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# Representations of bimeasures

by

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**Abstract.** Separately $\sigma$-additive and separately finitely additive complex functions on the Cartesian product of two algebras of sets are represented in terms of spectral measures and their finitely additive counterparts. Applications of the techniques include a bounded joint convergence theorem for bimeasure integration, characterizations of positive-definite bimeasures, and a theorem on decomposing a bimeasure into a linear combination of positive-definite ones.

**1. Introduction and notation.** Throughout this paper, $S_i$ is a nonempty set and $\Sigma_i$ an algebra (field) of subsets of $S_i$ for $i = 1, 2$. Unless specified otherwise, $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is an arbitrary bounded separately (finitely) additive function. In case $\beta$ is separately $\sigma$-additive (i.e., $\beta(X, \cdot)$ and $\beta(\cdot, Y)$ are countably additive for all $X \in \Sigma_1$, $Y \in \Sigma_2$), $\beta$ will be called a (complex) *bimeasure*. For the basic theory of bimeasures defined on products of $\sigma$-algebras we refer to [1] and [13]. The $C^*$-algebra theory we need may be found e.g. in [12].

All vector spaces will be complex. For any Hilbert space $H$, $(\cdot \mid \cdot)$ or $(\cdot \mid \cdot)_H$ denotes its inner product, and $L(H)$ the space of bounded linear operators on $H$.

Our main results depend on the Grothendieck inequality, "the fundamental theorem in the metric theory of tensor products" of Grothendieck [7]: For the spaces $C(\Omega_i)$ of continuous complex functions on compact Hausdorff spaces $\Omega_i$, $i = 1, 2$, and any bounded bilinear form $B : C(\Omega_1) \times C(\Omega_2) \to \mathbb{C}$ there are positive linear forms $\phi : C(\Omega_1) \to \mathbb{C}$ and $\psi : C(\Omega_2) \to \mathbb{C}$ such that

$$|B(f,g)|^2 \le \phi(|f|^2)\psi(|g|^2)$$

for all $f \in C(\Omega_1)$, $g \in C(\Omega_2)$. (We do not normalize $\phi$ and $\psi$, and do not display the Grothendieck constant.) As noted by several authors (see e.g. [5], [6], [10]), Grothendieck's theorem implies that $B$ can be expressed in terms of Hilbert space representations of the commutative $C^*$-algebras $C(\Omega_i)$: There

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are Hilbert spaces $H_1$ and $H_2$ with vectors $\xi \in H_1$, $\eta \in H_2$, $*$-representations $\pi_i : C(\Omega_i) \to L(H_i)$, and a bounded linear map $T : H_2 \to H_1$ such that

$$B(f,g) = (\pi_1(f)T\pi_2(g)\eta \mid \xi)$$

for all $f \in C(\Omega_1)$, $g \in C(\Omega_2)$.

A bounded separately additive function $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ extends uniquely to a bounded bilinear form on the Cartesian product of the commutative $C^*$-algebras generated by the characteristic functions of the sets in $\Sigma_i$, and we apply Grothendieck's theorem to this bilinear form. Another important tool is the Yosida–Hewitt decomposition of a finitely additive measure into the sum of a countably additive and a purely $\sigma$-additive one.

Section 2 contains the basic representation theorems. The separately finitely additive case in Lemma 2.1 is reduced to the representation recalled in the preceding paragraph, while the bimeasure case uses the Yosida–Hewitt decomposition. A dilation argument shows that if e.g. $\beta : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ is a bimeasure, then there is a Hilbert space $H$ with vectors $\zeta, \theta \in H$ and spectral measures $P_i : \Sigma_i \to L(H)$ satisfying $\beta(X,Y) = (P_1(X)P_2(Y)\zeta \mid \theta)$ for all $X \in \Sigma_1$, $Y \in \Sigma_2$ (Theorem 2.3). This result is applied to a bounded joint convergence theorem in Section 3. In the case of one algebra of sets a representation theorem involving only one spectral measure is proved (Theorem 2.4), and in Section 4 it yields the decomposition of a bimeasure into a linear combination of positive-definite ones.

