_Q w, \quad \text{for } p = 1,$$

for all cubes Q in $\mathbb{R}^n$. The smallest such C is called the A_p *norm* of w. We set $A_\infty = \bigcup_{p\ge1} A_p$.

For further information about A_p weights, we refer the reader to [15].

DEFINITION 2.2. We say that a collection of couples of weights W is *stable* if $(w,v) \in W$ implies that there is an $\varepsilon > 0$ such that $(w^{1+\varepsilon}, v^{1+\varepsilon}) \in W$.

We will also use stable collections of single weights by considering the pair (w,w). Thus, it follows from well-known results about A_p weights that the A_p spaces are stable for $1 \le p \le \infty$. Moreover, the A_p spaces defined with respect to other bases form stable classes of weights. Similarly, for a fixed p, the collection of weights $u(x) = w(x)|x|^a$, where $w \in A_p$ and $a \in \mathbb{R}$, is a stable class of weights.

Let $1 < p < \infty$. Consider the class W^p of pairs of weights (w,v) such that

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)^r\,dx\right)^{1/pr}\left(\frac{1}{|Q|}\int_Q v(x)^{(1-p')r}\,dx\right)^{1/p'r} \le C,$$

for some $1 < r < \infty$. The class W^p is stable [26].

Let $\mathcal{M}$ be the set of real-valued, Lebesgue measurable functions defined on $\mathbb{R}^n$ and let L^∞_0 be the subspace of those functions in $\mathcal{M}$ that are essentially bounded and have compact support. We assume that T is a linear operator in $\mathcal{M}$, with a domain of definition which contains every compactly supported function in a fixed L^p space, $p < \infty$. Given a real-valued function $b \in \mathrm{BMO}$, we can define a family of linear operators $\{T_z : L^\infty_0 \to \mathcal{M} : z \in \mathbb{C},\ |z| < r(b)\}$, where

$$T_z f = e^{zb}T(e^{-zb}f).$$

Our first result is the following theorem.

THEOREM 2.3. *Let $1 < p < \infty$, $1 < q \le \infty$, and let T be as above. Suppose that W is a stable class of pairs of weights. Assume that*

$$T \in \mathcal{L}(L^p(v), L^p(w)) \quad \text{for all } (w,v) \in W, \qquad T \in \mathcal{L}(L^p(u), L^p(u)) \quad \text{for all } u \in A_q. \tag{2.4}$$

Then, for each real-valued function $b \in \mathrm{BMO}$ and each pair $(w,v) \in W$, there is a $\delta > 0$, which also depends on p and q, such that

$$T_z \in \mathcal{L}(L^p(v), L^p(w))$$

for each $|z| < \delta$. Moreover, $\sup_{|z|<\delta}\|T_z\| < \infty$.

Proof. Fix $b \in \mathrm{BMO}$. There is a $\gamma > 0$ such that $e^{\gamma b} \in A_q$ (see, e.g., [15]). Thus, by (2.4),

$$T \in \mathcal{L}(L^p(e^{\gamma b}), L^p(e^{\gamma b})),$$

with norm M_1. In fact, γ and M_1 depend on the BMO norm of b. Since $b \in \mathrm{BMO}$ implies that $rb \in \mathrm{BMO}$ for $|r| \le 1$ with a smaller BMO norm, we see that

$$T \in \mathcal{L}(L^p(e^{tb}), L^p(e^{tb})) \quad \text{for } |t| \le \gamma,$$

with norm bounded by M_1.

Fix $(w,v) \in W$. Since W is stable, there is an $\varepsilon > 0$ such that $(w^{1+\varepsilon}, v^{1+\varepsilon}) \in W$. By (2.4),

$$T \in \mathcal{L}(L^p(v^{1+\varepsilon}), L^p(w^{1+\varepsilon})), \tag{2.6}$$

with norm M_2.

Given $z=\alpha+i\beta$, the operator T_z will belong to $\mathcal{L}(L^p(v),L^p(w))$ provided that

$$T\in\mathcal{L}(L^p(ve^{\alpha pb}),L^p(we^{\alpha pb}))\,. \tag{2.7}$$

Let $\delta=\gamma\varepsilon/p(1+\varepsilon)$ and suppose that $|z|<\delta$. Then $|\alpha|<\gamma\varepsilon/p(1+\varepsilon)$ or $|\alpha|p(1+\varepsilon)/\varepsilon<\gamma$. From (2.5), we obtain

$$T\in\mathcal{L}(L^p(e^{\alpha pb(1+\varepsilon)/\varepsilon}),L^p(e^{\alpha pb(1+\varepsilon)/\varepsilon}))\,.$$

