Contents of Volume 104, Number 3

STUDIA MATHEMATICA

Managing Editors: **Z. Ciesielski, A. Pełczyński, W. Żelazko**

The journal publishes original papers in English, French, German and Russian, mainly in functional analysis, abstract methods of mathematical analysis and probability theory. Usually 3 issues constitute a volume.

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STUDIA MATHEMATICA

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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TEX at the Institute
Printed and bound by
drukarnia
herman i herman
02-240 WARSZAWA, ul. Jakobinów 23

PRINTED IN POLAND

ISBN 83-85116-84-2 ISSN 0039-3223

On rank one and finite elements of Banach algebras

by

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Abstract. We give a spectral characterisation of rank one elements and of the socle of a semisimple Banach algebra.

In [6] J. Puhl defined rank one elements in a semiprime Banach algebra in such a way that it was possible to define a trace functional in a subalgebra of this algebra. Puhl also gave a characterisation of rank one elements ([6], Corollary 3.3). Our main aim in this paper is to give a spectral characterisation of rank one elements in a semisimple Banach algebra. Our paper is organised as follows: In Section 1 we collect some basic properties of rank one elements, Section 2 contains a spectral characterisation of rank one elements and Section 3 contains a spectral characterisation of the socle of a semisimple Banach algebra.

1. Basic properties of rank one elements. Throughout this paper A will denote a complex semiprime Banach algebra with invertible group A^{-1} and identity 1. The *rank* ≤ 1 *elements* of A are the members of the set

$$\mathcal{F}_1(A) := \{a \in A : L_a R_a \in A' \otimes a : A \to A\},$$

where A' is the dual space of A, and L_a, R_a denote respectively the left and right multiplication by a. We write $f \otimes x(y) := f(y)x$ for each $y \in X$. Thus $a \in \mathcal{F}_1(A)$ if and only if there is a bounded linear functional $\tau_a : A \to \mathbb{C}$ for which $axa = \tau_a(x)a$ for each $x \in A$. Evidently τ_a is unique if $a \neq 0$; if $a = 0$ we take $\tau_a = 0$. For example if $A := B(X, X)$ is the bounded linear operators on a Banach space X then the rank one elements are what they ought to be: $\mathcal{F}_1(A) = X' \otimes X$ ([6], Proposition 2.6). If instead, $A := C_\infty(\Omega)$ is the bounded continuous complex-valued functions on a completely regular Hausdorff space Ω, the rank one elements are the "delta functions", i.e., $\mathcal{F}_1(A) = \bigcup\{\mathbb{C}\delta_t : t \in \mathrm{iso}(\Omega)\}$, where $\mathrm{iso}(\Omega)$ denotes the set of isolated

1991 *Mathematics Subject Classification*: 46H05, 46H10.

points of Ω, and $\delta_t(s) = 1$ if $s = t$ and $\delta_t(s) = 0$ if $s \neq t$ ([6], remark preceding Lemma 2.7).

If A is a Banach algebra then the *centre* of A is the set $\text{centre}(A) := \{a \in A : xa = ax \text{ for every } x \in A\}$.

PROPOSITION 1.1. *Let A be a semiprime Banach algebra and $a \in A$.*

(1) *The rank one elements in A form a multiplicative ideal, i.e., $A\mathcal{F}_1(A)A \subset \mathcal{F}_1(A)$* ([6], Lemma 2.7).
(2) *If $L_a \in \mathcal{F}_1 B(A, A)$ or $R_a \in \mathcal{F}_1 B(A, A)$ then $a \in \mathcal{F}_1(A)$.*
(3) *If $a \in \mathcal{F}_1(A) \cap \text{centre}(A)$ then $L_a = R_a \in \mathcal{F}_1 B(A, A)$.*

Proof. (2) If $L_a = \phi \otimes b \in \mathcal{F}_1 B(A, A)$ then $a = \phi(1)b$ so that we can always take $b = a$. Observe that $L_a R_a = (\phi \otimes a) R_a = (\phi R_a) \otimes a$, giving $a \in \mathcal{F}_1(A)$ if L_a is rank one, and similarly if R_a is rank one.