## 2. Representation theorems.

The characteristic function of a set $A \subset S_i$ is denoted by $\chi_A$. We let $\mathcal{F}_i$ be the space of finite linear combinations of $\chi_A$ for $A \in \Sigma_i$, and write $\mathcal{C}_i$ for the closure of $\mathcal{F}_i$ in the space of bounded complex functions on $S_i$ equipped with the supremum norm. Then $\mathcal{C}_i$ is a commutative $C^*$-algebra.

Since $\beta$ is bounded and separately additive, its semivariation (for the definition, see [13, p. 120]) is finite (see [13, p. 121]), and an elementary argument shows that there is a unique bounded bilinear function $B : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathbb{C}$ such that

$$B(\chi_X, \chi_Y) = \beta(X,Y)$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. We write

$$B(f,g) = \int (f,g)\, d\beta$$

for $f \in \mathcal{C}_1$, $g \in \mathcal{C}_2$; the notation is consistent with [13], see [13, p. 126]. (See also Remark 3.2.)

Our proof of the following lemma generalizes to the not necessarily positive-definite case (and also simplifies by avoiding the use of a control measure of a vector measure) the approach of Chatterji in [2, pp. 271–273]. Chatterji's version can be obtained from ours by using the mapping $(X,Y) \mapsto (\mu(X) \mid \mu(Y))$ for a bounded additive Hilbert space valued function $\mu$ on an algebra of sets. Integration with respect to a finitely additive positive measure is understood in the sense of [4] or via a more elementary approach since we only consider bounded functions.



2.1. Lemma. *There are finitely additive measures $\mu : \Sigma_1 \to [0,\infty)$ and $\nu : \Sigma_2 \to [0,\infty)$ such that*

$$\Big| \int (f,g)\, d\beta \Big| \le \Big( \int |f|^2\, d\mu \Big)^{1/2} \Big( \int |g|^2\, d\nu \Big)^{1/2}$$

*for all $f \in \mathcal{C}_1$, $g \in \mathcal{C}_2$. If $\beta$ is separately $\sigma$-additive, then $\mu$ and $\nu$ may be taken to be $\sigma$-additive.*

Proof. Since $\mathcal{C}_i$ is a commutative $C^*$-algebra and thus representable as the space of continuous complex functions on a compact Hausdorff space, and $B$, with $B(f,g) = \int (f,g)\, d\beta$, is a bounded bilinear form on $\mathcal{C}_1 \times \mathcal{C}_2$, Grothendieck's theorem shows that there are positive linear forms $\phi : \mathcal{C}_1 \to \mathbb{C}$ and $\psi : \mathcal{C}_2 \to \mathbb{C}$ such that

$$|B(f,g)|^2 \le \phi(|f|^2)\psi(|g|^2)$$

for all $f \in \mathcal{C}_1$, $g \in \mathcal{C}_2$. The separately finitely additive case is thus settled by choosing $\mu(X) = \phi(\chi_X)$ and $\nu(Y) = \psi(\chi_Y)$.

Assume now that $\beta$ is separately $\sigma$-additive. Let $\phi$ and $\psi$ be as above and define $\lambda(X) = \phi(\chi_X)$ for $X \in \Sigma_1$. Then $\lambda$ is a finitely additive bounded nonnegative measure on $\Sigma_1$. Let $\lambda = \lambda_c + \lambda_p$ be the Yosida–Hewitt decomposition of $\lambda$, i.e. $\lambda_c : \Sigma_1 \to [0,\infty)$ is countably additive and $\lambda_p : \Sigma_1 \to [0,\infty)$ is purely finitely additive (see [16, p. 52]). Now fix $g \in \mathcal{C}_2$. The function $X \mapsto \int (\chi_X, g)\, d\beta$ is easily seen to be countably additive on $\Sigma_1$. Let $m : \Sigma_1 \to [0,\infty)$ be its total variation. Since $m$ is countably additive, Theorem 1.18 in [16] shows that for any integer $n \ge 1$ there is a set $A_n \in \Sigma_1$ such that $\lambda_p(A_n) < 1/n$ and $m(S_1 \setminus A_n) < 1/n$. Then for any $f \in \mathcal{C}_1$,

$$\begin{aligned}\Big| \int (\chi_{A_n} f, g)\, d\beta \Big|^2 &\le \phi(|\chi_{A_n} f|^2)\psi(|g|^2)\\ &\le \Big( \int |f|^2\, d\lambda_c + \int |\chi_{A_n} f|^2\, d\lambda_p \Big)\psi(|g|^2)\\ &\le \Big( \int |f|^2\, d\lambda_c + \frac{1}{n}\|f\|_\infty^2 \Big)\psi(|g|^2).\end{aligned}$$

