STUDIA MATHEMATICA 104 (3) (1993)

Pointwise estimates for densities of stable semigroups of measures

by

PAWEŁ GŁOWACKI and WALDEMAR HEBISCH (Wrocław)

Abstract. Let $\{\mu_t\}$ be a symmetric α-stable semigroup of probability measures on a homogeneous group $\mathcal{N}$, where $0 < \alpha < 2$. Assume that μ_t are absolutely continuous with respect to Haar measure and denote by h_t the corresponding densities. We show that the estimate

$$h_t(x) \le t\Omega(x/|x|)|x|^{-n-\alpha}, \quad x \neq 0,$$

holds true with some integrable function Ω on the unit sphere Σ if and only if the density of the Lévy measure of the semigroup belongs locally to the Zygmund class $L\log L(\mathcal{N}\setminus\{e\})$. The problem turns out to be related to the properties of the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} \frac{1}{t}\Big|\int_0^t h_{t-s} * f * h_s(x)\,ds\Big|$$

which, as is proved here, is of weak type $(1,1)$.

Introduction. Let $\{\mu_t\}$ be a symmetric α-stable semigroup of probability measures on $\mathbb{R}^n$, where $0 < \alpha < 2$. (Note that the Gaussian semigroups are excluded.) Assume that μ_t are absolutely continuous with respect to Lebesgue measure and denote by h_t the corresponding densities. It is known that the h_t are automatically continuous functions on $\mathbb{R}^n$ and vanish at infinity (cf. Głowacki [9]).

The question considered here is that of the rate of decay of the densities at infinity. Let us remark that in the ideal case when the Lévy measure is smooth away from the origin we have

$$\lim_{r\to\infty} r^{n+\alpha}h_1(r\bar{x}) = m(\bar{x}), \tag{1}$$

for $\bar{x} \in \Sigma = \{y \in \mathbb{R}^n : |y| = 1\}$, where m is the density of the Lévy measure (cf. Dziubański [5]). It is easily seen that, by homogeneity, (1) implies

$$h_t(x) \le Ctm(\bar{x})|x|^{-n-\alpha}, \qquad x \neq 0,\ t > 0, \tag{2}$$

1991 *Mathematics Subject Classification*: Primary 22E30.

where $\bar{x}$ denotes the unit vector $x/|x|$ and C is a positive constant.

However, as simple examples show, the densities h_t may very well exist even when the Lévy measure is singular. In addition, they always satisfy the estimate

$$h_t(x) \le t\Omega_\beta(\bar{x})|x|^{-n-\beta}, \quad x \ne 0,\ t > 0, \tag{3}$$

for any $0 < \beta < \alpha$, where Ω_β is a function on the unit sphere satisfying an integral Lipschitz condition (cf. Głowacki [9]). Therefore, the right question to ask seems to be: How is the estimate

$$h_t(x) \le t\Omega(\bar{x})|x|^{-n-\alpha}, \quad x \ne 0,\ t > 0, \tag{4}$$

with Ω enjoying certain regularity properties related to the regularity of the Lévy measure itself? Note that in (4) the exponent assumes the critical value α.

In this paper we consider this question in the general context of a nilpotent homogeneous group $\mathcal{N}$ of which $\mathbb{R}^n$ with the usual dilations is the simplest example. We show that for every symmetric α-stable semigroup $\{\mu_t\}$ with densities h_t the estimate (4) holds true with some Ω in $L^1(\Sigma)$ if and only if the density m of the Lévy measure locally belongs to the Zygmund class $L\log L(\mathcal{N}\setminus\{e\})$. If $1 < p \le \infty$ and $\Omega \in L^p(\mathcal{N}\setminus\{e\})$, this is so if and only if $m \in L^p_{\mathrm{loc}}(\mathcal{N}\setminus\{e\})$. (In the above formulas n is understood to be the homogeneous dimension of the group which in most cases is different from the topological dimension.) We also prove that if the Lévy measure is absolutely continuous relative to Haar measure, then (1) holds for a.e. $\bar{x}$ on the unit sphere Σ with m being the density of the Lévy measure restricted to Σ.

Our method is based on a perturbation formula. The problem turns out to be related to the properties of the maximal function

$$\mathcal{M}f(x) = \sup_{t>0} \frac{1}{t}\Big|\int_0^t h_{t-s} * f * h_s(x)\,ds\Big|$$

which, as is easily seen, coincides with the usual maximal function associated with the semigroup provided $\mathcal{N}$ is commutative. It is proved here that $\mathcal{M}$ is of weak type $(1,1)$ in the general case.

1. Preliminaries and notation.

Let $\mathcal{N}$ be a homogeneous Lie group endowed with a family of dilations $\{\delta_t\}$ and a homogeneous norm $x \mapsto |x|$ which is subadditive, that is, satisfies

$$|xy| \le |x| + |y|$$

for $x, y \in \mathcal{N}$. Such norms always exist, as is shown by Hebisch and Sikora [12]. Let dx denote Haar measure on $\mathcal{N}$ and Q the homogeneous dimension of $\mathcal{N}$. Let $B(r) = \{x \in \mathcal{N} : |x| < r\}$ denote the ball of radius $r > 0$ and let $\Sigma = \{x \in \mathcal{N} : |x| = 1\}$ be the unit sphere relative to the homogeneous norm. For $x \in \mathcal{N}\setminus\{e\}$, let

$$\bar{x} = \delta_{|x|^{-1}}x.$$

There exists a unique Radon measure $d\bar{x}$ on Σ such that for all continuous functions f on $\mathcal{N}$ with compact support

$$\int_{\mathcal{N}} f(x)\,dx = \int_0^\infty r^{Q-1}\int_\Sigma f(\delta_r\bar{x})\,d\bar{x}\,dr.$$

We denote by $C_c^\infty(\mathcal{N})$ the space of compactly supported C^∞-functions on $\mathcal{N}$.