Applying Stein's interpolation theorem [29] to this last result and (2.6) yields (2.7) with a norm bounded by $\max\{M_1,M_2\}$. This completes the proof of the theorem. ■

We would like to point out that if $q=1$, Theorem 2.3 is still true if we assume that $b\in\mathrm{VMO}$, the space of functions of vanishing mean oscillation. The comment applies to the later results in this section. ■

We will need a vector-valued version of Theorem 2.3 to obtain certain applications. Let $\mathbf{A}$ be a Banach space with norm $\|\ \|_{\mathbf{A}}$ and u a weight. For $1\le p<\infty$, define the Banach space $L^p_{\mathbf{A}}(u)$ to be the set of strongly measurable functions $f:\mathbb{R}^n\to\mathbf{A}$ such that $\int\|f(x)\|^p_{\mathbf{A}}u(x)\,dx<\infty$. We will use $L^\infty_0(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ to denote the vector-valued analogs of the spaces considered above. Repeating the proof of Theorem 2.3, we obtain the following vector-valued result.

THEOREM 2.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Banach spaces and suppose that* $T:L^\infty_0(\mathbf{A})\to\mathcal{M}(\mathbf{B})$ *is a linear operator. Let* $1<p<\infty$, $1<q\le\infty$, *and let* W *be a stable class of weights. Suppose that*

$$\begin{aligned}&T\in\mathcal{L}(L^p_{\mathbf{A}}(v),L^p_{\mathbf{B}}(w)) \quad \text{for all } (w,v)\in W\,,\\ &T\in\mathcal{L}(L^p_{\mathbf{A}}(u),L^p_{\mathbf{B}}(u)) \quad \text{for all } u\in A_q\,.\end{aligned} \tag{2.9}$$

Then, for each real-valued function $b\in\mathrm{BMO}$ *and each pair* $(w,v)\in W$, *there is a* $\delta>0$, *which also depends on* p *and* q, *such that*

$$T_z\in\mathcal{L}(L^p_{\mathbf{A}}(v),L^p_{\mathbf{B}}(w))$$

for each $|z|<\delta$. *Moreover,* $\sup_{|z|<\delta}\|T_z\|<\infty$. ■

We would like to show that the mapping $z\to T_z$, with values in $\mathcal{L}(L^p_{\mathbf{A}}(v),L^p_{\mathbf{B}}(w))$, is analytic near $z=0$, in order to identify the coefficients in its Taylor expansion with the iterated commutators of T and b. When $\mathbf{A}=\mathbf{B}=\mathbb{C}$, we are able to prove the analyticity of this map using a characterization stated in [21]. The proof relies on selecting appropriate dense sets in $L^p(v)$ and the dual of $L^p(w)$. We are unable to extend this proof to the vector-valued case without imposing some conditions on the space $\mathbf{B}$, such as the Radon–Nikodym condition. However, it is still possible to show the boundedness of the iterated commutators without proving the analyticity of the map $z\to T_z$. The rest of this section is devoted to showing this alternative proof. Let $D_\eta=\{z\in\mathbb{C}:|z|<\eta\}$.

LEMMA 2.10. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Banach spaces. Let* $1<p<\infty$, $1<q\le\infty$, *and let* W *be a stable class of pairs of weights. Suppose that* $T:L^\infty_0(\mathbf{A})\to\mathcal{M}(\mathbf{B})$ *is a linear operator which satisfies conditions* (2.9). *Then, for each real-valued function* $b\in\mathrm{BMO}$ *and each pair* $(w,v)\in W$, *there is an* $\eta>0$, *which also depends on* p *and* q, *such that for each* $f\in L^\infty_0(\mathbf{A})$, *the map* $z\to T_z(f)$ *is continuous from* D_η *into* $L^p_{\mathbf{B}}(w)$.

Proof. We need to find an $\eta>0$ such that for $z\in D_\eta$ and $\{z_n\}\subset D_\eta$, if $z_n\to z$ then $\|(T_{z_n}-T_z)f\|_{L^p_{\mathbf{B}}(w)}\to0$ as $n\to\infty$. Write

$$T_{z_n}f-T_zf=e^{z_nb}T((e^{-z_nb}-e^{-zb})f)+(e^{z_nb}-e^{zb})T(e^{-zb}f)=I+II\,.$$

Consider the $L^p_{\mathbf{B}}(w)$-norm of I. Let $\alpha_n=\mathrm{Re}(z_n)$. We have

$$\begin{aligned}\int_{\mathbb{R}^n}\|I(x)\|^p_{\mathbf{B}}w(x)\,dx&=\int_{\mathbb{R}^n}\|e^{z_nb(x)}T((e^{-z_nb}-e^{-zb})f)(x)\|^p_{\mathbf{B}}w(x)\,dx\\&=\int_{\mathbb{R}^n}e^{p\alpha_nb(x)}\|T((e^{-z_nb}-e^{-zb})f)(x)\|^p_{\mathbf{B}}w(x)\,dx\\&\le\int_{\mathbb{R}^n}\|T((e^{-z_nb}-e^{-zb})f)(x)\|^p_{\mathbf{B}}e^{p\eta|b(x)|}w(x)\,dx\,,\end{aligned}$$

for $|z_n|<\eta$.