(3) If $a \in \mathcal{F}_1(A)$ then $a^2 = \tau_a(1)a$, and if $a \in \text{centre}(A)$, then $\tau_a(1) \neq 0$: Suppose to the contrary that a is nilpotent (see Proposition 2.1(1)). Then the two-sided ideal Aa would satisfy $(Aa)^2 = \{0\}$, which is not possible because A is semiprime. Hence $\tau_a(1) L_a = L_{a^2} = L_a R_a = \tau_a \otimes a$, giving

$$L_a = \left(\frac{1}{\tau_a(1)} \tau_a\right) \otimes a. \ \blacksquare$$

The converse of 1.1(2) is not in general true. For example, let A be the algebra of all 3×3 matrices. Then

$$a := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is a rank one element in A while neither L_a nor R_a is a rank one element in $B(A, A)$.

A *homomorphism* $T : A \to B$ is a linear map such that $T(ab) = T(a)T(b)$ $(a, b \in A)$ and $T1 = 1$. Rank one elements are to a certain extent respected by homomorphisms.

PROPOSITION 1.2. *Let A and B both be semiprime Banach algebras and let $T : A \to B$ be a homomorphism.*

(1) *If T is one-one then $T^{-1}\mathcal{F}_1(B) \subset \mathcal{F}_1(A)$.*
(2) *If T is surjective then $T\mathcal{F}_1(A) \subset \mathcal{F}_1(B)$.*

Proof. (1) If $Ta \in \mathcal{F}_1(B)$ and $y \in B$ then $(Ta)y(Ta) = \phi(y)(Ta)$, and if $y = Tx$ then $axa - \phi(Tx)a \in T^{-1}(0) = \{0\}$.

(2) If $a \in \mathcal{F}_1(A)$ then there is a linear functional τ_a on A such that $axa = \tau_a(x)a$ for every $x \in A$. In order to show that $Ta \in \mathcal{F}_1(B)$ it is sufficient to show that $\dim(TaBTa) \leq 1$ ([6], Corollary 3.3). Indeed, $TaBTa = T(aAa)$, which implies that $\dim(TaBTa) \leq 1$. $\blacksquare$

We provide examples to show that the injective and surjective condition in the above result cannot be omitted.

EXAMPLES 1.3. (1) Let A be the Banach algebra $\mathbb{C}^n$ and let B be the Banach algebra $\mathbb{C}$. If $T : A \to B$ is defined by $T(\alpha_1, \ldots, \alpha_n) := \alpha_1$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$, then T is a homomorphism which is not injective. Furthermore, the element $(1, 1, 0, \ldots, 0)$ is not rank one in $\mathbb{C}^n$ while $T(1, 1, 0, \ldots, 0) = 1$ is rank one in $\mathbb{C}$.

(2) Let $D := \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$ and let $\mathcal{A}(D)$ denote the disc algebra. Consider the complex numbers $\mathbb{C}$ as an algebra over $\mathbb{C}$. If $T : \mathbb{C} \to \mathcal{A}(D)$ is the homomorphism defined by $T\alpha := \alpha 1$ $(\alpha \in \mathbb{C})$ where $1(z) := 1$ $(z \in D)$ is the identity in $\mathcal{A}(D)$, then T is not onto. Furthermore, the image of a rank one element is not a rank one element because every $0 \neq \alpha \in \mathbb{C}$ is rank one while $\mathcal{A}(D)$ contains no nonzero rank one elements (see the remarks following 2.1).

2. A spectral characterisation of rank one elements. In this section we give a spectral characterisation of rank one elements in semisimple Banach algebras. In order to do this we first take a look at some spectral properties of rank one elements.

The *spectrum* of an element $a \in A$ is the set $\sigma_A(a) := \{\lambda \in \mathbb{C} : \lambda - a \notin A^{-1}\}$ and if the spectrum of a in A is finite, $\#\sigma_A(a)$ will denote the number of elements in $\sigma_A(a)$. Rank one elements are *algebraic*, since $a(a - \tau_a(1)) = 0$, so that their spectrum can at most contain two points:

PROPOSITION 2.1. *Let $A \neq \mathbb{C}$ be a semiprime Banach algebra and let $a \in \mathcal{F}_1(A)$.*

(1) $\sigma_A(a) = \{0, \tau_a(1)\}$ ([6], Lemma 2.8).
(2) *If $b \in A^{-1}$ then $b + a \notin A^{-1}$ if and only if $\tau_a(b^{-1}) = -1$.*
(3) *If $b \in A$ and $s, t \in \mathbb{C}$ with $s \neq t$ then $\sigma_A(b+ta) \cap \sigma_A(b+sa) \subset \sigma_A(b)$.*

Proof. (1) Note that $0 \in \sigma_A(a)$ because $a \in \mathcal{F}_1(A)$ and $A \neq \mathbb{C}$.