Let $\{\mu_t\}$ be a continuous semigroup of probability measures on $\mathcal{N}$. It is assumed throughout the paper that $\{\mu_t\}$ is symmetric and α-stable with exponent $0 < \alpha < 2$, that is,

$$\langle f, \mu_t\rangle = \langle f \circ \delta_{t^{1/\alpha}}, \mu_1\rangle, \quad f \in C_c^\infty(\mathcal{N}),\ t > 0. \tag{1.1}$$

It is also assumed that the measures μ_t have densities h_t relative to Haar measure dx on $\mathcal{N}$. The densities h_t are automatically square-integrable, hence continuous and bounded (cf. Głowacki [9]). In terms of the densities, (1.1) is equivalent to

$$h_t(x) = t^{-Q/\alpha}h(\delta_{t^{-1/\alpha}}x), \quad x \in \mathcal{N},\ t > 0,$$

where $h = h_1$. It is well known that the generating functional P of such a semigroup is a symmetric real distribution on $\mathcal{N}$ homogeneous of degree $-Q-\alpha$. P is also dissipative, that is to say,

$$\langle f, P\rangle \le 0$$

for every real $f \in C_c^\infty(\mathcal{N})$ which takes on its largest value at the identity e. Any real dissipative distribution is a generating functional for some (uniquely determined) continuous semigroup of measures.

Since P is dissipative, it coincides on $\mathcal{N}\setminus\{e\}$ with a Radon measure ν. This measure is positive and bounded outside any neighbourhood of the identity. We shall refer to ν as the *Lévy measure* of the semigroup $\{\mu_t\}$. We have

$$\langle f, \nu\rangle = \lim_{t\searrow 0}\frac{1}{t}\langle f, h_t\rangle, \quad f \in C_c^\infty(\mathcal{N}\setminus\{e\}).$$

If the Lévy measure ν has a density k relative to Haar measure, then, by homogeneity, there exists a function $N \in L^1(\Sigma)$ such that

$$k(x) = N(\bar{x})|x|^{-Q-\alpha} \quad \text{for a.e. } x \in \mathcal{N}\setminus\{e\}.$$

Given a symmetric and bounded neighbourhood W of the identity, let $\psi = \psi_W$ be a nonnegative smooth function supported in W and equal to 1 in a neighbourhood of e. Then the distribution $\widetilde{P} = \psi P$ supported in W is dissipative as well. We also have

$$P = \widetilde{P} + \nu_0, \tag{1.2}$$

where ν_0 is a bounded positive measure equal to ν outside W. We choose and fix once for all a neighbourhood W such that $W^{2q} \subseteq B(1)$, where $q \geq Q/\alpha + 1$ is a fixed integer.

Denote by $\{\widetilde{\mu}_t\}$ the continuous semigroup of measures generated by $\widetilde{P}$. It is easy to see that $\widetilde{\mu}_t \geq 0$ and $\widetilde{\mu}_t(\mathcal{N}) \leq 1$. In addition, we have the following perturbation formula:

$$\mu_t = \widetilde{\mu}_t + \int_0^t \widetilde{\mu}_{t-s} * \nu_0 * \mu_s \, ds, \quad t > 0, \tag{1.3}$$

where the integral is understood to be convergent in the weak sense (see, e.g., Kato [15]). As is easily seen, (1.3) implies $\widetilde{\mu}_t \leq \mu_t$ so that the $\widetilde{\mu}_t$ are also absolutely continuous. Their densities will be denoted by $\widetilde{h}_t$. By homogeneity and the semigroup property,

$$\|\widetilde{h}_t\|_\infty \leq \|h_t\|_\infty \leq \|h_{t/2}\|_2^2 = Ct^{-Q/\alpha}, \quad t > 0, \tag{1.4}$$

where

$$\|f\|_\infty = \sup_{x\in\mathcal{N}} |f(x)| \quad \text{and} \quad C = 2^Q \|h\|_2^2 = 2^Q \int_{\mathcal{N}} h(x)^2 dx.$$

(1.5) Lemma. *For every positive integer j, there exists a constant $C_j > 0$ such that*

$$\widetilde{\mu}_t(\mathcal{N} \setminus W^j) \leq C_j t^j, \quad t > 0.$$

Proof. This is a direct application of Duflo [4]. ∎

(1.6) Corollary. *There exists a constant M such that*

$$\sup_{|x|\geq 1} \widetilde{h}_t(x) \leq Mt, \quad t > 0.$$

Proof. If $|x| \geq 1$, then

$$\begin{aligned}
\widetilde{h}_t(x) &= \int_{\mathcal{N}} \widetilde{h}_{t/2}(y^{-1}x)\widetilde{h}_{t/2}(y)\,dy \\
&\leq \int_{y\notin W^q} \widetilde{h}_{t/2}(y^{-1}x)\widetilde{h}_{t/2}(y)\,dy + \int_{y^{-1}x\notin W^q} \widetilde{h}_{t/2}(y^{-1}x)\widetilde{h}_{t/2}(y)\,dy \\
&\leq 2C_q(t/2)^{q-Q/\alpha} \leq Mt,
\end{aligned}$$

where we have made use of Lemma (1.5) and (1.4). ∎

We conclude this section with

(1.7) Proposition. *There exists a constant A such that*

$$|h(x) - h(e)| \leq A|x|^\alpha, \quad x \in \mathcal{N}.$$

Moreover, there exists $\Omega_0 \in L^1(\Sigma)$ such that

$$h(x) \leq \Omega_0(\bar{x})|x|^{-Q}, \quad x \in \mathcal{N} \setminus \{e\}.$$

Proof. This has been proved in Głowacki [9]. ∎

For a general account of homogeneous groups, see Folland and Stein [7]. As for an introduction to the theory of continuous semigroups of measures on Lie groups, we recommend Hulanicki [13] and Duflo [4]. See also Pazy [16].

