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Semigroups with nonquasianalytic growth

by

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Abstract. We study asymptotic behavior of C_0-semigroups $T(t)$, $t \geq 0$, such that $\|T(t)\| \leq \alpha(t)$, where $\alpha(t)$ is a nonquasianalytic weight function. In particular, we show that if $\sigma(A) \cap i\mathbb{R}$ is countable and $P\sigma(A^*) \cap i\mathbb{R}$ is empty, then $\lim_{t\to\infty} \frac{1}{\alpha(t)} \|T(t)x\| = 0$, $\forall x \in X$. If, moreover, f is a function in $L^1_\alpha(\mathbb{R}_+)$ which is of spectral synthesis in a corresponding algebra $L^1_{\alpha_1}(\mathbb{R})$ with respect to $(i\sigma(A)) \cap \mathbb{R}$, then $\lim_{t\to\infty} \frac{1}{\alpha(t)} \|T(t)\widehat{f}(T)\| = 0$, where $\widehat{f}(T) = \int_0^\infty f(t)T(t)\,dt$.

Analogous results are obtained also for iterates of a single operator. The results are extensions of earlier results of Katznelson–Tzafriri, Lyubich–Vũ Quôc Phóng, Arendt–Batty, ..., concerning contraction semigroups. The proofs are based on the operator form of the Tauberian Theorem for Beurling algebras with nonquasianalytic weight.

0. The classical Wiener Approximation Theorem is equivalent to the General Tauberian Theorem (see e.g. [5, 10]). In the papers [11], [12], we have shown that from this theorem one can obtain some new results (or new proofs of old results) on asymptotic behavior of iterates of power-bounded operators, and of trajectories of bounded C_0-semigroups (cf. [1, 2, 6, 8]). On the other hand, Beurling has introduced a more general class of Banach algebras $L^1_\alpha(\mathbb{R})$ with a weight α, and proved that the Tauberian Theorem remains valid if α satisfies a certain condition of nonquasianalyticity (see e.g. [10]). This theorem has then been used in [7] for spectral analysis of groups of operators which are dominated by nonquasianalytic weight functions.

In this paper, we extend the results of [1, 2, 6, 8, 11] to a more general case of semigroups $\mathcal{T} = \{T(t) : t \geq 0\}$ of bounded linear operators such that $\|T(t)\|$ is dominated by a weight function with nonquasianalytic growth. We use an operator-theoretical form of the Tauberian Theorem (Lemmas 5 and 6), and a modification of a construction (limit semigroups) in [11].

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1. Let $\alpha(t)$, $t \in \mathbb{R}$, be a real measurable locally bounded function satisfying

$$(1) \qquad 1 \le \alpha(t) < \infty, \quad \alpha(t+s) \le \alpha(t)\alpha(s).$$

Such a function is called a (*two-sided*) *weight function*. One can show that if $\alpha(t)$, $-\infty < t < \infty$, is a weight function, then the limits

$$(2) \qquad \omega_+(\alpha) = \lim_{t\to\infty} t^{-1}\log\alpha(t), \quad \omega_-(\alpha) = \lim_{t\to-\infty} t^{-1}\log\alpha(t)$$

exist and are finite, and that $\omega_-(\alpha) \le 0 \le \omega_+(\alpha)$.

A weight function $\alpha(t)$, $t \ge 0$, is said to be *nonquasianalytic* if

$$(3) \qquad \int_0^\infty \frac{\log\alpha(t)}{1+t^2}\,dt < \infty.$$

It is known that if $\alpha(t)$ is nonquasianalytic, then it has *zero exponential type*, i.e. $\omega_+(\alpha) = \omega_-(\alpha) = 0$. Let $\alpha(t), -\infty < t < \infty$, be a (two-sided) weight function, and $L^1_\alpha(\mathbb{R})$ the Banach algebra consisting of functions which are absolutely integrable with the weight $\alpha(t)$ and with the norm

$$\|f\|_{L^1_\alpha(\mathbb{R})} = \int_{-\infty}^\infty |f(t)|\alpha(t)\,dt < \infty.$$

This algebra is called the *Beurling algebra*. It is well known that if $\alpha(t)$ is nonquasianalytic, then $L^1_\alpha(\mathbb{R})$ is a regular Banach algebra (see e.g. [10]). If $\alpha(t)$ has zero exponential type, then the Gelfand space of $L^1_\alpha(\mathbb{R})$ coincides with $\mathbb{R}$. For each $f \in L^1_\alpha(\mathbb{R})$ we denote its Fourier transform by $\widehat{f}$. Let Δ be a closed subset of $\mathbb{R}$. A function $f \in L^1_\alpha(\mathbb{R})$ is said to be of *spectral synthesis* with respect to Δ if there exists a sequence g_n in $L^1_\alpha(\mathbb{R})$ such that, for each n, the Fourier transform $\widehat{g}_n$ is equal to 0 on a neighborhood $\mathcal{U}_n$ of Δ and $\|g_n - f\|_{L^1_\alpha(\mathbb{R})} \to 0$ as $n \to \infty$ (see e.g. [1, 5, 6]).