(2) If $b \in A^{-1}$ then $b + a = b(1 + b^{-1}a)$. Hence,

$$b + a \notin A^{-1} \Leftrightarrow -1 \in \sigma_A(b^{-1}a) \Leftrightarrow -1 \in \{0, \tau_a(b^{-1})\} \Leftrightarrow \tau_a(b^{-1}) = -1.$$

(3) If $\lambda \notin \sigma_A(b)$ then by (2), $\lambda \in \sigma_A(b + sa) \cap \sigma_A(b + ta)$ implies that $s\tau_a((\lambda - b)^{-1}) = -1 = t\tau_a((\lambda - b)^{-1})$ and so $s = t$. $\blacksquare$

If $A \neq \mathbb{C}$ is a Banach algebra with no nonzero divisors of zero then $\mathcal{F}_1(A) = \{0\}$. Indeed, if $0 \neq a \in \mathcal{F}_1(A)$ then by 2.1(1) the spectrum of a is either $\{0, \tau_a(1)\}$ or $\{0\}$. The first instance is not possible since it would imply the existence of nontrivial idempotents, which is impossible because in an algebra, with no nonzero zero divisors the only idempotents are 0 and 1.

The latter case is also not possible, because if it were then $a^2 - \tau_a(1)a = 0$ would imply the existence of nonzero divisors of zero.

Jafarian and Sourour ([5], Theorem 1) have shown that if $A = B(X,X)$ is the algebra of bounded linear operators on a Banach space X the condition 2.1(3) is sufficient for $a \in \mathcal{F}_1(A)$. Condition 2.1(3) is also sufficient for $a \in \mathcal{F}_1(A)$ when $A = C_\infty(\Omega)$ is the algebra of complex-valued bounded continuous functions on a completely regular Hausdorff space Ω: Suppose $a \notin \mathcal{F}_1(A)$. If there are $s, t \in \Omega$ with $0 \neq a(s) \neq a(t) \neq 0$, then 2.1(3) fails for $b = 0$ and $k := a(s)/a(t)$ since $\sigma_A(b+a) \cap \sigma_A(b+ka) \not\subset \sigma_A(b)$. This argument also shows in general that if A is a Banach algebra and $0 \neq a \in A$ satisfies 2.1(3) then the spectrum of a cannot contain two distinct nonzero points. We will show later that in actuality a stronger statement is true (see 2.6). If a is a nontrivial idempotent then there exists an open-closed subset Ω_1 of Ω consisting of more than one point. Let b be a bounded continuous function which is nonconstant on Ω_1 and which is invertible in A. It is then possible to choose complex numbers c_1 and c_2 such that $0 \in \sigma_A(b+c_1a) \cap \sigma_A(b+c_2a)$ and so $\sigma_A(b+c_1a) \cap \sigma_A(b+c_2a) \not\subset \sigma_A(b)$. If the spectrum of a consists of 0 and one other point a similar argument holds.

If a is a scalar α with $\alpha \neq 0$, pick an element b in A such that its spectrum contains 0 and α, but not 2α. Then $\sigma_A(b+a) \cap \sigma_A(b+2a) \not\subset \sigma_A(b)$.

Before we formulate our main theorem we point out that the spectral condition 2.1(3) is not always sufficient for membership in $\mathcal{F}_1(A)$: Let A be a commutative semiprime Banach algebra which is not semisimple. If $0 \neq a \in \operatorname{Rad} A$ then a satisfies the spectral condition 2.1(3), but a is not rank one. If it were, then the ideal Aa would satisfy $(Aa)^2 = 0$, which contradicts the fact that A is semiprime.

THEOREM 2.2. *Let A be a semisimple Banach algebra and $a \in A$. Then a is rank one if and only if a satisfies the following condition*:

$$
\text{(2.3)} \quad b \in A \text{ and } s_0, s_1 \in \mathbb{C},\ 0 \neq s_0 \neq s_1 \neq 0 \Rightarrow \sigma_A(b+s_0a) \cap \sigma_A(b+s_1a) \subset \sigma_A(b).
$$

As we have remarked, Jafarian and Sourour ([5], Theorem 1) proved 2.2 for the algebra $B(X,X)$. Our proof uses entirely different methods. Two important features of our proof are the use of minimal idempotents and an application of a scarcity lemma of Aupetit ([1], Theorem 3.2, p. 67).