2. Statement of the theorem. Let $\mathfrak{M}$ be a measure space. As usual, we denote by $L^p(\mathfrak{M})$ the totality of measurable functions f on $\mathfrak{M}$ such that $|f|^p$ is integrable. The Zygmund class $L\log L(\mathfrak{M})$ consists of all measurable functions f such that

$$\int_{\mathfrak{M}} |f|(1 + \log^+ |f|) < \infty.$$

It is a proper subspace of $L^1(\mathfrak{M})$.

The following theorem collects together the results of Proposition (3.1), Theorem (3.4), Theorem (5.1), and Theorem (7.6) below.

(2.1) Theorem. *The estimate*

$$h(x) \leq \Omega(\bar{x})|x|^{-Q-\alpha} \quad \textit{for a.e. } x \in \mathcal{N} \setminus \{e\}$$

with $\Omega \in L^p(\Sigma)$ holds true if and only if the Lévy measure ν is absolutely continuous with respect to Haar measure on $\mathcal{N} \setminus \{e\}$ and the corresponding density belongs locally to $L^p(\mathcal{N} \setminus \{e\})$ if $1 < p \leq \infty$ or to $L\log L(\mathcal{N} \setminus \{e\})$ if $p = 1$.

As a by-product, we get information on the a.e. pointwise asymptotic behaviour of the densities (cf. Dziubański [5], formula (4)).

(2.2) Theorem. *If the Lévy measure ν is absolutely continuous with respect to Haar measure on $\mathcal{N} \setminus \{e\}$, then*

$$\lim_{t\nearrow\infty} t^{Q+\alpha} h(\delta_t \bar{x}) = N(\bar{x}) \quad \textit{for a.e. } \bar{x} \in \Sigma,$$

where $N \in L^1(\Sigma)$ and $N(\bar{x})/|x|^{Q+\alpha}$ is the density of ν.

Proof. This follows by homogeneity from Corollary (4.5) below. ∎

3. The case $p > 1$. In this section a complete account is given of the easier case when $p > 1$.

(3.1) PROPOSITION. *Let $1 \le p \le \infty$. If there exists $\Omega \in L^p(\Sigma)$ such that*

$$h(x) \le \Omega(\bar{x})|x|^{-Q-\alpha} \quad \text{for a.e. } x \in \mathcal{N} \setminus \{e\},$$

then the Lévy measure ν of $\{h_t\}$ is absolutely continuous with respect to Haar measure on $\mathcal{N} \setminus \{e\}$ with a density in $L^p_{\text{loc}}(\mathcal{N} \setminus \{e\})$.

Proof. We have

$$\langle f, \nu \rangle = \lim_{t \searrow 0} \frac{1}{t} \langle f, h_t \rangle$$

for $f \in C_c^\infty(\mathcal{N} \setminus \{e\})$. By hypothesis,

$$h_t(x) \le t\Omega(\bar{x})|x|^{-Q-\alpha}, \quad t > 0,$$

for a.e. $x \in \mathcal{N} \setminus \{e\}$, whence

$$\langle f, \nu \rangle \le \int |f(x)|\Omega(\bar{x})|x|^{-Q-\alpha}\, dx,$$

which immediately implies our claim. ∎

Let us define

$$M_{\mathrm{R}} f(x) = \sup_{t>0} |f * h_t(x)| \quad \text{and} \quad M_{\mathrm{L}} f(x) = \sup_{t>0} |h_t * f(x)|$$

for $f \in C_c^\infty(\mathcal{N})$ and $x \in \mathcal{N}$. Recall from Głowacki [9] that both M_{R} and M_{L} are of type (p,p) for every $1 < p \le \infty$. Therefore, the maximal function

$$\mathcal{M} f(x) = \sup_{t>0} \frac{1}{t} \Big| \int_0^t h_{t-s} * f * h_s(x)\, ds \Big| \tag{3.2}$$

is also of type (p,p) as

$$\mathcal{M} f(x) \le M_{\mathrm{L}}(M_{\mathrm{R}} f)(x), \quad f \in C_c^\infty(\mathcal{N}),\ x \in \mathcal{N}. \tag{3.3}$$

(3.4) THEOREM. *Let $1 < p \le \infty$. If the Lévy measure ν is absolutely continuous and its density belongs locally to $L^p(\mathcal{N} \setminus \{e\})$, then there exists an Ω in $L^p(\Sigma)$ such that*

$$h(x) \le \Omega(\bar{x})|x|^{-Q-\alpha} \quad \text{for a.e. } x \in \mathcal{N}.$$

Proof. By homogeneity, it is sufficient to show that for some $r > 0$ there exists $\Omega \in L^p(\Sigma)$ such that

$$h_t(\delta_r \bar{x}) \le t\Omega(\bar{x}) \quad \text{for a.e. } \bar{x} \in \Sigma \text{ and } t > 0. \tag{3.5}$$