Setting $\kappa(X) = \int (\chi_X, g)\, d\beta$ for $X \in \Sigma_1$, we have $\int (\chi_X f, g)\, d\beta = \int_X f\, d\kappa$ for $f \in \mathcal{C}_1$ and $X \in \Sigma_1$. Thus $\lim_{n\to\infty} \int (\chi_{A_n} f, g)\, d\beta = \int (f,g)\, d\beta$, and so we have

$$\Big| \int (f,g)\, d\beta \Big| \le \Big( \int |f|^2\, d\mu \Big)^{1/2} (\psi(|g|^2))^{1/2}$$

where $\mu = \lambda_c$. The proof is completed by repeating the above argument for $\psi$ in place of $\phi$. ∎

If $H$ is a Hilbert space, a mapping $E : \Sigma_i \to L(H)$ is called a *finitely additive spectral measure* if $E(S_i) = I(= \mathrm{id}_H)$, $E(X) = E(X)^* = E(X)^2$ for all $X \in \Sigma_i$, and $E(X \cup Y) = E(X) + E(Y)$ for disjoint $X, Y \in \Sigma_i$. We call such an $E$ *strongly $\sigma$-additive* or simply a *spectral measure* if $E$ is countably additive with respect to the strong (or, equivalently, weak) operator topology.

2.2. Lemma. *There are Hilbert spaces $H_1$ and $H_2$ with vectors $\xi \in H_1$ and $\eta \in H_2$, finitely additive spectral measures $E_i : \Sigma_i \to L(H_i)$, and a bounded linear map $T : H_2 \to H_1$ such that*

$$\beta(X,Y) = (E_1(X)TE_2(Y)\eta \mid \xi)$$

*for all $X \in \Sigma_1$, $Y \in \Sigma_2$. If $\beta$ is separately $\sigma$-additive, then the $E_i$ can be taken to be strongly $\sigma$-additive.*

Proof. The finitely additive case follows immediately from the representation recalled in the introduction applied to $B$.

Assume now that $\beta$ is separately $\sigma$-additive. Thus the $\mu$ and $\nu$ in Lemma 2.1 may be taken to be $\sigma$-additive. Let $\widetilde{\Sigma}_i$ be the $\sigma$-algebra generated by $\Sigma_i$. Let $\widetilde{\mu} : \widetilde{\Sigma}_1 \to [0,\infty)$ be the $\sigma$-additive Hahn extension of $\mu$, and define $\widetilde{\nu}$ analogously. Set $H_1 = L^2(\widetilde{\mu})$ and $H_2 = L^2(\widetilde{\nu})$. Define the $\sigma$-additive spectral measure $E_1 : \Sigma_1 \to L(H_1)$ by the formula $E_1(X)f = \chi_X f$, and similarly $E_2 : \Sigma_2 \to L(H_2)$. Let $\xi$ be the constant 1 on $S_1$, and $\eta$ equal to 1 on $S_2$. Since e.g. for functions in $\mathcal{C}_1$ the integrals with respect to $\mu$ and $\widetilde{\mu}$ exist and are the same, from the inequality in Lemma 2.1 it follows that the mapping $(f_2, f_1) \mapsto \int (\bar{f}_1, f_2)\, d\beta$ on $\mathcal{C}_2 \times \mathcal{C}_1$ extends uniquely to a bounded sesquilinear form $\Phi : H_2 \times H_1 \to \mathbb{C}$. (It is well known and easily follows from the formula

$$\widetilde{\mu}(A) = \inf\Big\{\sum_{n=1}^{\infty} \mu(A_n) \,\Big|\, A \subset \bigcup_{n=1}^{\infty} A_n,\ A_n \in \Sigma_1 \text{ for all } n \in \mathbb{N}\Big\}$$

and the $\sigma$-additivity of $\widetilde{\mu}$ that e.g. $\mathcal{C}_1$ is dense in the Hilbert space $H_1 = L^2(\widetilde{\mu})$.) Thus there is a $T \in L(H_2, H_1)$ such that $\Phi(h,g) = (Th \mid g)$ for all $g \in H_1$, $h \in H_2$, and we have

$$\beta(X,Y) = \Phi(\chi_Y, \chi_X) = (TE_2(Y)\eta \mid E_1(X)\xi) = (E_1(X)TE_2(Y)\eta \mid \xi)$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. ∎

The dilation proof below is related to some techniques used in [15, p. 152].

