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On supportless absorbing convex subsets in normed spaces

by

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Abstract. It is proved that a separable normed space contains a closed bounded convex symmetric absorbing supportless subset if and only if this space may be covered (in its completion) by the range of a nonisomorphic operator.

According to the Hahn–Banach theorem every boundary point x of a solid (i.e. with nonempty interior) closed bounded convex subset A of a normed space X is a *support point*, i.e. there exists a linear functional $f \in X^* \setminus \{0\}$ such that $f(x) = \sup f(A)$. If A is not solid then the completeness of the space X begins to play a role. Namely, in 1958 V. Klee [3] gave an example of an (incomplete) dense subspace of ℓ_2 which contains a closed bounded convex absorbing subset with no support points (a *supportless subset*). In 1961 E. Bishop and R. Phelps [1] showed that in a Banach space such an example is impossible: the support functionals of closed bounded convex subsets of a Banach space are always dense in the dual space. In 1985 J. Borwein and D. Tingley [3], developing the ideas of V. Klee, constructed in every infinite-dimensional separable Banach space a dense linear subspace which contains a closed bounded convex absorbing supportless subset.

The purpose of this paper is a full description of the class of (incomplete) separable normed spaces which contain closed bounded convex absorbing supportless subsets.

We use the standard Banach space theory notation. By $U(E)$ we denote the unit ball of a normed space E; if $A \subset E$ then $[A]$ is the closed linear span of A and $\operatorname{lin} A$ is the linear span of A.

We begin with an auxiliary proposition.

PROPOSITION. *Let X be a separable Banach space, M be a dense subspace of X and $T : Y \to X$ be a one-to-one linear bounded operator from a Banach space Y into X such that $TY \supset M$, the inverse mapping T^{-1} is*

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unbounded and the subspace T^*X^* is norming. Then there exists a fundamental minimal system $\{x_n\}\subset M$ with conjugate system $\{h_n\}\subset X^*$ such that $\|x_n\|\le 2^{-n}$, $\|T^*h_n\|\le 2^{-n}$, $n=1,2,\ldots$

Proof. By the density of M in X we can find an M-basis $\{v_i\}$ of X such that

$$\{v_i\}\subset M\,,\quad \|v_i\|\le \frac{2^{-4i-1}}{i!}\,,\quad \|T^{-1}v_i\|\le 2^{-i-5}\,,\quad i=1,2,\ldots$$

Let $\{\varepsilon_i\}$ be a stability sequence of the M-basis $\{v_i\}$. Set $L=T^{-1}(M)$. It is easily verified that the restriction $T|_L$ is also nonisomorphic and hence by a standard process we can construct a basic sequence $\{w_i\}\subset S(L)$ with basis constant $C\le 3/2$ such that

$$\|Tw_i\|\le \frac{2^{-4i-1}}{i!}\varepsilon_i\,,\quad i=1,2,\ldots$$

Let $z_i'=w_i+T^{-1}v_i$, $i=1,2,\ldots$ Then by the Krein–Milman–Rutman stability theorem, $\{z_i'\}$ is a basic sequence with basis constant $C_1\le 2$ and $\{Tz_i'\}$ is an M-basis of X. Obviously, $\{Tz_i'\}\subset M$, $\|z_i'\|\le 1+1/32$ and $\{Tz_i'\|\le 2^{-4i}/i!$, $i=1,2,\ldots$ Denote by $\{h_i'\}$ a conjugate system in Y^* to the basic system $\{z_i'\}$ such that $\|h_i'\|\le 2C_1\le 4$ for all $i=1,2,\ldots$ Without loss of generality we can assume that T^*X^* is 1-norming and hence for all $n=1,2,\ldots$ there exists a functional $g_n'\in 4U(T^*X^*)$ such that

$$\|h_n'|_{[z_i']_1^n}-g_n'|_{[z_i']_1^n}\|\le \frac{2^{-7n-1}}{(n!)^2}\,. \tag{1}$$

Define

$$z_n=n!2^{4n}z_n'\,,\quad g_n=\frac{2^{-4n}}{n!}g_n'\,,\quad \Delta_n=\det(g_k(z_l))_{k,l=1,\ldots,n}\,.$$

Using (1) we get

$$|g_n(z_i)|\le \frac{2^{-7n}}{(n!)^2}\,,\quad 1\le i<n;\quad |g_n(z_n)-1|\le \frac{2^{-7n}}{(n!)^2}\,. \tag{2}$$