Now let $\alpha(t)$ be a locally bounded measurable function which is defined and satisfies (1)–(2) for $t \ge 0$. Such a function is called a *one-sided weight function*. Let $L^1_\alpha(\mathbb{R}_+)$ denote the Banach algebra of functions on $\mathbb{R}_+$ such that

$$\int_0^\infty |f(t)|\alpha(t)\,dt < \infty.$$

If $\omega_+(\alpha) = 0$, i.e. $\alpha(t)$ has zero exponential type, then the Gelfand space of $L^1_\alpha(\mathbb{R}_+)$ coincides with $\mathbb{C}_- = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$.

We assume, moreover, that $\alpha(t)$ satisfies

$$(4) \qquad \liminf_{t\to\infty} \frac{\alpha(t+s)}{\alpha(t)} \ge 1, \quad \forall s > 0.$$

(It is not difficult to see that condition (4) holds for every s if and only if it holds for a certain $s > 0$.) With each one-sided weight function $\alpha(t)$, $t \ge 0$, we also associate a one-sided weight function $\alpha_1(t)$, $t \ge 0$, defined by

$$\alpha_1(s) \equiv \limsup_{t\to\infty} \frac{\alpha(t+s)}{\alpha(t)}.$$

Clearly, $\alpha_1(t) \le \alpha(t)$ for each $t \ge 0$, and, thanks to (4), $\alpha_1(t)$ is a nondecreasing function. Note that $\alpha_1(t)$ may be essentially less than $\alpha(t)$. For example, if $\alpha(t) = (t+1)^k$, $k \ge 0$, or $\alpha(t) = \exp\{t^k\}$, $0 \le k < 1$, then $\alpha_1(t) \equiv 1$. We call $\alpha_1(t)$ the *reduced weight function* of $\alpha(t)$.

Since $\alpha_1(t)$ is nondecreasing, it can be extended to a two-sided weight function by putting $\alpha_1(t) = 1$ if $t < 0$.

Now let $\mathcal{T} = \{T(t) : t \ge 0\}$ be a strongly continuous semigroup (C_0-semigroup) of bounded linear operators in a Banach space X. We say that $\mathcal{T}$ is *dominated* by a (one-sided) weight function $\alpha(t)$ if $\|T(t)\| \le \alpha(t)$ for each $t \ge 0$. If $\mathcal{T}$ is dominated by a weight function which has zero exponential type, then the spectrum $\sigma(A)$ of its generator A is contained in the left half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le 0\}$. We define, for each f in $L^1_\alpha(\mathbb{R}_+)$, its *Laplace transform* $\widehat{f}(\mathcal{T})$ with respect to the semigroup $\mathcal{T}$ by

$$\widehat{f}(\mathcal{T}) = \int_0^\infty f(t)T(t)\,dt.$$

(The integral is a strongly convergent Bochner integral.)

We shall need the following preliminary lemmas.

LEMMA 1. *Suppose that $\mathcal{T} = \{T(t) : t \ge 0\}$ is a C_0-semigroup with generator A. Assume that $\|T(t)x\| \ge \|x\|$ for each x in X. Then*

$$(5) \qquad \|Ax - \lambda x\| \ge |\operatorname{Re}\lambda|\|x\|$$

for every λ with $\operatorname{Re}\lambda < 0$ and every x in X.

Proof. We write $\lambda = -\varrho + i\omega$, $\omega \in \mathbb{R}$, $\varrho > 0$, and consider the vector-valued function $u(t) = e^{-\lambda t}T(t)x$, $t \ge 0$. From the assumption we have

$$(6) \qquad \|u(t)\| \ge e^{\varrho t}\|x\|.$$

On the other hand,

$$u(t) = x + \int_0^t \frac{du(\tau)}{d\tau}\,d\tau = x + \int_0^t e^{-\lambda\tau}T(\tau)(Ax - \lambda x)\,d\tau.$$

Therefore

$$(7) \qquad \|u(t)\| \le \|x\| + \int_0^t e^{\varrho\tau}\,d\tau \sup_{0\le\tau\le t}\|T(\tau)(Ax - \lambda x)\|.$$

From (6) and (7) we obtain

$$(e^{\varrho t}-1)\|x\| \le \frac{e^{\varrho t}-1}{\varrho} \sup_{0\le\tau\le t} \|T(\tau)(Ax-\lambda x)\|\,,$$

or

$$\|x\| \le \frac{1}{\varrho} \sup_{0\le\tau\le t} \|T(\tau)(Ax-\lambda x)\|, \quad \forall t>0\,. \tag{8}$$

Since the semigroup $\mathcal{T}$ is strongly continuous, from (8) we get (5). ∎

From Lemma 1 we get the following sufficient condition for extendability of $\mathcal{T}$ to a C_0-group.