Fix $\gamma>0$ such that $e^{\gamma p|b|}\in A_q$. Then $e^{\gamma p|b|/2}\in A_q$ and by hypothesis, $T\in\mathcal{L}(L^p_{\mathbf{A}}(e^{\gamma p|b|/2}),L^p_{\mathbf{B}}(e^{\gamma p|b|/2}))$. Since $(w^{1+\varepsilon},v^{1+\varepsilon})\in W$ for some $\varepsilon>0$, we also have $T\in\mathcal{L}(L^p_{\mathbf{A}}(v^{1+\varepsilon}),L^p_{\mathbf{B}}(w^{1+\varepsilon}))$. Thus, by Stein's interpolation result, we obtain

$$T\in\mathcal{L}(L^p_{\mathbf{A}}(ve^{\gamma p|b|\varepsilon/2(1+\varepsilon)}),L^p_{\mathbf{B}}(we^{\gamma p|b|\varepsilon/2(1+\varepsilon)}))\,.$$

Hence, setting $\eta=\gamma\varepsilon/2(1+\varepsilon)$, we have

$$T\in\mathcal{L}(L^p_{\mathbf{A}}(ve^{\eta p|b|}),L^p_{\mathbf{B}}(we^{\eta p|b|}))\,.$$

It follows that the norm of I is bounded by a constant times

$$\int_{\mathbb{R}^n}\|(e^{-z_nb(x)}-e^{-zb(x)})f(x)\|^p_{\mathbf{A}}e^{p\eta|b(x)|}v(x)\,dx\,.$$

We claim that this integral approaches 0 as $n\to\infty$. Indeed, it is clear that the integrand converges to 0 pointwise. Furthermore,

$$\begin{aligned}&\|(e^{-z_nb(x)}-e^{-zb(x)})f(x)\|_{\mathbf{A}}e^{\eta|b(x)|}\\&\quad\le e^{|\alpha_n|\,|b(x)|}\|f(x)\|_{\mathbf{A}}e^{\eta|b(x)|}+e^{|\alpha|\,|b(x)|}\|f(x)\|_{\mathbf{A}}e^{\eta|b(x)|}\\&\quad\le 2\|f(x)\|_{\mathbf{A}}e^{2\eta|b(x)|}\,,\end{aligned}$$

and

$$\int_{\mathbb{R}^n} \|f(x)\|_{\mathbf{A}}^p e^{2p\eta|b(x)|} v(x)\,dx \le \|f\|_{L_0^\infty(\mathbf{A})}^p \int_{\operatorname{supp}(f)} e^{2p\eta|b(x)|} v(x)\,dx$$

$$\le C\|f\|_{L_0^\infty(\mathbf{A})}^p \Big(\int_{\operatorname{supp}(f)} e^{2p\eta|b(x)|(1+\varepsilon)/\varepsilon}\,dx\Big)^{\varepsilon/(1+\varepsilon)} \Big(\int_{\operatorname{supp}(f)} v(x)^{1+\varepsilon}\,dx\Big)^{1/(1+\varepsilon)}$$

$$= C\|f\|_{L_0^\infty(\mathbf{A})}^p \Big(\int_{\operatorname{supp}(f)} e^{\gamma p|b(x)|}\,dx\Big)^{\varepsilon/(1+\varepsilon)} \Big(\int_{\operatorname{supp}(f)} v(x)^{1+\varepsilon}\,dx\Big)^{1/(1+\varepsilon)}.$$

Since $e^{\gamma p|b|}$ and $v^{1+\varepsilon}$ are locally integrable and f has compact support, this last expression is finite. By the Lebesgue Dominated Convergence Theorem, $\int \|I\|_{\mathbf{B}}^p w\,dx$ goes to 0 as $n \to \infty$.