A *minimal idempotent* in an algebra A is a nonzero idempotent e such that eAe is a division algebra. It was remarked in ([6], Remarks 2.4, 2.5) that in a Banach algebra there is a close relationship between minimal idempotents, minimal ideals and rank one elements. It follows from these remarks that if a is a rank one element then Aa is a minimal left ideal and that there is a minimal idempotent $p \in Aa$, i.e., there exists $c \in A$ with $p = ca$. In our next result we will demonstrate an explicit way to obtain such a minimal idempotent.

To prove 2.2 we need the following results:

PROPOSITION 2.4. *Let A be a Banach algebra, $a \in A$ and suppose α is a nonzero isolated point of $\sigma_A(a)$. If p is the spectral idempotent associated with α then there exists a $c \in A$ such that $p = ac = ca$.*

Proof. Let Γ be a circle centered at α separating α from 0 and the rest of the spectrum of a. For $\lambda \in \Gamma$ we have

$$
(\lambda - a)^{-1} = \frac{1}{\lambda} + \frac{1}{\lambda} a(\lambda - a)^{-1}.
$$

So we have

$$
p = \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda}\, d\lambda + \frac{a}{2\pi i} \int_\Gamma \frac{1}{\lambda}(\lambda - a)^{-1}\, d\lambda.
$$

Since the first term is zero, $p = ac = ca$ with

$$
c := \frac{1}{2\pi i} \int_\Gamma \frac{1}{\lambda}(\lambda - a)^{-1}\, d\lambda. \ \blacksquare
$$

COROLLARY 2.5. *Let A be a semiprime Banach algebra and let $a \in A$ be a rank one element which is not quasinilpotent. Then the spectral idempotent p associated with the nonzero spectral point of a is a minimal idempotent p with $p = ca = ac$ for some $c \in A$.*

Proof. Since a is not quasinilpotent, $\tau_a(1) \neq 0$. In view of 2.4 the spectral idempotent associated with $\tau_a(1)$ has the property that $p = ca = ac$ for some $c \in A$. It remains to show that p is a minimal idempotent. Observe that Aa is a minimal left ideal of A with $\{0\} \neq Ap \subset Aa$. Hence $Ap = Aa$ and from ([3], Lemma 2, p. 154) we conclude that p is a minimal idempotent. $\blacksquare$

LEMMA 2.6. *Let A be a semisimple Banach algebra and let $b \in A^{-1}$. If $0 \neq a \in A$ satisfies (2.3) then the spectrum of $b^{-1}a$ cannot contain two distinct nonzero points.*

Proof. Let $b \in A^{-1}$ and suppose $\alpha_i \in \sigma_A(b^{-1}a)$ $(i = 0, 1)$ with the α_i nonzero and distinct. If we assume that $a \in A$ satisfies (2.3), then in particular, for $b \in A^{-1}$,

$$
b + s_0 a \in A^{-1} \quad \text{or} \quad b + s_1 a \in A^{-1}.
$$

This implies that

$$
-\frac{1}{s_0} \notin \sigma_A(b^{-1}a) \quad \text{or} \quad -\frac{1}{s_1} \notin \sigma_A(b^{-1}a).
$$

If we choose $s_i = -1/\alpha_i$ $(i = 0, 1)$ we obtain a contradiction. $\blacksquare$

LEMMA 2.7. *Let A be a Banach algebra and $0 \neq a \in A$. If $\#\sigma_A(ba) \leq n$ for every $b \in A^{-1}$ then $\#\sigma_A(ba) \leq n$ for every $b \in A$.*

Proof. Let $b \in A^{-1}$ and $x \in A$. The map

$$\lambda \mapsto [b + \lambda(x - b)]a$$

is an analytic function from $\mathbb{C}$ to Aa. Since A^{-1} is open, there exists an $r > 0$ such that $\#\sigma_A[(b + \lambda(x - b))a] \leq n$ for all λ with $|\lambda| < r$. By ([1], Theorem 3.2, p. 67 or by the subharmonicity of the log of the nth diameter of the spectrum [7]) one deduces that $\#\sigma[(b+\lambda(x-b))a] \leq n$ for all $\lambda \in \mathbb{C}$; in particular, for $\lambda = 1$. Hence the spectrum of xa has the required property. ∎