LEMMA 2. *Suppose that $\mathcal{T}=\{T(t):t\ge 0\}$ is a C_0-semigroup which is dominated by a weight function with zero exponential type. Assume that $\|T(t)x\|\ge\|x\|$ for each x in X, and, moreover, $\sigma(A)$ does not fill the whole imaginary axis. Then each $T(t)$ is invertible, so that $\mathcal{T}$ can be extended to a C_0-group.*

Proof. From Lemma 1 it follows that the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$ is contained in a regular component of the operator A (see [3]). It is known that the defect number $\delta(\lambda)\equiv\dim\{\phi\in\mathcal{D}(\mathcal{A}^*):\mathcal{A}^*\phi=\lambda\phi\}$ does not depend on λ belonging to the same regular component. Since $\mathcal{T}$ is dominated by a weight function with zero exponential type, the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}$ is contained in a regular component (in fact, the resolvent set) of A. From the existence of a point in $(i\mathbb{R})\cap\varrho(A)$ it follows that the defect number of A corresponding to the half-plane $\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}$ is zero. Thus we have

$$\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda<0\}\subset\varrho(A)\,. \tag{9}$$

From (9) and Lemma 1 it follows that the operator $-A$ satisfies

$$\{\lambda\in\mathbb{C}:\operatorname{Re}\lambda>0\}\subset\varrho(-A)\,,$$

and

$$\|[(-A)-\lambda]^{-1}\|\le\frac{1}{\operatorname{Re}\lambda}, \quad \forall\operatorname{Re}\lambda>0\,.$$

By the Hille–Yosida Theorem (see e.g. [9]), $-A$ is the generator of a strongly continuous semigroup of contractions $S(t)$, $t\ge 0$. It is easy to see that

$$\frac{d}{dt}\{T(t)S(t)x\}=T(t)S(t)Ax-T(t)S(t)Ax=0, \quad \forall x\in X\,,$$

so that $T(t)S(t)=I$ for each $t\ge 0$. Thus $T(t)^{-1}=S(t)$, and $\mathcal{T}$ can be extended to a C_0-group. ∎

LEMMA 3. *Let $\mathcal{T}=\{T(t):t\ge 0\}$ be a C_0-semigroup dominated by a weight function $\alpha(t)$, $t\ge 0$. Then there exist a Banach space E, a continuous homomorphism $\pi:X\to E$, with a dense range, and a semigroup $\mathcal{V}=\{V(t):t\ge 0\}$ in E, with generator S, such that $\mathcal{V}$ is dominated by the reduced weight function $\alpha_1(t)$ and the following properties hold:*

(i) *$\|V(t)z\|\ge\|z\|$ for each z in E and $t\ge 0$;*

(ii) *$V(t)\circ\pi=\pi\circ T(t)$ for each $t\ge 0$; thus $\pi(\mathcal{D}(A))\subset\mathcal{D}(S)$ and $S\circ\pi x=\pi\circ Ax$, $\forall x\in\mathcal{D}(A)$;*

(iii) *$\|\pi x\|=\limsup_{t\to\infty}\frac{1}{\alpha(t)}\|T(t)x\|$, $\forall x\in X$;*

(iv) *$\sigma(S)\subset\sigma(A)$, $P\sigma(S^*)\subset P\sigma(A^*)$, where $P\sigma$ denotes point spectrum.*

The Banach space E and semigroup $\mathcal{V}$ with the above properties are unique up to unitary equivalence.

Proof. We set, for each x in X,

$$l(x)=\limsup_{t\to\infty}\frac{1}{\alpha(t)}\|T(t)x\|\,. \tag{10}$$

Clearly, $l(x)$ is a seminorm in X and

$$l(x)\le\|x\| \tag{11}$$

for each x in X. Let $L=\ker l=\{x\in X:l(x)=0\}$. From (11) it follows that L is a closed subspace which is invariant with respect to every operator $T(t)$. Let $\widehat{X}$ denote the quotient space X/L and π the canonical homomorphism from X to $\widehat{X}$. Then l naturally induces a norm $\widehat{l}$ in $\widehat{X}$ by $\widehat{l}(\pi x)\equiv l(x)$ and each operator $T(t)$ induces an operator $\widehat{T}(t)$ in $\widehat{X}$ by $\widehat{T}(t)(\pi x)\equiv\pi(T(t)x)$. Since the seminorm $l(x)$ is dominated by the original norm in X by (11), the semigroup $\widehat{T}(t)$ is strongly continuous in the norm $\widehat{l}$. Let E be the completion of $\widehat{X}$ in the norm $\widehat{l}$ and $V(t)$ be the extension of $\widehat{T}(t)$ by continuity from $\widehat{X}$ to E. It is clear from the preceding remark that $\mathcal{V}=\{V(t):t\ge 0\}$ is a strongly continuous semigroup. From (10) it is easy to see that

$$l(T(s)x)\le\alpha_1(s)l(x)\,,$$

which implies that the semigroup $\mathcal{V}$ is dominated by the weight function $\alpha_1(t)$.