2.3. Theorem. *There is a Hilbert space $H$ with vectors $\zeta, \theta \in H$, and finitely additive spectral measures $P_i : \Sigma_i \to L(H)$ such that*

$$\beta(X,Y) = (P_1(X)P_2(Y)\zeta \mid \theta)$$



*for all $X \in \Sigma_1$, $Y \in \Sigma_2$. If $\beta$ is separately $\sigma$-additive, then the $P_i$ can be taken to be strongly $\sigma$-additive.*

Proof. Let $H_1$, $H_2$, $T$, $E_i$, $\xi$ and $\eta$ be as in Lemma 2.2. Without loss of generality we can assume that $\|T\| \le 1$. There is a unitary operator $U \in L(H_1 \oplus H_2)$ such that $T\phi = P_{H_1}U(0,\phi)$ for all $\phi \in H_2$, where $P_{H_1}$ is the projection onto $H_1$. (The proof of the Halmos dilation theorem in [8, pp. 126–127] or [9, pp. 177–178] also works in the case of two different Hilbert spaces.) Choose $a \in S_1$ and $b \in S_2$, and let $\delta_a : \Sigma_1 \to \{0,1\}$ and $\delta_b : \Sigma_2 \to \{0,1\}$ be the corresponding point measures. Let $I_i$ be the identity operator of $H_i$, $i = 1,2$. Define $P_1(X) = U^*(E_1(X) \oplus \delta_a(X)I_2)U$ for $X \in \Sigma_1$, and $P_2(Y) = \delta_b I_1 \oplus E_2(Y)$ for $Y \in \Sigma_2$. Clearly, $P_1$ and $P_2$ are finitely additive spectral measures ($\sigma$-additive, if $\beta$ is separately $\sigma$-additive), and writing $\zeta = (0,\eta)$ and $\theta = U^*(\xi, 0)$ we get

$$\begin{aligned}(P_1(X)P_2(Y)\zeta \mid \theta) &= (U^*(E_1(X) \oplus \delta_a(X)I_2)U(\delta_b I_1 \oplus E_2(Y))\zeta \mid \theta)_H\\ &= ((E_1(X) \oplus \delta_a(X)I_2)U(\delta_b I_1 \oplus E_2(Y))\zeta \mid U\theta)_H\\ &= ((E_1(X)TE_2(Y))\eta \mid \xi)_{H_1} = \beta(X,Y)\end{aligned}$$

for all $X \in \Sigma_1$, $Y \in \Sigma_2$. ∎

We now consider the case of only one algebra of sets. The following theorem will be applied in Section 4.

2.4. Theorem. *Let $S$ be a nonempty set and $\Sigma$ an algebra of subsets of $S$. Let $\beta : \Sigma \times \Sigma \to \mathbb{C}$ be a bounded separately additive function.*

(a) *There is a Hilbert space $H$ with a finitely additive spectral measure $E : \Sigma \to L(H)$, a bounded linear operator $T : H \to H$ and a vector $\xi \in H$ such that*

$$\beta(X,Y) = (E(X)TE(Y)\xi \mid \xi)$$

*for all $X, Y \in \Sigma$.*

(b) *If $\beta$ is separately $\sigma$-additive, then $E$ in* (a) *can be taken to be strongly $\sigma$-additive.*