To this end, note that, by hypothesis, the measure ν_0 in (1.2) has a density $k_0 \in L^p(\mathcal{N})$. By the perturbation formula (1.3),

$$\begin{aligned} h_t(x) &\le \widetilde{h}_t(x) + \int_0^t h_{t-s} * k_0 * h_s(x)\, ds \\ &\le \widetilde{h}_t(x) + t\mathcal{M} k_0(x). \end{aligned} \tag{3.6}$$

Therefore, by Corollary (1.6),

$$h_t(\delta_r \bar{x}) \le t(M + \mathcal{M} k_0(\delta_r \bar{x})), \quad r \ge 1,\ |\bar{x}| = 1.$$

Since $\mathcal{M}$ is of type (p,p), $\mathcal{M} k_0 \in L^p(\mathcal{N})$ and so, by Fubini's theorem, the function $\bar{x} \to \mathcal{M} k_0(\delta_r \bar{x})$ is in $L^p(\Sigma)$ for a.e. $r \ge 1$. Pick one such r and let

$$\Omega(\bar{x}) = M + \mathcal{M} k_0(\delta_r \bar{x}), \quad \bar{x} \in \Sigma.$$

Obviously, Ω satisfies (3.5). ∎

4. A maximal theorem. The *Zo norm* of an L^1 function φ on a homogeneous group $\mathcal{N}$ is defined by

$$\|\varphi\|_{\mathrm{Zo}} = \sup_{x \in \mathcal{N}} \int_{|y| \ge 2|x|} \sup_{t>0} |\delta_t \varphi(xy) - \delta_t \varphi(y)|\, dy,$$

where, by definition,

$$\delta_t \varphi(x) = t^{-Q} \varphi(\delta_{t^{-1}} x), \quad x \in \mathcal{N},\ t > 0.$$

The following result is well known as Zo's lemma (cf. Zo [18] as well as Stein [17], pp. 71–73).

(4.1) LEMMA. *Let $k \in L^1(\mathcal{N})$. If $\|k\|_{\mathrm{Zo}} < \infty$, then the maximal operator K defined by*

$$K f(x) = \sup_{t>0} |f * k_t(x)|$$

is of weak type $(1,1)$.

The main result of this section is

(4.2) PROPOSITION. *The maximal operator $\mathcal{M}$ as defined by (3.2) is of weak type $(1,1)$.*

Proof. By a simple change of variable, for $0 \le f \in L^1(\mathcal{N})$,

$$\mathcal{M} f = \sup_{t>0} \int_0^1 h_{\theta t} * f * h_{(1-\theta)t}\, d\theta \le \mathcal{M}_0 f + \mathcal{M}_\infty f,$$

where

$$\mathcal{M}_0 f = \sup_{t>0} \int_0^{1/2} h_{(1-\theta)t} * f * h_{\theta t}\, d\theta\,,$$

$$\mathcal{M}_\infty f = \sup_{t>0} \int_0^{1/2} h_{\theta t} * f * h_{(1-\theta)t}\, d\theta\,.$$

We are going to show that $\mathcal{M}_\infty$ is of weak type $(1,1)$. The proof for $\mathcal{M}_0$ is entirely "symmetric". To this end, let $\mathcal{G} = \mathcal{N}\times\mathcal{N}$ be the direct product of $\mathcal{N}$ by itself. $\mathcal{G}$ acts on $\mathcal{N}$ by

$$(x,y)z = x^{-1}zy, \qquad (x,y)\in\mathcal{G},\ z\in\mathcal{N}\,,$$

and $\mathcal{M}_\infty$ is obtained by transference by means of this action from the maximal function on $\mathcal{G}$

$$MF(x,y) = \sup_{t>0} F * H_t, \qquad 0\le F\in L^1(\mathcal{G})\,,$$

where

$$H_t(x,y) = \int_0^{1/2} h_{\theta t}(x) h_{(1-\theta)t}(y)\, d\theta$$

is an integrable function on $\mathcal{G}$ for every $t>0$. By the transference principle (Emerson [6], see also Calderón [1] as well as Coifman and Weiss [2], [3]), it is therefore sufficient to show that M is of weak type $(1,1)$. Let us make $\mathcal{G}$ into a homogeneous group by introducing the product dilations

$$\Delta_t(x,y) = (\delta_t x, \delta_t y)\,,$$

and the homogeneous norm

$$|(x,y)| = |x| + |y| \tag{4.3}$$

for $(x,y)\in\mathcal{G}$ and $t>0$. Then

$$H_t = \Delta_{t^{1/\alpha}}(H_1), \qquad t>0\,,$$

and, by Zo's lemma (Lemma (4.1)), it is enough to show that the Zo norm of $H_1\in L^1(\mathcal{G})$ is finite.

We have

$$\|H_1\|_{\mathrm{Zo}} \le \int_0^{1/2} \|h_\theta \otimes h_{1-\theta}\|_{\mathrm{Zo}}\, d\theta = \int_0^{1/2} \|H_1^\theta\|_{\mathrm{Zo}}\, d\theta\,,$$

where

$$H_t^\theta = h_t \otimes h_{(1/\theta-1)t}, \qquad 0<\theta\le 1/2,\ t>0\,.$$

The last equality is due to the invariance of the Zo norm under dilations.