Property (i) follows from (10), and (ii) and (iii) follow easily from the construction. To prove $\sigma(S)\subset\sigma(A)$ let us first remark that if R is any bounded operator commuting with $T(t)$ for each $t\ge 0$, then R induces naturally a bounded operator $\widehat{R}$ in E. This is a simple consequence of the invariance of L with respect to R and of the obvious inequality

$$l(Rx)\le\|R\|l(x)\,.$$

It is well known (see e.g. [9]) that each λ such that $\operatorname{Re}\lambda>\omega_+(\alpha)$ is contained in the resolvent set $\varrho(A)$ of A and

$$R(\lambda;A)x\equiv(A-\lambda)^{-1}x=-\int_0^\infty e^{-\lambda t}T(t)x\,dt, \quad \forall x\in X\,.$$

Therefore

$$\widehat{R}(\lambda; A)\pi x = \pi(R(\lambda; A)x) = -\int_0^\infty e^{-\lambda t} V(t)\pi x\, dt\,,$$

or

$$\widehat{R}(\lambda; A) = R(\lambda; S)\,.$$

Let $\mu \in \varrho(A)$. From the Hilbert identity

$$R(\lambda; A) - R(\mu; A) = (\lambda - \mu) R(\lambda; A) R(\mu; A)$$

it follows that

$$\widehat{R}(\mu; A)[I + (\lambda - \mu)\widehat{R}(\lambda; A)] = \widehat{R}(\lambda; A)\,,$$

or

$$\widehat{R}(\mu; A)[I + (\lambda - \mu)R(\lambda; S)] = R(\lambda; S) = R(\mu; S)[I + (\lambda - \mu)R(\lambda; S)]\,,$$

which implies that $\mu \in \varrho(S)$ (and $R(\mu; S) = \widehat{R}(\mu; A)$). Thus we have shown $\varrho(A) \subset \varrho(S)$, or $\sigma(S) \subset \sigma(A)$.

Furthermore, suppose that $\lambda \in P\sigma(S^*)$ and ϕ is a corresponding eigenfunctional such that $S^*\phi = \lambda\phi$. We put $f(x) \equiv \phi(\pi x)$ for each x in X. From (11) it follows that f is a nonzero continuous functional; moreover,

$$\begin{aligned}(A^* f)(x) &= f(Ax) = \phi(\pi \circ Ax) = \phi(S \circ \pi x) \\ &= (S^*\phi)(\pi x) = \lambda\phi(\pi x) = \lambda f(x), \qquad \forall x \in X\,,\end{aligned}$$

so that $A^* f = \lambda f$, i.e. $\lambda \in P\sigma(A^*)$.

Finally, it is easy to see that such a semigroup $\mathcal{V}$ is unique up to unitary equivalence. Indeed, if E' is another Banach space, π' a homomorphism from X to E' with a dense range, and $\mathcal{V}' = \{V'(t) : t \geq 0\}$ a C_0-semigroup with the same properties as (i)–(iv), then the operator $U : E \to E'$ defined by $U\pi x = \pi' x$ (first for all vectors from $\pi(X)$ and then extended by continuity to the whole E) will serve as unitary transformation of $\mathcal{V}$ to $\mathcal{V}'$. ■

The semigroup $\mathcal{V}$ with the properties (i)–(iv) will be called the *limit semigroup* of $\mathcal{T}$.

From Lemmas 3 and 1 it follows that the left half-plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$ is contained in one regular component of S. Lemma 2 shows that if $\mathcal{T}$ is dominated by a sequence $\alpha(t)$ which has zero exponential type, and if $\sigma(A)$ does not fill the imaginary axis, then $\mathcal{V}$ can be extended to a group $\{V(t) : -\infty < t < \infty\}$ such that $\|V(-t)\| \leq 1$ for each $t \geq 0$.

The following result is a slightly improved version of the Gelfand Theorem (see [4, p. 128]). We formulate it in a ready form for later use.

LEMMA 4. *If T is an invertible linear contraction in a Banach space X such that the spectrum of T consists of a single point $\{\lambda_0\}$, and if $\|T^{-n}\| = O(n^k)$ for $n = 1, 2, \ldots$, with some $k \geq 0$, then $T = \lambda_0 I$.*

Proof. We can assume, without loss of generality, that $\lambda_0 = 1$. Proceeding as in [4, Theorem 4.10.1], we see that the function $R(\lambda; T) \equiv (T - \lambda)^{-1}$ is an entire function of $(1 - \lambda)^{-1}$ which has the following two Laurent expansions:

$$R(\lambda; T) = \sum_{n=0}^{\infty} T^{-n-1}\lambda^n \qquad \text{for } |\lambda| < 1\,,$$

$$R(\lambda; T) = -\sum_{n=0}^{\infty} T^n \lambda^{-n-1} \qquad \text{for } |\lambda| > 1\,,$$

with the coefficients satisfying $\|T^{-n}\| = o(n^{k+1})$. By [4, Theorem 3.13.5], $R(\lambda; T)$ is a polynomial in $(1 - \lambda)^{-1}$ of some degree N, say $R(\lambda; T) = \sum_{j=0}^{N} (\lambda - 1)^{-j} A_{N-j}$, $A_0 \neq 0$. On the other hand, since T is a contraction, we have $\|R(\lambda; T)\| \leq (\lambda - \|T\|)^{-1}$ $(\lambda > 1)$. Therefore

$$\|A_0\| = \lim_{\lambda \to 1} \|(\lambda - 1)^N R(\lambda; T)\| \leq \lim_{\lambda \to 1} (\lambda - 1)^N (\lambda - \|T\|)^{-1}\,.$$