Proof. (a) Let $B$ be as before, when $S = S_1 = S_2$, $\Sigma = \Sigma_1 = \Sigma_2$, and $\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2$. Grothendieck's theorem yields positive linear forms $\phi, \psi : \mathcal{C} \to \mathbb{C}$ such that

$$|B(f,g)|^2 \le \phi(|f|^2)\psi(|g|^2)$$

for all $f, g \in \mathcal{C}$. Set $\lambda = \phi + \psi$. Defining $B_0(f,g) = B(\overline{g}, f)$ we get a sesquilinear form $B_0 : \mathcal{C} \times \mathcal{C} \to \mathbb{C}$ such that $|B_0(f,g)|^2 \le \lambda(|f|^2)\lambda(|g|^2)$ for $f, g \in \mathcal{C}$. Write $N = \{f \in \mathcal{C} \mid \lambda(|f|^2) = 0\}$. Then $N$ is a vector subspace of $\mathcal{C}$, and setting $(f + N \mid g + N) = \lambda(\overline{g}f)$ we get a well-defined inner product in $\mathcal{C}/N$. Let $H$ be the Hilbert space completion of $\mathcal{C}/N$. Since $|B_0(f,g)|^2 \le \lambda(|f|^2)\lambda(|g|^2)$, there exists a bounded sesquilinear form $\widetilde{B}_0 : H \times H \to \mathbb{C}$

such that $\widetilde{B}_0(f+N, g+N) = B_0(f,g)$ for $f, g \in \mathcal{C}$, so there is a bounded linear operator $T: H \to H$ such that $\widetilde{B}_0(z,w) = (Tz \mid w)$. Let $\{\pi, H, \xi\}$ be the cyclic representation induced by $\lambda$, i.e., $\lambda(f) = (\pi(f)\xi \mid \xi)$ for $f \in \mathcal{C}$, $\pi(f)(g+N) = fg + N$ for $f, g \in \mathcal{C}$ and $\xi = 1 + N$. Then

$$(\pi(f)T\pi(g)\xi \mid \xi) = \widetilde{B}_0(g+N, \bar{f}+N) = B_0(g, \bar{f}) = B(f,g)$$

for all $f, g \in \mathcal{C}$. We may thus, in the separately finitely additive case, choose $E(X) = \pi(\chi_X)$ for $X \in \Sigma$.

(b) Suppose now that $\beta$ is separately $\sigma$-additive. Let $\mathcal{C}$ be as above. Lemma 2.1 yields countably additive measures $\mu : \Sigma \to [0,\infty)$ and $\nu : \Sigma \to [0,\infty)$ such that

$$\left| \int (f,g)\, d\beta \right| \le \left( \int |f|^2\, d\mu \right)^{1/2} \left( \int |g|^2\, d\nu \right)^{1/2}$$

for all $f, g \in \mathcal{C}$. Define $\lambda = \mu + \nu$. Let $\widetilde{\Sigma}$ be the $\sigma$-algebra generated by $\Sigma$, and $\widetilde{\lambda} : \widetilde{\Sigma} \to [0,\infty)$ the Hahn extension of $\lambda$. Since $\mathcal{C}$ is dense in the Hilbert space $H = L^2(\widetilde{\lambda})$ (see the proof of Lemma 2.2) and

$$\left| \int (f,g)\, d\beta \right| \le \left( \int |f|^2\, d\lambda \right)^{1/2} \left( \int |g|^2\, d\lambda \right)^{1/2},$$

we obtain a uniquely defined bounded sesquilinear form $B_0 : H \times H \to \mathbb{C}$ such that $B_0(f,g) = \int (\bar{g}, f)\, d\beta$ for all $f, g \in \mathcal{C}$, and hence a bounded linear map $T : H \to H$ satisfying $B_0(f,g) = (Tf \mid g)$, $f, g \in H$. Define the (strongly $\sigma$-additive) spectral measure $E : \Sigma \to L(H)$ by the formula $E(X)f = \chi_X f$. If $\xi \in H$ is the constant function 1, we have

$$\begin{aligned}(E(X)TE(Y)\xi \mid \xi) &= (TE(Y)\xi \mid E(X)\xi) \\ &= B_0(\chi_Y, \chi_X) = \beta(X,Y)\end{aligned}$$

for all $X, Y \in \Sigma$. ∎

**3. A bounded joint convergence theorem.** In this section we assume that $\Sigma_1$ and $\Sigma_2$ are $\sigma$-algebras and $\beta$ is a bimeasure, i.e., separately $\sigma$-additive. Now $\mathcal{C}_i$ is the space of bounded $\Sigma_i$-measurable functions on $S_i$.