It is clear that for each $\theta\in(0,1/2]$, $\{H_t^\theta\}$ is an α-stable symmetric semigroup of measures on the homogeneous group $\mathcal{G}$ with $P_\theta = P\otimes\delta + \delta\otimes (1/\theta-1)P$ being its infinitesimal generator. It has been shown in Głowacki [9] that for every symmetric α-stable semigroup of measures with absolutely continuous densities h_t, the Zo norm of h_1 is finite. Therefore, in particular,

$$\|H_1^{1/2}\|_{\mathrm{Zo}} = C_1 < \infty\,.$$

For $X\in\mathcal{G}$ put

$$|X|_\theta = |\varrho_\theta X|\,,$$

where $\varrho_\theta(x,y) = (x, \delta_{\tilde\theta^{-1}} y)$, and $\tilde\theta = (1/\theta - 1)^{1/\alpha}$, so that

$$H_t^\theta(X) = \tilde\theta^{-Q} H_t^{1/2}(\varrho_\theta X)\,,$$

and, by our choice of the homogeneous norm (4.3),

$$\tilde\theta^{-1}|X| \le |X|_\theta \le |X|, \qquad X\in\mathcal{G}\,.$$

For fixed $X\in\mathcal{G}$ and $2^{1/\alpha}\le\tilde\theta<\infty$ let

$$I(X) = \int_{|Y|\ge 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)|\, dY\,.$$

Then

$$\begin{aligned}
I(X) &= \int_{2\tilde\theta|X|\ge|Y|\ge 2|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)|\, dY \\
&\quad + \int_{|Y|\ge 2\tilde\theta|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)|\, dY \\
&\le 2 \int_{4\tilde\theta|X|\ge|Y|\ge|X|} \sup_{t>0} H_t^\theta(Y)\, dY \\
&\quad + \int_{|Y|\ge 2\tilde\theta|X|} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)|\, dY \\
&\le 2 \int_{4\tilde\theta^2|X|_\theta\ge|Y|_\theta\ge\tilde\theta^{-1}|X|_\theta} \sup_{t>0} H_t^\theta(Y)\, dY \\
&\quad + \int_{|Y|_\theta\ge 2|X|_\theta} \sup_{t>0} |H_t^\theta(XY) - H_t^\theta(Y)|\, dY\,.
\end{aligned}$$

Let us remark that we have taken advantage of the fact that the homogeneous norm on $\mathcal{N}$, and consequently that on $\mathcal{G}$, is subadditive. By the change of variable $V = \varrho_\theta Y$,

$$(4.4)\qquad I(\varrho_{\theta^{-1}}X) \le 2 \int_{4\bar\theta^2|X|\ge|V|\ge\bar\theta^{-1}|X|} \sup_{t>0} H_t^{1/2}(V)\,dV + \int_{|V|\ge 2|X|} \sup_{t>0} |H_t^{1/2}(XV) - H_t^{1/2}(V)|\,dV\,.$$

The second term is of course bounded by $C_1 = \|H_1^{1/2}\|_{Z_0}$. By Proposition (1.7) and homogeneity,

$$|H_t^{1/2}(V)| \le \Omega_0(\overline{V})|V|^{-2Q}, \quad V \ne (e,e),\ t > 0\,,$$

for some Ω_0 integrable on the unit sphere in $\mathcal{G}$, where $2Q$ is the homogeneous dimension of $\mathcal{G}$. Therefore the first term of (4.4) is less than or equal to

$$\int_{4\bar\theta^2|X|\ge|V|\ge\bar\theta^{-1}|X|} \Omega_0(\overline{V})|V|^{-2Q}\,dV \le C_2|\log\theta|\,,$$

which implies that

$$\|H_1^\theta\|_{Z_0} = \sup_{X\in\mathcal{G}} I(X) = \sup_{X\in\mathcal{G}} I(\varrho_{\theta^{-1}}X) \le C|\log\theta|\,,$$

and, consequently,

$$\|H_1\|_{Z_0} \le \int_0^{1/2} \|H_1^\theta\|_{Z_0} d\theta \le C\int_0^{1/2} |\log\theta| d\theta < \infty\,,$$

which completes the proof of Proposition (4.2). ∎

(4.5) COROLLARY. *If the Lévy measure ν has a density $N(\bar x)/|x|^{Q+\alpha}$, where $N \in L^1(\Sigma)$, then*

$$\lim_{t\searrow 0} \frac{1}{t} h_t(\bar x) = N(\bar x) \quad \textit{for a.e. } \bar x \in \Sigma\,.$$

Proof. This is derived, by a routine argument, from (1.3), Corollary (1.6), and Proposition (4.2). ∎

(4.6) COROLLARY. *For every $f \in L\log L(\mathcal{N})$, $\mathcal{M}f$ is locally integrable on $\mathcal{N}$.*

Proof. This is a well known property of weak type (1, 1) operators (see, e.g., Guzmán [11], Theorem (6,1)). ∎

5. The case $p = 1$: sufficiency

(5.1) THEOREM. *If ν has a density which is locally in the Zygmund class on $\mathcal{N}\setminus\{e\}$, then there exists $\Omega \in L^1(\Sigma)$ such that*

$$h(x) \le \Omega(\bar x)|x|^{-Q-\alpha}, \quad x \in \mathcal{N}\setminus\{e\}\,.$$

Proof. The argument is quite analogous to that in the proof of Proposition (3.1). The only difference is that now k_0 is in the Zygmund class $L\log L(\mathcal{N})$, which, by Corollary (4.6), implies that for some $r \ge 1$,

$$\Omega(\bar x) = M + \mathcal{M}k_0(\delta_r\bar x), \quad \bar x \in \Sigma\,,$$

is integrable on Σ and satisfies the desired inequality. ∎

The remaining part of the paper is devoted to the demonstration of the inverse implication.