Since $R(\lambda; T)$ is not constant, we have $N = 1$ and $R(\lambda; T) = A_0(\lambda - 1)^{-1} + A_1$. Comparing with the corresponding Laurent expansion of $R(\lambda; T)$ we obtain $T = I$. ■

For the convenience of the reader we also formulate the following two lemmas, which can be derived from the Tauberian Theorem and regularity of the Banach algebra $L^1_\alpha(\mathbb{R})$, with a nonquasianalytic weight [7].

LEMMA 5. *Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}\}$ be a C_0-group which is dominated by a nonquasianalytic weight function, and A be its generator. Then $\sigma(A)$ is nonempty.*

LEMMA 6. *Let $\mathcal{T} = \{T(t) : t \in \mathbb{R}\}$ be a C_0-group which is dominated by a nonquasianalytic weight function, and A be its generator. Assume that f is a function in $L^1_\alpha(\mathbb{R})$ such that its Fourier transform $\widehat{f}$ vanishes on a neighborhood of $i\sigma(A)$. Then $\widehat{f}(\mathcal{T}) = 0$.*

Now we are in a position to prove the following theorems.

THEOREM 7. *Suppose that $\mathcal{T}$ is a C_0-semigroup which is dominated by a weight function $\alpha(t)$, $t \geq 0$, such that $\alpha_1(t) = O(t^k)$ for some $k \geq 0$. Assume, moreover, that the imaginary spectrum of A, $\sigma(A) \cap i\mathbb{R}$, is countable, and the imaginary point spectrum of A^*, $P\sigma(A^*) \cap i\mathbb{R}$, is empty. Then $\lim_{t\to\infty} \frac{1}{\alpha(t)} \|T(t)x\| = 0$ for each x in X.*

Proof. The statement means that the limit semigroup $\mathcal{V}$ of $\mathcal{T}$ is equal to 0. Assume that $E \neq 0$. Since $\sigma(A) \cap i\mathbb{R}$ is countable, from Lemma 3(iv) it follows that $\sigma(S) \cap i\mathbb{R}$ is countable. From Lemma 3(i) and Lemma 2 it follows that $\mathcal{V}$ can be extended to a group $\{V(t) : -\infty < t < \infty\}$. Thus

$\{V(t) : -\infty < t < \infty\}$ is a nonquasianalytic group which satisfies

$$\|V(-t)\| \le 1, \quad \|V(t)\| = O(t^k) \quad (t > 0). \tag{12}$$

In particular, the spectrum of S lies on the imaginary axis, hence $\sigma(S)$ itself is countable. Moreover, it is a nonempty set, by Lemma 5, so there exists an isolated point $i\omega$ in $\sigma(S)$. Let Q be the projection in E which corresponds to this isolated point according to the Riesz–Dunford functional calculus. Then Q commutes with each $V(t)$, so that the image $\operatorname{Im} Q$ is an invariant subspace on which the restriction of $V(t)$ is a uniformly continuous semigroup whose generator has one-point spectrum at $i\omega$. By the spectral mapping theorem (valid for all uniformly continuous semigroups), each operator $V(t)|\operatorname{Im} Q$ has one-point spectrum at $e^{i\omega t}$. From (12) it follows that, for a fixed $t_0 > 0$, the operator $V(-t_0)$, and hence $V^*(-t_0)$, satisfies the conditions of Lemma 4. Consequently, there exists a nonzero eigenfunctional ϕ in E^* such that $V^*(t_0)\phi = e^{i\omega t_0}\phi$. Hence $S^*\phi = i\omega\phi$. By Lemma 3(iv), $i\omega \in P\sigma(A^*)$, which is a contradiction. ∎

THEOREM 8. *Let $\mathcal{T} = \{T(t) : t \ge 0\}$ be a C_0-semigroup dominated by a weight function $\alpha(t)$, $t \ge 0$, such that $\alpha_1(t)$ is nonquasianalytic. Assume that f is a function in $L^1_\alpha(\mathbb{R}_+)$ which is of spectral synthesis in the algebra $L^1_{\alpha_1}(\mathbb{R})$ with respect to the set $(i\sigma(A))\cap\mathbb{R}$. Then $\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)\widehat{f}(\mathcal{T})\| = 0$.*