3.1. THEOREM. *Let $(f_n)$ and $(g_n)$ be uniformly bounded sequences in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. If $f(x) = \lim_{n\to\infty} f_n(x)$ and $g(y) = \lim_{n\to\infty} g_n(y)$ for all $x \in S_1$, $y \in S_2$, then*

$$\lim_{n\to\infty} \int (f_n, g_n)\, d\beta = \int (f,g)\, d\beta\,.$$

Proof. Let $H$, $P_i$, $\zeta$, and $\theta$ be as in Theorem 2.3. Define $m_1(X) = P_1(X)\theta$ and $m_2(Y) = P_2(Y)\zeta$ for $X \in \Sigma_1$, $Y \in \Sigma_2$. Now, e.g. $\int (f,g)\, d\beta$ can be expressed in terms of integration with respect to the vector measures $m_1$ and $m_2$:

$$\int (f,g)\, d\beta = \left( \int g\, dm_2 \,\middle|\, \int \bar{f}\, dm_1 \right).$$

The dominated convergence theorem for vector measures [4, p. 328] combined with the joint continuity of the inner product in $H$ completes the proof. ∎



3.2. Remark. As in the above proof, it follows from Theorem 2.3 (or already from Lemma 2.2) that the assumption on a bimeasure $\nu$ made in Lemma 6.2 in [11, p. 285] is automatically satisfied. It may be observed that the conclusion of [11, Lemma 6.2] for an arbitrary bimeasure also follows from [13, Corollary 5.7], independently of the Grothendieck inequality.

3.3. Remark. It follows from the dominated convergence theorem that in the situation of the above theorem both the iterated limits $\lim_{m\to\infty}\lim_{n\to\infty}\int (f_m, g_n)\, d\beta$ and $\lim_{n\to\infty}\lim_{m\to\infty}\int (f_m, g_n)\, d\beta$ exist, and both are equal to $\int (f,g)\, d\beta$. This last observation was also made by Chang and Rao in [1, p. 46] but they do not consider joint convergence.

3.4. Remark. Let $\Sigma$ be the algebra generated in the power set of $S_1 \times S_2$ by the rectangles $X \times Y$ where $X \in \Sigma_1$, $Y \in \Sigma_2$. The function $X \times Y \mapsto \beta(X,Y)$ for $X \in \Sigma_1$, $Y \in \Sigma_2$ has a unique additive extension $\widetilde{\beta} : \Sigma \to \mathbb{C}$. Although it is clear from Theorem 3.1 that $\lim_{n\to\infty} \widetilde{\beta}(A_n) = 0$ whenever $(A_n)$ is a decreasing sequence of *rectangles* with empty intersection, this does not remain true if the $A_n$ are allowed to be general sets in $\Sigma$. For otherwise $\widetilde{\beta}$ could be extended to a $\sigma$-additive function on the $\sigma$-algebra generated by $\Sigma$, and this is known not always to be the case (see e.g. [1, pp. 12–15, 17–19]).

**4. Positive-definite bimeasures.** In this section we consider only one algebra $\Sigma$ of subsets of a nonempty set $S$. A bounded separately finitely additive function $\beta : \Sigma \times \Sigma \to \mathbb{C}$ is said to be *positive-definite* if it is a positive-definite kernel in the usual sense, i.e.,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \bar{c}_j \beta(X_i, X_j) \ge 0$$

for all finite sequences of $X_1, \ldots, X_n \in \Sigma$, $c_1, \ldots, c_n \in \mathbb{C}$. It is easily seen to be equivalent to require that

$$\int (f, \bar{f})\, d\beta \ge 0$$

for every $f$ in the space $\mathcal{C}$ of functions that can be expressed as uniform limits of linear combinations of characteristic functions of sets in $\Sigma$.

4.1. THEOREM. *Let $S$ be a nonempty set and $\Sigma$ an algebra of subsets of $S$. Let $\beta : \Sigma \times \Sigma \to \mathbb{C}$ be a bounded separately additive function.*

(a) *The following conditions are equivalent*:

(i) $\beta$ *is positive definite*;

(ii) *there is a Hilbert space* $H$ *with a finitely additive spectral measure* $E:\Sigma\to L(H)$, *an orthogonal projection* $P:H\to H$ *and a vector* $\xi\in H$ *such that*

$$\beta(X,Y)=(E(X)PE(Y)\xi\mid\xi)$$

*for all* $X,Y\in\Sigma$;

(iii) *there is a Hilbert space* $H$ *with a finitely additive spectral measure* $E:\Sigma\to L(H)$, *a positive operator* $T\in L(H)$, *and a vector* $\xi\in H$ *such that*

$$\beta(X,Y)=(E(X)TE(Y)\xi\mid\xi)$$

*for all* $X,Y\in\Sigma$.