We will use similar arguments to estimate the norm of II. Since

$$\int_{\mathbb{R}^n} \|II\|_{\mathbf{B}}^p w(x)\,dx = \int_{\mathbb{R}^n} \|(e^{z_n b(x)} - e^{-zb(x)})T(e^{-zb}f)(x)\|_{\mathbf{B}}^p w(x)\,dx,$$

it is clear that the integrand converges to 0. This last integral is bounded by $2\int \|T(e^{-zb}f)(x)\|_{\mathbf{B}}^p e^{p\eta|b(x)|} w(x)\,dx$, so using the same interpolation argument as above, we see that

$$\int_{\mathbb{R}^n} \|II\|_{\mathbf{B}}^p w(x)\,dx \le C\int_{\mathbb{R}^n} \|e^{-zb(x)}f(x)\|_{\mathbf{A}}^p e^{p\eta|b(x)|} v(x)\,dx$$

$$\le C\int_{\mathbb{R}^n} \|f(x)\|_{\mathbf{A}}^p e^{p\eta|b(x)|} v(x)\,dx.$$

As before, we can conclude that $\int \|II\|_{\mathbf{B}}^p w\,dx$ goes to 0 as $n \to \infty$. It follows that $\|(T_{z_n} - T_z)f\|_{L_{\mathbf{B}}^p(w)} \to 0$ as $n \to \infty$, which completes the proof of the lemma. ∎

Remark 2.11. Let $\delta > 0$ and $\eta > 0$ be as in Theorem 2.8 and Lemma 2.10, respectively. By taking $\eta < \delta$ if necessary, we can assume that

$$M = \sup_{D_\eta} \|T_z\| < \infty,$$

where the norm denotes the operator norm in $\mathcal{L}(L_{\mathbf{A}}^p(v), L_{\mathbf{B}}^p(w))$.

Let $0 < r < \eta$ and $\partial D_r = \{z \in \mathbb{C} : |z| = r\}$, oriented counterclockwise. By the previous lemma, the Bochner integral

$$\frac{n!}{2\pi i}\int_{\partial D_r} \frac{T_z(f)}{z^{n+1}}\,dz \tag{2.12}$$

exists for each $n = 0, 1, 2, \ldots$, and yields an operator, C_n, which is densely defined in $L_{\mathbf{A}}^p(v)$ with values in $L_{\mathbf{B}}^p(w)$. Moreover,

$$\|C_n(f)\|_{L_{\mathbf{B}}^p(w)} \le M\frac{n!}{r^n}\|f\|_{L_{\mathbf{A}}^p(v)},$$

thus showing $C_n \in \mathcal{L}(L_{\mathbf{A}}^p(v), L_{\mathbf{B}}^p(w))$ with an operator norm bounded by $Mn!r^{-n}$.

We are now ready to prove the continuity of the iterated commutators.

THEOREM 2.13. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be Banach spaces and suppose that* $T : L_0^\infty(\mathbf{A}) \to \mathcal{M}(\mathbf{B})$ *is a linear operator. Let* $1 < p < \infty$, $1 < q \le \infty$, *and* W *be a stable class of pairs of weights. Suppose that*

$$T \in \mathcal{L}(L_{\mathbf{A}}^p(v), L_{\mathbf{B}}^p(w)) \quad \text{for all } (w,v) \in W,$$
$$T \in \mathcal{L}(L_{\mathbf{A}}^p(u), L_{\mathbf{B}}^p(u)) \quad \text{for all } u \in A_q.$$

Then, given $b \in \mathrm{BMO}$ *and* $(w,v) \in W$, *the* n*-th commutator of* T *and* b, *defined pointwise as* $T((b(x)-b(\cdot))^n f(\cdot))(x)$ *for* $f \in L_0^\infty(\mathbf{A})$, *coincides with the operator* C_n *given by* (2.12). *Hence, for each* $n = 1, 2, 3, \ldots$, *the* n*-th commutator belongs to* $\mathcal{L}(L_{\mathbf{A}}^p(v), L_{\mathbf{B}}^p(w))$.

Proof. For $N \in \mathbb{N}$ and $f \in L_0^\infty(\mathbf{A})$, set

$$S_N(x,z) = \sum_{j=0}^N \frac{z^j}{j!} b(x), \qquad T_z^{(N)}(f)(x) = S_N(x,z)T(S_N(\cdot,-z)f)(x).$$

Since $S_N(x,z) \to e^{zb(x)}$ and $|S_N(x,z)| \le e^{\eta|b(x)|}$ for all $z \in D_\eta$, the same arguments used to prove Lemma 2.10 show that $T_z^{(N)}(f) \to T_z(f)$ in $L_{\mathbf{B}}^p(w)$ with $\|T_z^{(N)}(f)\|_{L_{\mathbf{B}}^p(w)}$ uniformly bounded for $z \in D_\eta$. Using the Dominated Convergence Theorem for the Bochner integral (see [32]), we see that

$$\lim_{N\to\infty} \frac{n!}{2\pi i}\int_{\partial D_r} \frac{T_z^{(N)}(f)(x,z)}{z^{n+1}}\,dz = \frac{n!}{2\pi i}\int_{\partial D_r} \frac{T_z(f)}{z^{n+1}}\,dz$$