Proof. We can assume that the function f is not identically zero, which implies that $(i\sigma(A)) \cap \mathbb{R}$ is a proper subset of $\mathbb{R}$. Consider the limit semigroup $\mathcal{V}$ of $\mathcal{T}$, and its generator S. By Lemma 3(i), (iv) and Lemma 2, $\mathcal{V}$ can be extended to a group $\{V(t) : -\infty < t < \infty\}$ such that $\|V(t)\| \le 1$ for each $t < 0$. Therefore, $\{V(t) : -\infty < t < \infty\}$ is dominated by a nonquasianalytic weight, namely by the (extended) reduced weight function $\alpha_1(t)$, $-\infty < t < \infty$. Moreover, $\sigma(S)$ lies on the imaginary axis and is a subset of $\sigma(A) \cap i\mathbb{R}$. Therefore, the function f is of spectral synthesis in $L^1_{\alpha_1}(\mathbb{R})$ with respect to $i(\sigma(S))$. If g is a function in $L^1_{\alpha_1}(\mathbb{R})$ such that $\widehat{g}$ is equal to 0 in a neighborhood of $i(\sigma(S))$, then $\widehat{g}(\mathcal{V}) = 0$, by Lemma 6. Hence $\widehat{f}(\mathcal{V}) = 0$, or equivalently, $\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)\widehat{f}(\mathcal{T})x\| = 0$ for each x in X.

So far we have established the strong convergence of $T(t)\widehat{f}(\mathcal{T})$ to 0, as $t \to \infty$, for each semigroup satisfying the conditions of Theorem 8. In order to get from this the convergence in the operator norm, let us consider the Banach space $L(X)$ of all bounded linear operators in X, and define a semigroup $\widehat{\mathcal{T}} = \{\widehat{T}(t) : t \ge 0\}$ on $L(X)$ by $\widehat{T}(t)Y = T(t)Y$ for each $Y \in L(X)$. The semigroup $\widehat{\mathcal{T}}$ in general is not strongly continuous, but we can consider the subspace $L_0(X)$ of $L(X)$ consisting of $Y \in L(X)$ such that $\|T(t)Y - Y\| \to 0$ as $t \to 0$. Since $\|T(t)\|$ is locally bounded, the subspace $L_0(X)$ is a closed subspace of $L(X)$ which is invariant with respect to $\widehat{T}(t)$ for each $t \ge 0$, and the restriction of $\widehat{\mathcal{T}}$ to $L_0(X)$ is strongly continuous. Moreover, $L_0(X)$ contains every operator of the form $\widehat{g}(\mathcal{T})$ for each g in $L^1_\alpha(\mathbb{R})$. This follows from the well known Lebesgue Lemma which asserts that the translation group is strongly continuous (in $L^1(\mathbb{R})$ but this fact remains true in $L^1_\alpha(\mathbb{R})$, see [10, p. 15]). It is also easy to see that the spectrum of the generator A does not change under this construction. Therefore, applying the proved above to the restriction of the semigroup $\widehat{\mathcal{T}}$ to $L_0(X)$, we have

$$\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)\widehat{f}(\mathcal{T})\widehat{g}(\mathcal{T})\| = 0$$

for each g in $L^1_\alpha(\mathbb{R})$. Now from the existence of the approximate identity in $L^1_\alpha(\mathbb{R})$ (see [10]) it follows that $\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)\widehat{f}(\mathcal{T})\| = 0$. ∎

It is known that the Wiener approximation theorem remains true for Beurling algebras with nonquasianalytic weight (see [10]). In other words, this means that the empty set is a set of spectral synthesis in $L^1_{\alpha_1}(\mathbb{R})$ (i.e. with respect to any function g in $L^1_{\alpha_1}(\mathbb{R})$, hence to any $f \in L^1_\alpha(\mathbb{R})$). Thus we have the following corollary.

COROLLARY 9. *Let $\mathcal{T}$ be a C_0-semigroup dominated by a weight function $\alpha(t)$ such that the corresponding reduced weight function $\alpha_1(t)$ is nonquasianalytic. Assume that the intersection of the spectrum of the generator A with the imaginary axis is empty. Then $\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)\widehat{f}(\mathcal{T})\| = 0$ for each f in $L^1_\alpha(\mathbb{R}_+)$. In particular,*

$$\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t)A^{-1}\| = 0 .$$

If the function $\alpha_1(t)$ satisfies

$$\alpha_1(t) = O(1+|t|)^\beta$$

with $0 \le \beta < 1$, then the corresponding Beurling algebra $L^1_{\alpha_1}(\mathbb{R})$ satisfies the Ditkin condition (see [10, p. 132]). In particular, in this algebra every closed countable subset of $\mathbb{R}$ is a set of spectral synthesis. Thus, applying Theorem 8 to the function $f = (e^{-\lambda t} * \delta_s - e^{-\lambda t})$ (where δ_s is the Dirac measure concentrated at s), which is of spectral synthesis with respect to the set $\{0\}$ by the Wiener–Ditkin theorem, we get the following assertion:

$$\lim_{t\to\infty} \frac{1}{\alpha(t)}\|T(t+s)R(\lambda; A) - T(t)R(\lambda; A)\| = 0$$

for each semigroup $\mathcal{T}$ dominated by $\alpha(t)$ such that $\sigma(A) \cap i\mathbb{R} \subset \{0\}$.