In particular,

$$|g_k(z_l)|\le 2\,,\quad k,l=1,2,\ldots \tag{3}$$

Let us expand the determinant Δ_n according to the last line:

$$\Delta_n=\sum_{l=1}^{n}g_n(z_l)(-1)^{n+l}M_{nl}=\sum_{l=1}^{n-1}g_n(z_l)(-1)^{n+l}M_{nl}+g_n(z_n)\Delta_{n-1}\,.$$

Hence using (2) and (3) we have for all $n=1,2,\ldots,$

$$|\Delta_n-\Delta_{n-1}|\le (n-1)\frac{2^{-7n}}{(n!)^2}(n-1)!\frac{2^{-7n}}{(n!)^2}+\frac{2^{-7n}}{(n!)^2}(n-1)!2^{n-1}\le 2^{-6n}\,.$$

But it is obvious that $1-2^{-7}<\Delta_1<1+2^{-7}$ and thus for all $n=1,2,\ldots,$

$$1/2\le \Delta_n\le 3/2\,. \tag{4}$$

Define $y_1=z_1$,

$$y_n=\frac{1}{\Delta_{n-1}}\begin{vmatrix} & & & g_1(z_n)\\ & \Delta_{n-1} & & g_2(z_n)\\ & & & \vdots\\ & & & g_{n-1}(z_n)\\ z_1 & z_2 & \ldots & z_n\end{vmatrix}\,, \tag{5}$$

$$f_n=\frac{1}{\Delta_n}\begin{vmatrix} & & g_1\\ & \Delta_{n-1} & \vdots\\ & & g_{n-1}\\ g_n(z_1) & \ldots\quad g_n(z_{n-1}) & g_n\end{vmatrix}\,,$$

and estimate $\|Ty_n\|$ expanding the determinant (5) according to the last line:

$$\|Ty_n\|=\Big\|T\Big(\frac{1}{\Delta_{n-1}}\sum_{k=1}^{n}(-1)^{n+k}M_{nk}z_k\Big)\Big\|\le \frac{1}{|\Delta_{n-1}|}n2^{n-1}(n-1)!\le n!2^n$$

(we have used (3), (4) and $\|Tz_k\|\le 1$, $k=1,2,\ldots$). For f_n we have

$$\begin{aligned} f_n&=\frac{1}{\Delta_n}\sum_{i=1}^{n-1}(-1)^{n+i}g_n(z_i)M_{ni}+\frac{\Delta_{n-1}}{\Delta_n}g_n\\ &=\frac{1}{\Delta_n}\sum_{i=1}^{n-1}(-1)^{n+i}g_n(z_i)\sum_{k=1}^{n-1}(-1)^{k+n-1}\widehat{M}_{k,n-1}g_k+\frac{\Delta_{n-1}}{\Delta_n}g_n\\ &=\frac{1}{\Delta_n}\sum_{k=1}^{n-1}\Big(\sum_{i=1}^{n-1}(-1)^{2n+k+i-1}g_n(z_i)\widehat{M}_{k,n-1}\Big)g_k+\frac{\Delta_{n-1}}{\Delta_n}g_n\,;\end{aligned}$$

hence,

$$\|f_n\|\le \frac{1}{\Delta_n}(n-1)(n-1)\frac{2^{-7n}}{(n!)^2}(n-2)!2^{n-2}+3\frac{2^{-4n}}{n!}\le \frac{2^{-4n+3}}{n!}\,.$$

Let

$$x_n=\frac{2^{-2n}}{n!}Ty_n\,,\quad h_n=n!2^{2n}T^{*-1}f_n\,,\quad n=1,2,\ldots$$

Then $\|x_n\|\le 2^{-n}$ and $\|T^*h_n\|\le 2^{-n}$. It is easily verified that $\{x_n\}$ is a fundamental minimal system with conjugate system $\{h_n\}$. The proof is complete.