exists in $L_{\mathbf{B}}^p(w)$. But since T is linear, for all $N > n$, we have

$$C_n(f)(x) = \frac{n!}{2\pi i}\int_{\partial D_r} \frac{T_z^{(N)}(f)(x,z)}{z^{n+1}}\,dz$$

$$= n!\sum_{j=0}^N\sum_{l=0}^N \frac{b(x)^j}{j!} T\Big(\frac{(-b(\cdot))^l}{l!} f(\cdot)\Big)(x)\frac{n!}{2\pi i}\int_{\partial D_r} z^{j+l-n-1}\,dz$$

$$= n!\sum_{j+l=n} \frac{b(x)^j}{j!} T\Big(\frac{(-b(\cdot))^l}{l!} f(\cdot)\Big)(x) = T((b(x)-b(\cdot))^n f(\cdot))(x).$$

We can conclude that the nth commutator $T((b(x)-b(\cdot))^n f(\cdot))(x)$ coincides with the operator C_n and, thus, it belongs to $\mathcal{L}(L^p_{\mathbf{A}}(v), L^p_{\mathbf{B}}(w))$. This completes the proof of the theorem. ∎

3. Applications. In this section we present applications of Theorem 2.13. As we mentioned in the introduction, one of the main tools for obtaining weighted norm inequalities for the operator T involves the sharp function of C. Fefferman and E. M. Stein. Let Mf be the Hardy–Littlewood maximal function of f and set $M_p(f) = (M(|f|^p))^{1/p}$. Now, suppose that for some p, $1 < p < \infty$, there is a constant $C = C_p > 0$ such that for all $x \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$,

$$(Tf)^\#(x) \le CM_pf(x).$$

Then T is bounded from $L^q(u)$ into $L^q(u)$ for $p < q < \infty$ and $u \in A_{q/p}$ (see for example [22, 27]). If T is linear, then we can apply Theorem 2.13 to prove that $[b, T]$ is also in $\mathcal{L}(L^q(u), L^q(u))$ for $p < q < \infty$ and $u \in A_{q/p}$. Several of the applications below arise as a consequence of this situation.

3.1. *Convolution kernels.* Let $k(x) = \Omega(x)/|x|^n$. For $1 < r < \infty$ and $0 < \delta \le 1$, set

$$\omega_r(\delta) = \sup_{|\varrho|\le\delta}\Big(\int_{\Sigma_{n-1}} |\Omega(x) - \Omega(\varrho x)|^r\, d\sigma(x)\Big)^{1/r},$$

where the supremum is taken over all rotations ϱ of the unit sphere Σ_{n-1} with $|\varrho| = \sup_{x\in\Sigma_{n-1}} |x - \varrho x| \le \delta$. We say that Ω satisfies the *L^r-Dini condition* if $\Omega \in L^r(\Sigma_{n-1})$, $\int_{\Sigma_{n-1}} \Omega\, d\sigma = 0$, and $\int_0^1 (\omega_r(\delta)/\delta)\, d\delta < \infty$.

Define the singular integral operator T by $Tf(x) = \mathrm{pv}\int k(y)f(x-y)\,dy$, for $f \in C_0^\infty(\mathbb{R}^n)$. As shown in [23], $(Tf)^\#(x) \le CM_{r'}f(x)$.

When $r = \infty$ and ω_∞ is defined in terms of the $L^\infty(\Sigma_{n-1})$-norm, we essentially get the classical singular integral operators. Since these operators are known to be bounded on $L^p(u)$ for $1 < p < \infty$ and $u \in A_p$, Theorem 2.13 implies that the commutator $[b, T]$ is also bounded on $L^p(u)$ for $b \in \mathrm{BMO}$, $1 < p < \infty$, and $u \in A_p$.

3.2. *Calderón–Zygmund operators.* Let $\Delta = \{(x, x) : x \in \mathbb{R}^n\}$ be the diagonal of $\mathbb{R}^n \times \mathbb{R}^n$. We define a *standard kernel* to be a locally integrable function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ which satisfies:

$$|k(x,y)| \le C|x-y|^{-n},$$
$$|k(x,y) - k(z,y)| + |k(y,x) - k(y,z)| \le C|x-z|^\varepsilon |y-z|^{-(n+\varepsilon)}$$

for $2|x-z| < |y-z|$ and some $0 < \varepsilon \le 1$.

Let $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ be a continuous linear operator. Then T is called a *Calderón–Zygmund operator* in the sense of Coifman and Meyer [7] if T extends to a continuous operator on $L^2(\mathbb{R}^n)$ and it is associated with a standard kernel k, which means

$$Tf(x) = \int k(x,y)f(y)\,dy$$

for $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin \mathrm{supp}(f)$.