For a more general class of weight functions $\alpha(t)$ we can prove the following theorem basically by the same method.

THEOREM 10. *Let $\mathcal{T}$ be a C_0-semigroup with generator A, which is dominated by a weight function $\alpha(t)$ such that $\alpha_1(t) = O(t^k)$ for some $k \ge 0$.*

Assume that $\sigma(A)\cap i\mathbb{R}\subset\{0\}$. *Then* $\lim_{t\to\infty}\frac{1}{\alpha(t)}\|T(t)(T(s)-I)R(\lambda;A)\|=0$ *for each* $s>0$ *and* $\lambda\in\varrho(A)$.

Proof. Again consider the limit semigroup $\mathcal{V}$ of $\mathcal{T}$. We first prove that

$$\lim_{t\to\infty}\frac{1}{\alpha(t)}\|T(t)(T(s)-I)x\|=0$$

for each $s>0$ and each x in X. Thus we can assume, without loss of generality, that E (and hence $\mathcal{V}$) is different from zero. Proceeding as in the proof of Theorem 7, we see that $\mathcal{V}$ can be extended to a group $\{V(t):-\infty<t<\infty\}$ with the same generator S such that the spectrum of S is at most one point $\{0\}$. Since $\mathcal{V}$ is dominated by a nonquasianalytic weight function, $\sigma(S)$ is nonempty, so that $\sigma(S)=\{0\}$. Consequently, $\sigma(V(s))=\{1\}$ for each s. Since $\mathcal{V}$ is dominated by $\alpha_1(t)$ with $\alpha_1(t)=O(t^k)$, we can apply Lemma 4 to get $V(s)-I=0$ for each $s\geq0$, or equivalently, $\lim_{t\to\infty}\frac{1}{\alpha(t)}\|T(t)(T(s)-I)x\|=0$ for each $s\geq0$ and $x\in X$.

So far we have proved the strong convergence of $T(t)(T(s)-I)$ to 0. In order to get from this the convergence in the operator norm of $T(t)(T(s)-I)R(\lambda;A)$ to zero as $t\to\infty$, we can consider, as in the proof of Theorem 8, the semigroup $\widehat{\mathcal{T}}$ in $L_0(X)$, noting that $R(\lambda;A)\in L_0(X)$ for each $\lambda\in\varrho(A)$. ■

2. Theorems 7 and 8 are also true, with corresponding simplifications in the proofs, for discrete semigroups of operators $\{T^n:n=1,2,\ldots\}$. We restrict ourselves to statements of the results and an outline of the proofs.

A sequence $\alpha(n)$, $n=1,2,\ldots$, is called a *weight sequence* if

$$\alpha(n)\geq1,\quad\forall n=1,2,\ldots;\qquad\alpha(n+m)\leq\alpha(n)\alpha(m),\quad\forall n,m.$$

A weight sequence $\alpha(n)$ is said to be *nonquasianalytic* if

$$\sum_{n=1}^{\infty}\frac{\log\alpha(n)}{1+n^2}<\infty.$$

Assume that also

$$\liminf_{n\to\infty}\frac{\alpha(n+1)}{\alpha(n)}\geq1.$$

With a weight sequence $\alpha(n)$ we also associate its *reduced* weight sequence $\alpha_1(n)$ by

$$\alpha_1(n)\equiv\limsup_{m\to\infty}\frac{\alpha(n+m)}{\alpha(m)}.$$

We say that a bounded linear operator T in a Banach space X is *dominated* by a weight sequence $\alpha(n)$ if $\|T^n\|\leq\alpha(n)$ for each $n=1,2,\ldots$ Below we denote by Γ the unit circle.

THEOREM 11. *Suppose that* T *is a bounded linear operator in* X *which is dominated by a weight sequence* $\alpha(n)$ *such that* $\alpha_1(n)=O(n^k)$ *for some* $k\geq0$. *Assume that the peripheral spectrum of* T, $\sigma(T)\cap\Gamma$, *is countable, and the peripheral point spectrum of* T^*, $P\sigma(T^*)\cap\Gamma$, *is empty. Then*

$$\lim_{n\to\infty}\frac{1}{\alpha(n)}\|T^nx\|=0$$

for each x *in* X.