LEMMA 2.8. *Let A be a Banach algebra and let B be a subalgebra of A such that $\sigma_A(b)$ consists of 0 and possibly one other point for every $b \in B$. Then there are no orthogonal idempotents in B.*

Proof. Suppose p and q are orthogonal idempotents in B. Then $\{1, 2\} \subset \sigma_A(p + 2q)$, which is impossible. ∎

LEMMA 2.9. *Let A be a semisimple Banach algebra and let $0 \neq a \in A$ be such that the spectrum of xa consists of 0 and possibly one other point for every $x \in A$. Then there exists a minimal idempotent $p \in Aa$.*

Proof. If every element of Aa is quasinilpotent then $a \in \operatorname{Rad} A = \{0\}$. Hence there exists an $xa \in Aa$ with a nonzero isolated point in its spectrum. Let p be the spectral idempotent associated with this point. By 2.4, $p = cxa$ for some $c \in A$ and hence $p \in Aa$. It remains to show that p is a minimal idempotent. Let $B = pAp$. Then $B \subset Aa$ and hence $\sigma_B(pyp)$ consists of at most two points for every $pyp \in pAp$. However, if $\sigma_B(pyp)$ contains two points then there exists a nontrivial idempotent pzp in pAp. But pzp and $p - pzp$ are orthogonal idempotents in Aa, which yields a contradiction with the fact that the spectrum of every element of Aa consists of 0 and possibly one other point. This together with the fact that pAp is semisimple implies $pAp = \mathbb{C}p$ ([2], BA 3.9, p. 108) and hence p is a minimal idempotent. ∎

Proof of Theorem 2.2. ($\Rightarrow$) This is 2.1(3).

($\Leftarrow$) Suppose $a \neq 0$ satisfies condition (2.3). There are two cases to be considered: (1) $a \in A^{-1}$ or (2) $a \notin A^{-1}$.

(1) If $a \in A^{-1}$ then $Aa = A$ and by 2.6 every element of $A^{-1}a$ has exactly one point in its spectrum, while by 2.7 we deduce that every element of A has a one point spectrum. This together with the fact that A is semisimple yields $A = \mathbb{C}1$ and hence $a \in \mathcal{F}_1 A$.

(2) If $a \notin A^{-1}$ then $Aa \neq A$ or $aA \neq A$. We only consider the first case. Since $Aa \neq A$ the spectrum of every element of Aa contains 0. It follows from 2.6 and 2.7 that the spectrum of every element of Aa consists of 0 and possibly one other point. By 2.9 there exists a minimal idempotent $p \in Aa$. If we suppose $ap \neq a$ then

$$A(ap - a) \subset Aa$$

and by 2.9 there exists a minimal idempotent $q \in A(ap - a)$. Clearly $qp = 0$ and if $w = q - pq$ then $qw = q \neq 0$, which shows that $w \neq 0$. Furthermore, $w^2 = w$ and $pw = wp = 0$. This is in contradiction with the fact that the spectrum of every element of Aa consists of 0 and possibly one other point. Since $p \in \mathcal{F}_1(A)$ and $a = ap$ the result follows. ∎

3. Finite elements. We conclude this paper by extending the theory developed in Section 2 to the set of finite-dimensional elements of a semisimple Banach algebra.

The *finite elements* of A, denoted by $\mathcal{F}(A)$, is the set of all $a \in A$ of the form

$$a = \sum_{i=1}^{n} a_i \quad \text{with } a_i \in \mathcal{F}_1(A).$$

In the case of a semiprime algebra the set of finite elements coincides with the socle of A, i.e. $\operatorname{Soc} A = \mathcal{F}(A)$. Since $\mathcal{F}_1(A)A, A\mathcal{F}_1(A) \subset \mathcal{F}_1(A)$, $\operatorname{Soc} A$ is a two-sided ideal in A.

The following theorem is our main result of this section. It is the analog of Theorem 2.2 for elements of the socle of a semisimple Banach algebra.

THEOREM 3.1. *Let A be a semisimple Banach algebra and $a \in A$. Then a is a finite element if and only if there exists a positive integer n such that for every $b \in A$ and every set of nonzero distinct scalars s_i $(i = 0, 1, \ldots, n)$,*

$$\sigma_A(b + s_0 a) \cap \sigma_A(b + s_1 a) \cap \ldots \cap \sigma_A(b + s_n a) \subset \sigma(b). \tag{3.2}$$

Before we prove Theorem 3.1 we extend some of the results of Section 2. Since the proofs of these results are easy modifications of those given in Section 2 some of them will be omitted.