Given a Calderón–Zygmund operator T and $1 < p < \infty$, there exists $C = C_p > 0$ such that

$$(Tf)^\#(x) \le CM_pf(x) \tag{3.3}$$

(see [7]). Such an inequality fails for $p = 1$, as can be seen by considering the Hilbert transform [7].

3.4. *Weakly-strongly singular Calderón–Zygmund operators.* There are singular integral operators which enjoy properties similar to those of the Calderón–Zygmund operators, while the kernels are more singular near the diagonal than in the standard case. The model for these operators is the multiplier operator $T_{\alpha\beta}$ defined by

$$(T_{\alpha\beta})^\wedge(\xi) = \frac{e^{i|\xi|^\alpha}}{|\xi|^\beta}\theta(\xi)\widehat{f}(\xi),$$

where $0 < \alpha < 1$, $\beta > 0$, and θ is a standard cutoff function. This operator was named weakly-strongly singular by C. Fefferman [13]. The convolution kernel of $T_{\alpha\beta}$ turns out to be essentially the function $\exp(i|x|^{\alpha'})/|x|^{n+\lambda}$, with $1/\alpha + 1/\alpha' = 1$ and $\lambda = (n\alpha/2 - \beta)/(1-\alpha)$. The non-convolution case is modeled after this example as follows.

Let $T : C_0^\infty(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n)$ be a continuous linear operator. Such a T is called a *weakly-strongly singular Calderón–Zygmund* operator if there is an α, $0 < \alpha < 1$, so that T extends to a continuous operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, for some $1 < p, q < 1$ with $p/q \le \alpha$ [3].

For $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin \mathrm{supp}(f)$,

$$Tf(x) = \int k(x,y)f(y)\,dy$$

where the distribution kernel coincides with a locally integrable function $k : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C}$ which satisfies

$$|k(x,y) - k(z,y)| \le C|x-z|^\varepsilon |y-z|^{-(n+\varepsilon/\alpha)}$$

for $2|x-z| < |y-z|$ and some $0 < \varepsilon \le 1$ and $0 < \alpha < 1$. Given a weakly-strongly singular Calderón–Zygmund operator and given r, $p < r < \infty$, there exists a $C = C_r > 0$ such that $(Tf)^\#(x) \le CM_rf(x)$.

3.5. *Pseudo-differential operators.* We next consider pseudo-differential operators in the Hörmander class $L^m_{\varrho,\delta}$ [17]. Let $m \in \mathbb{R}$, $0 \le \delta \le 1$, and $0 \le \varrho \le 1$. An operator $T \in L^m_{\varrho,\delta}$ if it has the representation

$$Tf(x) = \int e^{-2\pi i x\cdot\xi} p(x,\xi)\widehat{f}(\xi)\,d\xi$$

for $f \in C_0^\infty(\mathbb{R}^n)$, where $p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies the estimates

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \le C_{\alpha\beta}(1+|\xi|)^{m-\varrho|\beta|+\delta|\alpha|}.$$

The function p, which is uniquely determined by T, is called the *symbol* of the operator.

In [7] it is shown that pseudo-differential operators in $L_{1,0}^0$ are Calderón–Zygmund operators. Thus, the pointwise estimate (3.3) holds. This estimate has been proved by alternate means by N. Miller [24]. More generally, operators in $L_{\varrho,\delta}^m$ with $0 \le \delta < 1$, $0 < \varrho \le 1$, and $m \le -(n+1)(1-\varrho)$ are Calderón–Zygmund operators [2] and thus satisfy estimate (3.3). Similarly, pseudo-differential operators in $L_{\varrho,\delta}^m$ with $0 \le \delta < 1$, $0 < \varrho \le 1$, and $m \le n(1-\varrho)$ are weakly-strongly singular Calderón–Zygmund operators, with $p=2$, $q=2/\varrho$, and $\alpha = 1/\varrho$ [2]. Such operators satisfy (3.3) with $r > 2$.

S. Chanillo and A. Torchinsky [6] have proved that pseudo-differential operators in the class $L_{\varrho,\delta}^{-n(1-\varrho)/2}$, $0 \le \delta < \varrho \le 1$, satisfy (3.3) with $p=2$. It is an open problem whether the same pointwise inequality holds with $r < 2$. The following result is a partial answer to this problem [2].