(b) *If* $\beta$ *is separately* $\sigma$*-additive, then* $E$ *in* (a) *can be taken to be strongly* $\sigma$*-additive.*

Proof. We prove (a) and (b) simultaneously. Assume first (i). Let $\mathcal{C}$ be as above and define $\widetilde{B}(f,g)=\int(g,f)\,d\beta$. Since $\widetilde{B}$ is a bounded bilinear form and $\widetilde{B}(f,\bar f)\ge 0$, a standard construction yields a Hilbert space $K$ and a bounded linear map $\Phi:\mathcal{C}\to K$ such that $\Phi(\mathcal{C})$ is dense in $K$ and $\widetilde{B}(f,g)=(\Phi(f)\mid\Phi(\bar g))$ for all $f,g\in\mathcal{C}$. (Set $N=\{f\in\mathcal{C}\mid\widetilde{B}(f,\bar f)=0\}$, take for $K$ the completion of $\mathcal{C}/N$ equipped with the inner product $(f+N\mid g+N)=\widetilde{B}(f,\bar g)$, and write $\Phi(f)=f+N$, $f\in\mathcal{C}$.) There is a positive linear form $\phi:\mathcal{C}\to\mathbb{C}$ such that $\|\Phi(f)\|^2\le\phi(|f|^2)$ for all $f\in\mathcal{C}$, and in part (b) we may, moreover, assume that $X\mapsto\mu(X)=\phi(\chi_X)$ is $\sigma$-additive on $\Sigma$. (Apply Lemma 2.1 to the mapping $(X,Y)\mapsto(\Phi(\chi_X)\mid\Phi(\chi_Y))$ on $\Sigma\times\Sigma$ or Lemma 2 in [2, p. 271]; in the latter case note that if $\beta$ is separately $\sigma$-additive, the map $X\mapsto\Phi(\chi_X)$ is weakly, or equivalently strongly, $\sigma$-additive, since it is bounded, and $X\mapsto(\Phi(\chi_X)\mid\xi)$ is $\sigma$-additive for each $\xi$ in the dense subspace $\Phi(\mathcal{C})$ of $K$.) Using the GNS-construction, we get in part (a) a Hilbert space $H_1$ and a $*$-representation $\pi_1:\mathcal{C}\to L(H_1)$ with a cyclic vector $\xi_1$ such that $(\pi_1(f)\xi_1\mid\xi_1)=\phi(f)$ for all $f\in\mathcal{C}$. In part (b), put $H_1=L^2(\widetilde{\mu})$, where $\widetilde{\mu}$ is the Hahn extension of $\mu$ (introduced above) to the $\sigma$-algebra $\widetilde{\Sigma}$ generated by $\Sigma$, write $\pi_1(f)\eta=f\eta$ for all $f\in\mathcal{C}$, $\eta\in H_1$, and take $\xi_1$ identically 1 on $S$. Then $\phi(f)=(\pi_1(f)\xi_1\mid\xi_1)$ for $f\in\mathcal{C}$, and $X\mapsto\pi_1(\chi_X)$ is strongly $\sigma$-additive in (b). In both (a) and (b), write $\Psi(f)=\pi_1(f)\xi_1$ for $f\in\mathcal{C}$. Since

$$\|\Psi(f)\|^2=(\pi_1(f)\xi_1\mid\pi_1(f)\xi_1)=(\pi_1(|f|^2)\xi_1\mid\xi_1)=\phi(|f|^2)\ge\|\Phi(f)\|^2,$$