LEMMA 3.3. *If A is a semisimple Banach algebra and if $0 \neq a \in A$ satisfies (3.2) then every element of $A^{-1}a$ can have at most n distinct nonzero points in its spectrum.*

LEMMA 3.4. *Let A be a semisimple Banach algebra and let $0 \neq a \in A$. If the spectrum of every element of Aa consists of 0 and at most n other distinct nonzero points then there exists a minimal idempotent $p \in Aa$.*

Proof. Choose $b \in Aa$ which has a nonzero isolated point λ in its spectrum. Such an element exists because in view of the semisimplicity of A not every element of Aa is quasinilpotent. If p is the spectral idempotent associated with λ and b, then by Proposition 2.4, $p \in Aa$. From our hypothesis we deduce that every element in the semisimple subalgebra $pAp \subset Aa$ has a finite spectrum. By [4], pAp is finite-dimensional and so there is a minimal idempotent $q \in Ap \subset Aa$ ([6], Lemma 3.2). ∎

Proof of Theorem 3.1. ($\Rightarrow$) If $0 \neq a \in \operatorname{Soc} A$ then $\dim aAa \leq m$, say. We claim that every element of aA has at most $2m$ distinct nonzero points in its spectrum: For every $b \in A$, $\dim(abAab) \leq \dim(aAa)$ and hence it suffices to show that the spectrum of a contains at most $2m$ distinct nonzero points. Let B be a maximal commutative subalgebra of A containing a. Then $\sigma_A(b) = \sigma_B(b)$ for every $b \in B$. The linear operator $L_{a^2} : B \to B$ defined by $L_{a^2}x := a^2x$ $(x \in B)$ is a finite rank operator with rank $\leq m$. Since every nonzero spectral value of L_{a^2} is an eigenvalue and the eigenvectors associated with different eigenvalues are linearly independent, it follows that L_{a^2} has at most m distinct nonzero points in its spectrum. The fact that $\sigma(L_{a^2}) = \sigma_B(a^2)$ and the spectral mapping theorem completes the argument. It follows readily that the spectral condition is satisfied with $n = 2m$.

($\Leftarrow$) Suppose $a \neq 0$ satisfies (3.2). As in the proof of Theorem 2.2 there are two cases to be considered: (1) $a \in A^{-1}$ or (2) $a \notin A^{-1}$.

(1) If $a \in A^{-1}$ then $Aa = A$ and by 3.3 every element of $A^{-1}a$ has at most n distinct points in its spectrum, while by 2.7 every element of A has at most n distinct points in its spectrum. This together with the fact that A is semisimple and [4] yields A is finite-dimensional and hence $A = \operatorname{Soc} A$ ([6], Corollary 3.5).

(2) We only consider the case where $Aa \neq A$. By 3.3 and by 2.7 every element of Aa has at most n distinct nonzero points in its spectrum. We infer from 3.4 that there exists a minimal idempotent $p \in Aa$. Analogous to 2.8 we conclude that there are at most n distinct orthogonal idempotents in Aa. Suppose $\{p_i : i = 1, \ldots, k\}$ is a maximal set of orthogonal minimal idempotents in Aa. If we put $p = \sum_{i=1}^{k} p_i$, then we claim that $a = ap$ and so $a \in \operatorname{Soc} A$ since every minimal idempotent is a rank one element. If we suppose to the contrary that $a \neq ap$ then in view of 3.4 the proof is a modification of case (2) in the proof of 2.2. ∎

Remarks. 1. If X is a Banach space, Theorem 3.1 gives a new characterisation of finite rank operators on X because the socle of the algebra of bounded linear operators on X is the finite rank operators.

2. Lemma 3.4 can also be proved directly, without the use of [4] and ([6], Lemma 3.2), by a modification of the proof of 2.9.

3. In the case of a commutative Banach algebra the proofs of 2.2 and 3.1 would be considerably shorter.

4. Our argument also yields a duality between the radical and the socle of a semisimple Banach algebra. By an easy modification of the proof of Lemma 2.7 it can be shown that $\operatorname{Rad} A = \{a : A^{-1}a \subset \mathrm{QN}(A)\}$ ($\mathrm{QN}(A)$ denotes the set of quasinilpotent elements of A).

The authors would like to thank Prof. Robin Harte and the referee for their helpful suggestions.

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Received September 11, 1991 (2834)
Revised version January 4, 1993