6. A space of homogeneous type.

Given $x, y \in \mathcal{N}$, we denote by $d(x,y)$ the greatest lower bound of the subset of all real $r > 0$ such that

$$z_1 x = y z_2$$

for some z_1, z_2 in the ball $B(r)$. It is not hard to see that, due to the subadditivity of the homogeneous norm, d is a metric on $\mathcal{N}$.

Let

$$A_x(r) = \{y \in \mathcal{N} : d(x,y) < r\}$$

be the open ball of radius r relative to d centred at $x \in \mathcal{N}$. Let also

$$\Phi_r^{(x)} = \chi_{B(r)} * \delta_x * \chi_{B(r)}\,,$$

where δ_x stands for the Dirac point mass located at $x \in \mathcal{N}$.

(6.1) LEMMA. *For every $r > 0$ and every $x \in \mathcal{N}$,*

$$\Phi_r^{(x)}(y) \le \Phi_{3r}^{(x)}(z) \quad \textit{if } d(x,z) \le r\,.$$

Proof. If $\Phi_r^{(x)}(y) > 0$, then $d(x,y) < r$ and there exist $x_1, x_2 \in B(r)$ such that

$$(6.2)\qquad \begin{aligned}\Phi_r^{(x)}(y) &= \Phi_r^{(x)}(x_1^{-1}xx_2)\\ &= (\delta_{x_1} * \chi_{B(r)}) * \delta_x * (\chi_{B(r)} * \delta_{x_2})(x)\\ &\le \chi_{B(2r)} * \delta_x * \chi_{B(2r)}(x) = \Phi_{2r}^{(x)}(x)\,.\end{aligned}$$

In a similar fashion it is shown that $d(x,z) \le r$ implies $\Phi_{2r}^{(x)}(x) \le \Phi_{3r}^{(x)}(z)$, which combined with (6.2) completes the proof. ∎

(6.3) LEMMA. *For every $r > 0$ and every $x \in \mathcal{N}$,*

$$|B(r)|^2 \le \|\Phi_r^{(x)}\|_\infty |A_x(r)| \le |B(3r)|^2\,.$$

Proof. By Lemma (6.1), we have

$$\|\Phi_r^{(x)}\|_\infty |A_x(r)| \le \int_{A_x(r)} \Phi_{3r}^{(x)}(y)\,dy\,.$$

Consequently,

$$|B(r)|^2 = \int \Phi_r^{(x)}(y)\,dy \le \|\Phi_r^{(x)}\|_\infty |A_x(r)| \le \int_{A_x(r)} \Phi_{3r}^{(x)}(y)\,dy \le |B(3r)|^2 ,$$

which completes the proof. ■

(6.4) PROPOSITION. *The measure space $(\mathcal{N}, dx)$ endowed with the metric d is a space of homogeneous type.*

Proof. It is sufficient to show that there exists a constant $C > 0$ such that for every $r > 0$ and every $x \in \mathcal{N}$,

$$|A_x(2r)| \le C|A_x(r)|$$

(see Coifman and Weiss [2]). In fact, by Lemma (6.3),

$$|A_x(2r)| \le \frac{|B(6r)|^2}{\|\Phi_{2r}^{(x)}\|_\infty} \le 6^{2Q}\frac{|B(r)|^2}{\|\Phi_r^x\|_\infty} \le 6^{2Q}|A_x(r)| . \ ■$$

For $r > 0$, let

$$\chi_r(x) = \frac{1}{|B(r)|}\chi_{B(3r)}(x), \quad x \in \mathcal{N} .$$

(6.5) LEMMA. *For every $r > 0$ and every $x \in \mathcal{N}$,*

$$\frac{1}{|A_x(r)|}\int_{A_x(r)} f(z)\,dz \le 6^{2Q}\chi_r * f * \chi_r(x), \quad f \ge 0 . \tag{6.6}$$

Proof. In fact, by Proposition (6.4),

$$\frac{1}{|A_x(r)|}\int_{A_x(r)} f(z)\,dz \le \frac{6^{2Q}}{|A_x(2r)|}\int_{A_x(r)} f(z)\,dz$$

and, by Lemma (6.3),

$$\frac{1}{|A_x(2r)|}\int_{A_x(r)} f(z)\,dz \le \int_{A_x(r)} \frac{f(z)}{|A_z(r)|}\,dz \le \frac{1}{|B(r)|^2}\int_{A_x(r)} f(z)\|\Phi_r^{(z)}\|_\infty\,dz \le \chi_r * f * \chi_r(x)$$

for $r > 0$, $x \in \mathcal{N}$. ■

Denote by $\mathrm{Mes}^+(\mathcal{N})$ the space of positive measurable functions on $\mathcal{N}$. For f in $\mathrm{Mes}^+(\mathcal{N})$, let

$$\Phi^* f(x) = \sup_{0<r\le 1} \chi_r * f * \chi_r(x) ,$$

$$f^*(x) = \sup_{r>0} \frac{1}{|A_x(r)|}\int_{A_x(r)} f(z)\,dz, \quad x \in \mathcal{N} .$$

Of course f^* is the usual Hardy–Littlewood maximal function for f on the space $(\mathcal{N}, dx, d)$ of homogeneous type.