Given $T \in L_{\varrho,\delta}^m$, $0 \le \delta < 1$ and $0 < \varrho \le 1$, and given r, $1 < r < \infty$, there is a $C = C_r > 0$ such that $(Tf)^\#(x) \le CM_rf(x)$ for $f \in C_0^\infty(\mathbb{R}^n)$ provided that $\lambda = \max\{0,(\delta-\varrho)/2\}$, $0 < \varrho \le \frac{1}{2}(1-2n\lambda/(n+2))$, and $m \le -n(1-\varrho)-\mu$, where

$$2\mu = \{1+n(\varrho+\lambda) - \sqrt{\{(1+n(\varrho+\lambda))\}^2 - 4n\lambda}\}.$$

One should note that $\mu = 0$ if $\lambda = 0$.

3.6. *Multipliers.* The pointwise conditions imposed on the kernels or symbols of our operators are at one end of a scale of integral conditions [27]. We will now consider a class of operators whose symbols satisfy a Hörmander type condition.

Let m be a bounded, measurable function on $\mathbb{R}^n$. Define the multiplier operator $T = T_m$ by $(Tf)^\wedge(\xi) = m(\xi)\widehat{f}(\xi)$. Given $1 \le s < \infty$ and $l \in \mathbb{N}$, we say that $m \in M(s,l)$ if

$$\sup_{R>0}\sup_{|\beta|\le l}\Big(R^{s|\beta|-n}\int_{\{\xi:R<|\xi|<2R\}}|D^\beta m(\xi)|^s\,d\xi\Big)^{1/s} < \infty$$

(see [23]). For $l > n/2$, $M(2,l)$ is the usual Hörmander multiplier condition. If $m \in M(s,l)$ with $1 < s \le 2$ and $n/s < l \le n$, then for $r > n/l$ there is a constant $C = C_r > 0$ so that $(T_Nf)^\#(x) \le CM_rf(x)$. Here, T_N is defined by a smooth cutoff of m, T_N converges to T as $N \to \infty$, and the constant C is independent of N.

The $M(s,l)$ condition has been generalized to include cases where l is not an integer. Let λ be a nonnegative real number. Set $M(s,\lambda) = M(s,l)$ if λ is an integer. If λ is not an integer, let $l = [\lambda]$ and $\gamma = \lambda - l$. Then $m \in M(s,\lambda)$ if $m \in M(s,l)$ and

$$\sup_{R>0}\sup_{0<|\eta|\le r/2}\max_{|\beta|=l}\bigg(\Big(\frac{r}{|\eta|}\Big)^{\gamma s}R^{s|\beta|-n} \times \int_{\{\xi:R<|\xi|<2R\}}|D^\beta m(\xi) - D^\beta m(\xi-\eta)|^s\,d\xi\bigg)^{1/s} < \infty.$$

Such multipliers have been studied in [25] in one dimension and in [9, 28, 30] in higher dimensions. The associated multiplier operators are bounded on $L_w^p(\mathbb{R}^n)$ for certain A_∞ weights which are not necessarily in A_p. The weights in question are a product of an A_p weight and a function of the form $(1+|x|)^a\prod_{j=1}^J|x-p_j|^{a_j}$. The conditions on the A_p weight and the exponents a and a_j create a stable class of weights.

3.7. *Rough singular integrals.* There are two standard techniques for proving weighted inequalities for classical singular integrals. One involves the so-called "good-λ" inequalities and the other is based on the sharp function. Both methods require fairly strong regularity conditions on the kernel of the operator.

Using the method of rotations, Calderón and Zygmund obtained L^p estimates for the principal value operator

$$Tf(x) = \operatorname{pv}\int\frac{\Omega(y)}{|y|^n}f(x-y)\,dy$$

assuming only that the function Ω is homogeneous of degree zero, $\Omega \in L^\infty(\Sigma_{n-1})$ and $\int_{\Sigma_{n-1}}\Omega\,d\sigma = 0$.

The method of J. Duoandikoetxea and J. L. Rubio de Francia [11] gives A_p weighted inequalities for operators such as the T above. Thus, Theorem 2.13 implies that $[b,T] \in \mathcal{L}(L^p(u), L^p(u))$ for $b \in \mathrm{BMO}$, $1 < p < \infty$, and $u \in A_p$.

Recently, D. K. Watson [31] and J. Duoandikoetxea [10] have considered kernels satisfying the weaker assumption $\Omega \in L^r(\Sigma_{n-1})$. They proved, independently, that $T \in \mathcal{L}(L^p(u), L^p(u))$ for p in an appropriate range depending on r and $u \in A_q$, with q depending on p and r, and also for weights u such that $u^q \in A_p$, with q again depending on p and r. Since all of these classes of weights are stable, we can invoke Theorem 2.13 to obtain the corresponding weighted estimates for the commutators $[b,T]$ with $b \in \mathrm{BMO}$.