The proof is analogous to the case of continuous semigroups. Namely, if the assertion does not hold, then we can associate with the operator T a nonzero *limit operator* V of T in a Banach space E with the following properties: there exists a continuous homomorphism π from X to E, with a dense range, such that $V\circ\pi=\pi\circ T$, $\|Vz\|\geq\|z\|$ for each z in E, $\sigma(V)\subset\sigma(T)$, $P\sigma(V^*)\subset P\sigma(T^*)$ and V is dominated by the weight sequence $\alpha_1(n)$. In particular, the spectrum of V lies in the closed unit disc. From $\|Vz\|\geq\|z\|$ it follows that the spectrum of V is either the whole unit disc, or a subset of the unit circle. Since $\sigma(V)\subset\sigma(T)$, we see that $\sigma(V)\cap\Gamma$ is countable; thus V is invertible, in particular the spectrum of V is countable and hence contains an isolated point λ_0, and $\|V^{-1}\|\leq1$. Consider the spectral subspace corresponding to $\{\lambda_0\}$, and the restriction V_0 of V to it. Then V_0, and hence V_0^*, satisfies the conditions of Lemma 4. Consequently, $V_0^*=\lambda_0$. Thus V^*, and therefore T^*, has an eigenvalue on the unit circle, which is a contradiction. The proof is complete.

Proceeding in the same way, we see that if the spectrum of T does not fill the unit circle, then the limit operator V is invertible and, moreover, $\|V^n\|\leq1$ for each $n=-1,-2,\ldots$ Therefore, V is dominated by a two-sided nonquasianalytic sequence $\alpha_1(n)$ ($\alpha_1(n)=1$ for $n=-1,-2,\ldots$). Now we can consider the subalgebra $\mathcal{A}_\alpha(\Gamma)$ of the algebra of absolutely convergent Fourier series $f(z)=\sum_{n=-\infty}^{\infty}a_nz^n$ such that $\|f\|_{\mathcal{A}_\alpha(\Gamma)}=\sum_{n=-\infty}^{\infty}|a_n|\alpha(n)<\infty$. For each $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{A}_\alpha(\Gamma)$ we can define the operator $f(T)=\sum_{n=0}^{\infty}a_nT^n$. A function $f\in\mathcal{A}_\alpha(\Gamma)$ is said to be of *spectral synthesis* in the algebra $\mathcal{A}_{\alpha_1}(\Gamma)$ with respect to a closed subset Δ of Γ if it can be approximated (in $\mathcal{A}_{\alpha_1}(\Gamma)$-norm) by functions which vanish in a neighborhood of Δ. From the spectral theory of invertible operators which are dominated by two-sided nonquasianalytic sequences (an analog of Lemma 6) it follows that if a function $g\in\mathcal{A}_{\alpha_1}(\Gamma)$ vanishes in a neighborhood of $\sigma(V)$ then $g(V)=0$. Therefore, $f(V)=0$ for each function f which is of spectral synthesis with respect to $\sigma(V)$, or equivalently, $\lim_{n\to\infty}\frac{1}{\alpha(n)}\|T^nf(T)x\|=0$ for each x in X. Applying this assertion to the operator $\widehat{T}$ in $L(X)$ defined by $\widehat{T}Y=TY$ for $Y\in L(X)$, we have $\lim_{n\to\infty}\frac{1}{\alpha(n)}\|T^nf(T)\|=0$. Thus we have proved the following theorem.

THEOREM 12. *Let $\alpha(n)$, $n = 1, 2, \ldots$, be a weight sequence such that its reduced weight sequence $\alpha_1(n)$ is nonquasianalytic. Suppose that T is a bounded operator dominated by $\alpha(n)$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{A}_\alpha(\Gamma)$ is a function which is of spectral synthesis as an element of $\mathcal{A}_{\alpha_1}(\Gamma)$ with respect to the peripheral spectrum of T, then $\lim_{n\to\infty} \frac{1}{\alpha(n)} \|T^n f(T)\| = 0$.*

The analog of Corollary 9 for discrete semigroups makes no interest because if the spectrum of T is inside the unit disc, then obviously $\|T^n\| \to 0$. The following analog of Theorem 8 is a direct generalization of a theorem of Y. Katznelson and L. Tzafriri [6].

THEOREM 13. *Let $\alpha(n)$ be a weight sequence such that $\alpha_1(n) = O(n^k)$ for some $k \geq 0$. Suppose that T is a bounded operator dominated by $\alpha(n)$ such that the peripheral spectrum of T consists of at most the point $\lambda = 1$. Then $\lim_{n\to\infty} \frac{1}{\alpha(n)} \|T^n(T - I)\| = 0$.*

The proof follows the same method. Indeed, if we consider the limit operator V of T, then V is invertible, dominated by $\alpha_1(n)$, $\|V^{-n}\| \leq 1$ for each $n = 1, 2, \ldots$, and the spectrum of V consists of one point $\lambda = 1$ (if V is not zero). By Lemma 4, we obtain $V - I = 0$, which means that $\lim_{n\to\infty} \frac{1}{\alpha(n)} \|T^n(T - I)\| = 0$ (first strongly, and then in the operator norm by passing to the operator $\widehat{T}$ in $L(X)$).

Finally, we remark that Theorem 12 for the case $\lim_{n\to\infty} \alpha(n+1)/\alpha(n) = 1$ (thus $\alpha_1(n) \equiv 1$) was obtained earlier in [1].

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