THEOREM. *Let M be an incomplete separable normed space and X be the completion of M. Suppose there exists a linear bounded one-to-one operator $A:Z\to X$ from some Banach space Z into X such that $AZ\supset M$ and* *the inverse mapping A^{-1} is unbounded (i.e. $AZ \neq X$). Then M contains a closed bounded convex symmetric absorbing supportless subset. Conversely, if a normed space M contains a closed bounded convex absorbing supportless subset then there exists a linear bounded one-to-one operator $A : Z \to X$ from some Banach space Z into the completion X of M such that $AZ \supset M$ and the inverse mapping A^{-1} is unbounded.*

Proof. Since A^{-1} is unbounded the set $V = \mathrm{cl}_X(AU(Z) \cap M)$ has empty interior. Denote by Y the Banach space $\mathrm{lin}\, V$ with V as unit ball and let $T : Y \to X$ be the natural injection. It is obvious that $TY \supset M$, $TU(Y) = V$, $\mathrm{cl}_X(V \cap M) = V$ and it is easily verified (with the help of the Hahn–Banach theorem) that T^*X^* is a 1-norming subspace. Then by the Proposition there exist sequences $\{x_k\} \subset M$ and $\{h_k\} \subset X^*$ such that $\|x_k\| \le 2^{-k}$, $\|T^*h_k\| \le 2^{-k}$, $k = 1, 2, \ldots$ Let us introduce a compact operator $F : c_0 \to X$ by $Fu_k = x_k$, $k = 1, 2, \ldots$, where $\{u_k\}$ is the canonical basis of c_0. Set $Q = \{\sum a_k x_k : \{a_k\} \in U(\ell_\infty)\}$.

We now verify the inclusion

$$Q \cap TY \subset F(U(c_0)) . \tag{6}$$

Let $x = \sum a_k x_k \in Q \cap TY$. Then $a_k = h_k(x)$, $x = Ty$, $y \in Y$. So for all $k = 1, 2, \ldots,$

$$|a_k| = |h_k(Ty)| = |(T^*h_k)(y)| \le \|T^*h_k\| \cdot \|y\| \le 2^{-k}\|y\| .$$

Hence $\sum a_k x_k \in F(U(c_0))$.

Now set $E = (Y \oplus c_0)_\infty$ and define an operator $B : E \to X$ by $B(y+u) = Ty + Fu$, $y \in Y$, $u \in c_0$. It is clear that $E^* = (Y^* \oplus \ell_1)_1$ and that $K = (U(Y^*) + B^*U(X^*)) \cap \ell_1$ is symmetric, convex and compact. Indeed, let $f \in K$; then $f = \sum \xi_i e_i = g + h$, $g \in U(Y^*)$, $h = B^*t$, $t \in U(X^*)$, where $\{e_i\}$ is the canonical basis of ℓ_1; so for all $i = 1, 2 \ldots,$

$$|\xi_i| = |f(u_i)| = |h(u_i)| = |(B^*t)(u_i)| = |t(x_i)| \le 2^{-i} .$$

It is obvious that $(Y^* + B^*X^*) \cap \ell_1 = \bigcup_{n=1}^{\infty} nK$ and by a category argument there exists $e_1' \in \ell_1$ with $\|e_1' - e_1\| \le 2^{-1}$ such that $[e_1'] \cap (Y^* + B^*X^*) = 0$. Let $K_1 = \mathrm{co}\{\pm e_1', K\}$. Then using a category argument again one can find $e_2' \in \ell_1$ with $\|e_2' - e_2\| \le 2^{-2}$ such that $[e_2'] \cap \bigcup_{n=1}^{\infty} nK_1 = 0$, and hence, $[e_1', e_2'] \cap (Y^* + B^*X^*) = 0$. Continuing in this way we construct a sequence $\{e_i'\} \subset \ell_1$ with the following properties:

$$\begin{gathered} \|e_i' - e_i\| \le 2^{-i} , \quad i = 1, 2, \ldots, \\ [e_i']_1^n \cap (Y^* + B^*X^*) = 0 , \quad n = 1, 2, \ldots \end{gathered} \tag{7}$$

Define $U_1^* = w^*\text{-}\mathrm{cl}\,\mathrm{co}\{\pm e_i'\}_1^\infty$ and let U_1 be the corresponding unit ball of c_0. It is clear that $\frac{1}{2}U(c_0) \subset U_1 \subset 2U(c_0)$, where $U(c_0)$ is the unit ball of c_0 in the usual norm.