Now, consider the kth Calderón commutator T_k defined by

$$T_k f(x) = \mathrm{pv} \int \frac{\Omega(y)}{|y|^{n+k}}(a(x) - a(y))^k f(x-y)\, dy\,,$$

where a is a Lipschitz function, Ω is homogeneous of degree zero, $\Omega \in L^\infty(\Sigma_{n-1})$, and for $|\beta| = k$, $\int_{\Sigma_{n-1}} x^\beta \Omega\, d\sigma = 0$. S. Hofmann [16] has proved weighted results for $1 < p < \infty$ with A_p weights for these operators, so that Theorem 2.13 applies to the commutators $[b, T_k]$.

3.8. *Vector-valued operators.* Given a function $\Phi \in L^1(\mathbb{R}^n)$, we can consider the approximation to the identity $\{\Phi_t\}_{t>0}$, where $\Phi_t(x) = t^{-n}\Phi(x/t)$, and its associated maximal function M_Φ defined by

$$M_\Phi f(x) = \sup_{t>0} |\Phi_t * f(x)|\,.$$

When Φ is the characteristic function of the unit ball in $\mathbb{R}^n$, we get the Hardy–Littlewood maximal function. Under fairly mild conditions on Φ, such as

$$|\Phi(x-y) - \Phi(x)| \le C|y|/|x|^{n+1} \quad \text{for } |x| > 2|y|\,,$$

M_Φ is bounded on $L^p(w)$ for $w \in A_p$. Since the convolutions are linear, M_Φ is realized as the l^∞ norm of a linear operator. Thus, the commutator

$$\sup_{t>0} |b(\Phi_t * f) - \Phi_t * (bf)|$$

is bounded on $L^p(w)$ for $w \in A_p$. In particular, we extend the result of Coifman, Rochberg, and Weiss to weighted L^p spaces with more general kernels. This same idea applies to maximal singular integrals and the Carleson maximal function [19, 27].

Let T be a sublinear operator which satisfies the conditions of Theorem 2.3. In many important instances, T can be realized as a Banach space norm of a linear operator S; in other words, $Tf(x) = \|Sf(x)\|_{\mathbf{B}}$. The weighted norm inequalities for T imply that S satisfies Theorem 2.13, so that the commutator $[b, S] \in \mathcal{L}(L^p(v), L^p(w))$. However, since the difference of norms is bounded by the norm of the difference, we have the following inequality:

$$\begin{aligned}\|[b,S](f)(x)\|_{\mathbf{B}} &= \|b(x)Sf(x) - S(bf)(x)\|_{\mathbf{B}}\\ &\ge \big|\,\|b(x)Sf(x)\|_{\mathbf{B}} - \|S(bf)(x)\|_{\mathbf{B}}\big|\\ &= \big|\,|b(x)|Tf(x) - T(bf)(x)\big|\,.\end{aligned}$$

It follows that for a nonnegative function $b \in \mathrm{BMO}$,

$$\|[b,S](f)(x)\|_{\mathbf{B}} \ge |[b,T](f)(x)|\,.$$

Thus, when b is nonnegative, the commutator results for S apply to T.

The main application of this situation that we have in mind is to square functions.

For the second application, let Ψ be a Schwartz function with integral mean value zero, $\int \Psi = 0$. We define the Littlewood–Paley operators by

$$g[\Psi](f)(x) = \Big(\int_0^\infty |\Psi_t * f(x)|^2 t^{-1}\, dt\Big)^{1/2},$$

$$S[\Psi](f)(x) = \Big(\iint_{\{(z,t)\in\mathbb{R}^{n+1}_+ : |z-x|<t\}} |\Psi_t * f(z)|^2 t^{-1-n}\, dz\, dt\Big)^{1/2},$$

and

$$g^*_\lambda[\Psi](f)(x) = \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-z|}\Big)^{\lambda n} |\Psi_t * f(z)|^2 t^{-1-n}\, dz\, dt\Big)^{1/2}, \quad 1 < \lambda\,.$$

In the classical situation, $\Psi_t(x) = t\nabla P(x,t)$, where P is the Poisson kernel. Each of these operators can be realized as the weighted L^2 norm of a linear operator; for example,

$$S[\Psi](f)(x) = \|\chi_{\Gamma_t(x)}(\Psi_t * f)(z)\|\,,$$

where $\Gamma_t(x)$ is the cone $\{(z,t) \in \mathbb{R}^{n+1}_+ : |z-x| < t\}$ and the L^2 norm is taken with respect to the measure $t^{-1-n}\, dz\, dt$. The condition on Ψ guarantees that these operators are bounded on $L^p(w)$ for $w \in A_p$. Thus, Theorem 2.13 applies to show that the commutators of these operators with nonnegative BMO functions are bounded operators. We would expect this to hold for all BMO functions, though our methods will not yield such a result.

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