(6.7) LEMMA. *For every nonnegative $f \in L^1(\mathcal{N})$ with $\|f\|_1 \le |B(1)|$ and every $\lambda \ge 1$,*

$$\{x \in \mathcal{N} : f^*(x) > \lambda\} \subseteq \{x \in \mathcal{N} : \Phi^* f(x) > 6^{-2Q}\lambda\} .$$

Proof. This is an immediate consequence of (6.6) and the fact that

$$\frac{1}{|A_x(r)|}\int_{A_x(r)} f(y)\,dy \le \frac{|B(1)|}{|B(r)|} \le 1$$

for $r \ge 1$. The first inequality holds because $xB(r) \subset A_x(r)$. ■

Recall that the Hardy–Littlewood maximal function f^* on a space of homogeneous type satisfies the inverse Kolmogorov inequality

$$\frac{1}{\lambda}\int_{f^*>\lambda} f(y)\,dy \le C|\{x \in \mathcal{N} : f^* > c\lambda\}| \tag{6.8}$$

for every $f \in \mathrm{Mes}^+(\mathcal{N})$, every $\lambda > 0$, and some constants $C, c > 0$. We may assume that $c \le 1$.

(6.9) PROPOSITION. *For every $f \in \mathrm{Mes}^+(\mathcal{N})$ with $\|f\|_1 \le |B(1)|$,*

$$\int f(x)\log^+ f(x)\,dx \le \frac{6^{2Q}C}{c}\int \Phi^* f(x)\,dx + \log\frac{1}{c}\int f(x)\,dx . \tag{6.10}$$

Proof. In fact,

$$\int f(x)\log^+ f(x)\,dx = \int_{f(x)>1} f(x)\log f(x)\,dx = \int_1^\infty \frac{1}{\lambda}\int_{f(x)>\lambda} f(x)\,dx\,d\lambda \le \log\frac{1}{c}\|f\|_1 + \int_{1/c}^\infty \frac{1}{\lambda}\int_{f(x)>\lambda} f(x)\,dx\,d\lambda .$$

The last term is, by (6.8) and Lemma (6.7), estimated by

$$C\int_{1/c}^{\infty}|\{x\in\mathcal{N}: f^*(x)>c\lambda\}|\,d\lambda \le C\int_{1/c}^{\infty}|\{x\in\mathcal{N}:\Phi^*f(x)>6^{-2Q}c\lambda\}|\,d\lambda$$
$$\le \frac{6^{2Q}C}{c}\int_{\mathcal{N}}\Phi^*f(x)\,dx\,. \blacksquare$$

For $\eta>0$ and $f\in \mathrm{Mes}^+(\mathcal{N})$, let

$$\Phi_\eta^*f(x)=\sup_{0<r<\eta}\chi_r*f*\chi_r(x),\qquad x\in\mathcal{N}\,.$$

(6.11) COROLLARY. *Let $f\in\mathrm{Mes}^+(\mathcal{N})$. If $\Phi_\eta^*f\in L^1(\mathcal{N})$ for some η, then $f\in L\log L(\mathcal{N})$.*

Proof. If $\eta=1$, then $\Phi_\eta^*=\Phi^*$ and our claim is a simple consequence of (6.10). In the general case

$$\Phi_\eta^*f(x)=[\Phi^*(f\circ\delta_\eta)](\delta_{\eta^{-1}}x),\qquad x\in\mathcal{N}\,,$$

which takes us back to the case $\eta=1$. ■

7. The case $p=1$: necessity

(7.1) LEMMA. *There exist constants $C,\varepsilon,\delta>0$ such that*

$$\widetilde{h}_t(x)\ge C\chi_{(\varepsilon t)^{1/\alpha}}(x),\qquad x\in\mathcal{N},\ 0<t<\delta\,. \tag{7.2}$$

Proof. By Proposition (1.7),

$$|h_t(x)-h_t(e)|\le At^{-Q/\alpha-1}|x|^\alpha,\qquad x\in\mathcal{N},\ 0<t\le 1\,. \tag{7.3}$$

By the perturbation formula (1.3) and (1.4),

$$\|h_t-\widetilde{h}_t\|_\infty\le Bt^{-Q/\alpha+1},\qquad t>0\,. \tag{7.4}$$

We also have

$$h_t(e)=Dt^{-Q/\alpha},\qquad t>0\,. \tag{7.5}$$

In the above formulas, A, B, and D are positive constants independent of x and t.

Now, let $|x|\le(\varepsilon t)^{1/\alpha}$, where ε is to be chosen later. Then, by (7.3)–(7.5),

$$\widetilde{h}_t(x)\ge h_t(x)-Bt^{-Q/\alpha+1}\ge h_t(e)-A|x|^\alpha t^{-Q/\alpha-1}-Bt^{-Q/\alpha+1}$$
$$\ge Dt^{-Q/\alpha}-(A\varepsilon+Bt)t^{-Q/\alpha}\ge\tfrac12 Dt^{-Q/\alpha}\ge Ct^{-Q/\alpha}$$

for sufficiently small ε and t, where $C=\frac12 D$. Thus our assertion is proved. ■

(7.6) THEOREM. *If h satisfies the estimate*

$$h(x)\le\Omega(\bar{x})|x|^{-Q-\alpha}\qquad \textit{for a.e. } x\in\mathcal{N}\setminus\{e\}$$

with $\Omega\in L^1(\Sigma)$, then the Lévy measure ν of $\{\mu_t\}$ is absolutely continuous and its density belongs to the Zygmund class $L\log L$ locally on $\mathcal{N}\setminus\{e\}$.

Proof. It is sufficient to show that the measure ν_0, as defined in (1.2), has a density in $L\log L$. By Proposition (3.1), ν_0 does have a density k_0 in $L^1(\mathcal{N})$. To prove that $k_0\in L\log L$, we shall show that $\Phi_\eta^*k_0\in L^1(\mathcal{N})$ for some $0<\eta\le1$ and then invoke Corollary (6.11). Note that, by homogeneity, we may assume that the support of k_0 is far away from the identity so that $\chi_1*k_0*\chi_1$ vanishes in a neighbourhood of e.