there is a linear contraction $\theta:H_1\to K$ such that $\theta(\Psi(f))=\Phi(f)$ for all $f\in\mathcal{C}$. Thus we can find a Hilbert space $H$ with isometric linear maps $\alpha:H_1\to H$ and $V:K\to H$ such that $\theta=V^*\circ\alpha$ (see e.g. [2, p. 271] or [14, p. 379], and references mentioned in those papers). We identify $H_1$ via $\alpha$ with its image $\alpha(H_1)$ in $H$. Let $H_2$ be the orthogonal complement of $H_1$ in $H$; we interpret $H=H_1\oplus H_2$. Fix $s\in S$ and define $\pi_2(f)=f(s)I_{H_2}$. Then $\pi=\pi_1\oplus\pi_2:\mathcal{C}\to L(H)$ is a $*$-representation preserving the identity, and so $X\mapsto E(X)=\pi(\chi_X)$ is a finitely additive spectral measure, which in case (b) is strongly $\sigma$-additive. Moreover, writing $\xi=(\xi_1,0)\in H=H_1\oplus H_2$, for all $X,Y\in\Sigma$ we get

$$\begin{aligned}\beta(X,Y)&=(\Phi(\chi_Y)\mid\Phi(\chi_X))=(\theta(\Psi(\chi_Y))\mid\theta(\Psi(\chi_X)))\\&=(V^*(\pi_1(\chi_Y)\xi_1)\mid V^*(\pi_1(\chi_X)\xi_1))=(V^*(\pi(\chi_Y)\xi)\mid V^*(\pi(\chi_X)\xi))\\&=(E(X)VV^*E(Y)\xi\mid\xi),\end{aligned}$$

and so (ii) holds, since $P=VV^*$ is an orthogonal projection.

Obviously (ii) implies (iii). Finally, a straightforward calculation shows that (iii) implies (i). ∎



4.2. THEOREM. *Let* $S$ *be a nonempty set and* $\Sigma$ *an algebra of subsets of* $S$. *Let* $\beta:\Sigma\times\Sigma\to\mathbb{C}$ *be a bounded separately additive function.*

(a) *There are four positive-definite bounded separately additive functions* $\beta_1,\dots,\beta_4$, *such that*

$$\beta=\beta_1-\beta_2+i(\beta_3-\beta_4).$$

(b) *If* $\beta$ *is separately* $\sigma$*-additive, then the* $\beta_1,\dots,\beta_4$ *in* (a) *can be taken to be separately* $\sigma$*-additive.*

Proof. In Theorem 2.4, write $T=T_1-T_2+i(T_3-T_4)$, where each $T_i$ is a positive bounded linear operator on $H$. Theorem 4.1 shows that we get the positive-definite separately additive (or separately $\sigma$-additive) functions we want. ∎

4.3. Remark. Part (a) in the above theorem also follows from some results (proved in a much more general setting with a heavy machinery) in [3]: see page 157 and Corollaries 4.3 and 5.6.

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# On supportless absorbing convex subsets in normed spaces

by

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**Abstract.** It is proved that a separable normed space contains a closed bounded convex symmetric absorbing supportless subset if and only if this space may be covered (in its completion) by the range of a nonisomorphic operator.

According to the Hahn–Banach theorem every boundary point $x$ of a solid (i.e. with nonempty interior) closed bounded convex subset $A$ of a normed space $X$ is a *support point*, i.e. there exists a linear functional $f \in X^* \setminus \{0\}$ such that $f(x) = \sup f(A)$. If $A$ is not solid then the completeness of the space $X$ begins to play a role. Namely, in 1958 V. Klee [3] gave an example of an (incomplete) dense subspace of $\ell_2$ which contains a closed bounded convex absorbing subset with no support points (a *supportless subset*). In 1961 E. Bishop and R. Phelps [1] showed that in a Banach space such an example is impossible: the support functionals of closed bounded convex subsets of a Banach space are always dense in the dual space. In 1985 J. Borwein and D. Tingley [3], developing the ideas of V. Klee, constructed in every infinite-dimensional separable Banach space a dense linear subspace which contains a closed bounded convex absorbing supportless subset.

The purpose of this paper is a full description of the class of (incomplete) separable normed spaces which contain closed bounded convex absorbing supportless subsets.

We use the standard Banach space theory notation. By $U(E)$ we denote the unit ball of a normed space $E$; if $A \subset E$ then $[A]$ is the closed linear span of $A$ and $\operatorname{lin} A$ is the linear span of $A$.

We begin with an auxiliary proposition.

PROPOSITION. *Let $X$ be a separable Banach space, $M$ be a dense subspace of $X$ and $T : Y \to X$ be a one-to-one linear bounded operator from a Banach space $Y$ into $X$ such that $TY \supset M$, the inverse mapping $T^{-1}$ is*

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