Let us show that $W = (V + F(U_1)) \cap M$ is closed in M. Let $v_n + q_n \in V + F(U_1)$, $\lim_n (v_n + q_n) = x \in M$. By a compactness argument we can assume that the sequence $\{q_n\}$ converges to some q_0. Since $F(U_1) \subset 2F(U(c_0)) \subset 2Q$ we have $q_0 \in 2Q$. So $\{v_n\}$ also converges to some $v_0 \in V$ (since V is closed). Thus $x = v_0 + q_0$. But by assumption $x \in M$ and $M \subset TY$, so $q_0 = x - v_0 \in TY$. Consequently, by the inclusions $q_0 \in 2Q$ and (6) we have $q_0 \in 2F(U(c_0))$. It is easily verified that $w\text{-}\lim F^{-1}q_n = F^{-1}q_0$ and by the weak closedness of the ball U_1 we have $F^{-1}q_0 \in U_1$. So $x \in V + F(U_1)$ and the closedness of W in M is proved.

It is obvious that $[W] = X$. Let us show that $\mathrm{supp}\, W = \emptyset$. Assume to the contrary that there exists a linear functional $f \in X^* \setminus \{0\}$ and $x \in W$ such that $f(x) = \sup f(W) \neq 0$. Since $\mathrm{cl}_X(M \cap V) = V$ and $\mathrm{cl}_X(M \cap F(U_1)) \supset F(U_1)$ we have $\mathrm{cl}_X W \supset V + F(U_1)$ and so $\sup f(W) = \sup f(V + F(U_1))$. Thus the functional B^*f attains its supremum on the subset $V + U_1 \subset E$ and therefore $B^*f = g + \sum_{i=1}^m \xi_i e_i'$, $g \in Y^*$. Consequently,

$$\sum_{i=1}^m \xi_i e_i' = g - B^*f \in Y^* + B^*X^*$$

and by (7), $\xi_i = 0$, $i = 1, \ldots, m$. Then $B^*f = g$; but $B^*f|_{c_0} = 0$, which gives $f(x_i) = 0$, $i = 1, 2, \ldots$, and by the completeness of the system $\{x_i\}$ we have $f = 0$. This contradicts $\sup f(W) \neq 0$ and completes the proof of the first part of the theorem.

Let us prove the second part. Let W be a closed bounded convex absorbing supportless subset of a normed space M, and let X be the completion of M. It is obvious $\mathring{W} = \emptyset$ (otherwise by the Hahn–Banach theorem $\mathrm{supp}\, W \neq \emptyset$) and by the Bishop–Phelps theorem M is incomplete, i.e. $M \neq X$. Let $W_1 = \mathrm{cl}_X W$, $V = W_1 \cap -W_1$. It is easily verified that $\mathring{V} = \emptyset$ and $\mathrm{lin}\, V \supset M$. Denote by Z the Banach space $\mathrm{lin}\, V$ with V as unit ball and let $A : Z \to X$ be the canonical injection. Then $AZ \supset M$ and (by $\mathring{V} = \emptyset$) the inverse mapping A^{-1} is unbounded. The proof of the theorem is complete.

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Commutators based on the Calderón reproducing formula

by

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Abstract. We prove the Schatten–Lorentz ideal criteria for commutators of multiplications and projections based on the Calderón reproducing formula and the decomposition theorem for the space of symbols corresponding to commutators in the Schatten ideal.

1. Introduction and summary. This paper is devoted to the study of commutators of multiplications and projections based on the Calderón reproducing formula. Commutators of multiplications and projections are usually defined in the context of a Hilbert space with reproducing kernel. Let $H \subset L^2(X, d\mu)$ be such a space, $P : L^2 \to H$ the orthogonal projection from L^2 onto H, and let b be a function defined on X. The *commutator* of M_b and P is

$$C_b = [M_b, P] = M_b P - P M_b\,,$$

where M_b denotes the operator of multiplication by b. The function b is called the *symbol* of C_b. The commutator C_b is closely related to the Hankel operator

$$H_b = (I - P) M_b P\,,$$

namely $H_b = C_b P$, $C_b = H_b - H_{\bar{b}}^*$.

The Calderón reproducing formula defines a class of Hilbert spaces with reproducing kernels. For a particular choice of the wavelet function which appears in the Calderón reproducing formula (and after a minor modification) the commutators based on the Calderón reproducing formula are unitarily equivalent to the commutators on weighted Bergman spaces on the upper half-plane. There has been an extensive study of Hankel operators on Bergman spaces ([Ax], [AFP], [BBCZ], [J], [Str], [Z2]). We refer to Zhu's book [Z1] for the background and more references. Our method of studying commutators does not rely on complex analytic tools nor on the formulas for reproducing kernels as is often done in the case of Bergman

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