Now, by Lemma (7.1) and the perturbation formula (1.3), there exist constants A,ε, and δ such that

$$\chi_{(\varepsilon t)^{1/\alpha}}*k_0*\chi_{(\varepsilon t)^{1/\alpha}}(x)\le\frac{A}{t}\int_{t/3}^{2t/3}\widetilde{h}_{t-s}*k_0*h_s(x)\,ds$$
$$\le\frac1t Ah_t(x)\le A\Omega(\bar{x})|x|^{-Q-\alpha}$$

for $x\ne e$ and $0<t<\delta$. Taking into account the fact that the supports of all the functions $\chi_r*k_0*\chi_r$, where $0<r\le1$, are uniformly cut off from the identity e, we conclude that $\Phi_\eta^*k_0$ is integrable for some $0<\eta\le1$, which completes the proof. ■

References

[1] A. Calderón, *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sci. U.S.A. 59 (1968), 349–353.

[2] R. R. Coifman et G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math. 242, Springer, Berlin 1971.

[3] —, *Transference Methods in Analysis*, CBMS Regional Conf. Ser. in Math. 31, Amer. Math. Soc., Providence 1977.

[4] M. Duflo, *Représentations de semi-groupes de mesures sur un groupe localement compact*, Ann. Inst. Fourier (Grenoble) 28 (3) (1978), 225–249.

[5] J. Dziubański, *Asymptotic behaviour of densities of stable semigroups of measures*, Probab. Theory Related Fields 87 (1991), 459–467.

[6] W. Emerson, *The pointwise ergodic theorem for amenable groups*, Amer. J. Math. 96 (1974), 472–487.

[7] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Princeton Univ. Press, Princeton 1982.

[8] P. Głowacki, *Stable semi-groups of measures as commutative approximate identities on non-graded homogeneous groups*, Invent. Math. 83 (1986), 557–582.

[9] —, *Lipschitz continuity of densities of stable semigroups of measures*, Colloq. Math., to appear.

[10] P. Głowacki and A. Hulanicki, *A semi-group of probability measures with non-smooth differentiable densities on a Lie group*, Colloq. Math. 51 (1987), 131–139.

[11] M. de Guzmán, *Differentiation of Integrals in $\mathbb{R}^n$*, Springer, Berlin 1975.

[12] W. Hebisch and A. Sikora, *A smooth subadditive homogeneous norm on a homogeneous group*, Studia Math. 96 (1990), 231–236.
[13] A. Hulanicki, *A class of convolution semi-groups of measures on a Lie group*, in: Lecture Notes in Math. 882, Springer, Berlin 1980, 82–101.
[14] A. Hulanicki, *Subalgebra of $L_1(G)$ associated with laplacian on a Lie group*, Colloq. Math. 31 (1974), 259–287.
[15] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin 1966.
[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin 1983.
[17] E. M. Stein, *Boundary behavior of harmonic functions on symmetric spaces: maximal estimates for Poisson integrals*, Invent. Math. 74 (1983), 63–83.
[18] F. Zo, *A note on approximation of the identity*, Studia Math. 55 (1976), 111–122.

MATHEMATICAL INSTITUTE
UNIVERSITY OF WROCŁAW
PL. GRUNWALDZKI 2/4
50-384 WROCŁAW, POLAND

Received March 31, 1992
Revised version November 23, 1992 (2921)

STUDIA MATHEMATICA 104 (3) (1993)

Markov's property of the Cantor ternary set

by

LEOKADIA BIAŁAS (Kraków)
and ALEXANDER VOLBERG (East Lansing, Mich.)

Abstract. We prove that the Cantor ternary set E satisfies the classical Markov inequality (see [Ma]): for each polynomial p of degree at most n ($n = 0, 1, 2, \ldots$)

$$\text{(M)} \qquad |p'(x)| \le Mn^m \sup_E |p| \quad \text{for } x \in E,$$

where M and m are positive constants depending only on E.

0. Introduction. In 1889 A. A. Markov proved (M) for $E = [-1; 1]$. Since that time it has become the object of extensive research (see e.g. [R-S] in the one-dimensional case and [Pa-Pl 1] for $\mathbb{R}^n$). In particular, it has appeared that the inequality plays an important role in the approximation and extension of C^∞ functions defined on compact subsets of $\mathbb{R}^n$ to C^∞ functions on the whole space (see [Pa-Pl 1], [Pa-Pl 2] and [Pl 4]).

The question about Markov's property for the Cantor ternary set has remained unanswered for many years. J. Siciak [Si 3] showed that there exists a Cantor type set $E \subset \mathbb{R}$ such that Leja's extremal function L_E (see [Lj 2], p. 261) has the following Hölder continuity property:

$$\text{(HCP)} \qquad L_E(z) \le 1 + M\delta^m \quad \text{if } \operatorname{dist}(z, E) \le \delta \le 1,$$

with some positive constants $M > 0$ and $m > 0$ depending only on E, which, by Cauchy's integral formula, is sufficient for E to preserve Markov's inequality (M) (see [Si 2], Remark after Lemma 1 and [Pl 1], Lemma 3.1). On the other hand, Pleśniak [Pl 2] constructed a Cantor type set E such that Leja's extremal function is continuous on $\mathbb{C}$ but E does not satisfy (M).

These results have given no answer to the question of whether the Cantor ternary set has Markov's property. In this paper we prove that the answer is affirmative. Actually, we show that this set even has the (HCP) property.

1991 *Mathematics Subject Classification*: Primary 41A17.