[Klei] D. C. Kleinecke, *On operator commutators*, Proc. Amer. Math. Soc. 8 (1957), 535–536.
[M–M] M. Mathieu and G. J. Murphy, *Derivations mapping into the radical*, Arch. Math. (Basel) 57 (1991), 469–474.
[M–R] M. Mathieu and V. Runde, *Derivations mapping into the radical, II*, Bull. London Math. Soc. 24 (1992), 485–487.
[Pos] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. 8 (1957), 1093–1100.
[Rick] C. E. Rickart, *General Theory of Banach Algebras*, The University Series in Higher Mathematics, D. van Nostrand, 1960.
[Run] V. Runde, *Automatic continuity of derivations and epimorphisms*, Pacific J. Math. 147 (1991), 365–374.
[Shi] F. V. Shirokov, *Proof of a conjecture of Kaplansky*, Uspekhi Mat. Nauk 11 (1956), 167–168 (in Russian).
[Sin 1] A. M. Sinclair, *Continuous derivations on Banach algebras*, Proc. Amer. Math. Soc. 20 (1967), 166–170.
[Sin 2] —, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge University Press, 1976.
[S–W] I. M. Singer and J. Wermer, *Derivations on commutative normed algebras*, Math. Ann. 129 (1955), 260–264.
[Tho 1] M. P. Thomas, *The image of a derivation is contained in the radical*, Ann. of Math. 128 (1988), 435–460.
[Tho 2] —, *Primitive ideals and derivations on noncommutative Banach algebras*, preprint, 1991.
[Vuk 1] J. Vukman, *On derivations in prime rings*, Proc. Amer. Math. Soc. 116 (1992), 877–884.
[Vuk 2] —, *A result concerning derivations on Banach algebras*, ibid., 971–976.
[Yoo] B. Yood, *Continuous homomorphisms and derivations on Banach algebras*, in: F. Greenleaf and D. Gulick (eds.), Banach Algebras and Several Complex Variables, Contemp. Math. 32, Amer. Math. Soc., 1984, 279–284.

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On some conjecture concerning Gaussian measures of dilatations of convex symmetric sets

by

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Abstract. The paper deals with the following conjecture: if μ is a centered Gaussian measure on a Banach space F, $\lambda > 1$, $K \subset F$ is a convex, symmetric, closed set, $P \subset F$ is a symmetric strip, i.e. $P = \{x \in F : |x'(x)| \le 1\}$ for some $x' \in F'$, such that $\mu(K) = \mu(P)$ then $\mu(\lambda K) \ge \mu(\lambda P)$.

We prove that the conjecture is true under the additional assumption that K is "sufficiently symmetric" with respect to μ, in particular it is true when K is a ball in a Hilbert space. As an application we give estimates of Gaussian measures of large and small balls in a Hilbert space.

I. Introduction. Let us recall that a measure μ defined on Borel subsets of a separable Banach space F is called *Gaussian* if for each $x' \in F'$ the measure $x'(\mu)$ coincides with the Gaussian measure $N(a,\sigma)$ on $\mathbb{R}^1$ for some a and σ which depend on x' (σ may be 0 as well). If $a = 0$ for each $x' \in F'$ then the measure is called *centered*.

A sequence of independent random variables ξ_i, $i = 1, 2, \ldots$, such that each ξ_i is distributed by the law $N(0,1)$ is called *canonical Gaussian*. In this case the distribution of the random vector $(\xi_1, \ldots, \xi_n)$ will be denoted by γ_n and it will be called the *canonical Gaussian measure* on $\mathbb{R}^n$.

If μ is a centered Gaussian measure on a separable Banach space F then there exists a sequence x_i, $i = 1, 2, \ldots$, in F such that the series $\sum_{i=1}^{\infty} x_i \xi_i$ is a.s. convergent in F and μ is the distribution of its sum; here ξ_i, $i = 1, 2, \ldots$, is a canonical Gaussian sequence. Each such sequence (x_i) will be called a *representing sequence* for μ. For all unexplained facts about Gaussian measures which will be used in this paper we refer to one of the books [5] or [8].

A sequence x_i, $i = 1, 2, \ldots$, in F is said to be a 1-*unconditional basis* for a symmetric convex set $K \subset F$ if for each $x \in K$ there exists a unique sequence α_i, $i = 1, 2, \ldots$, of numbers such that $\sum_{i=1}^{\infty} \alpha_i x_i$ is convergent to

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x and such that for each sequence of signs $\varepsilon_i = \pm 1$, $i = 1, 2, \ldots$, the series $\sum_{i=1}^{\infty} \varepsilon_i \alpha_i x_i$ is convergent to an element in K.

A set $P \subset F$ is said to be a *symmetric strip* if it is of the form $\{x \in F : |x'(x)| \le 1\}$ for some $x' \in F'$.

Moreover, let

$$\Psi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-s^2/2}\, ds \quad \text{for } t \in \mathbb{R}^1,$$

$$\Phi(t) = \sqrt{\frac{2}{\pi}} \int_{0}^{t} e^{-s^2/2}\, ds \quad \text{for } t \in \mathbb{R}^1_+,$$

i.e. Ψ is the distribution function of ξ and Φ the distribution function of $|\xi|$ where ξ is a canonical Gaussian random variable, i.e. distributed by $N(0,1)$. For convenience we extend Ψ and Φ by $\Psi(+\infty) = \Phi(+\infty) = 1$ and $\Psi(-\infty) = 0$.

To the best of our knowledge the following conjecture is open: if μ is a centered Gaussian measure on a separable Banach space F and $K \subset F$ is convex, symmetric and closed then

(1) $\mu(\lambda K) \ge \mu(\lambda P)$ for each $\lambda > 1$ and each symmetric strip P in F such that $\mu(P) = \mu(K)$.

Since for a symmetric strip $P \subset F$ we have $\mu(\lambda P) = \Phi(\lambda \Phi^{-1}(\mu(P)))$ the above conjecture is equivalent to the following: if μ is a Gaussian measure on a separable Banach space F and K is a symmetric, convex, closed subset of F then

(2) $$\mu(\lambda K) \ge \Phi(\lambda \Phi^{-1}(\mu(K))) \quad \text{for each } \lambda > 1.$$

The conjecture has been known since the appearance, in 1969, of a preprint (unpublished) by L. Shepp on the existence of strong exponential moments of a Gaussian measure on a Banach space. However, it seems that in a published paper it was stated for the first time by S. Szarek [7]. We will call it *Conjecture S*.

Let us observe that in the formulation of Conjecture S we may equivalently require that for each $\lambda < 1$ the reverse inequality holds in (1) (resp. in (2)), i.e. $\mu(\lambda K) \le \mu(\lambda P)$ (resp. $\mu(\lambda K) \le \Phi(\lambda \Phi^{-1}(\lambda \mu(K)))$).

More generally, if μ is a measure on F, not necessarily Gaussian, and C is a Borel subset of F, not necessarily convex, then we will say that C, μ *support Conjecture S* if for each $t > 0$ the set $K = tC$ satisfies (1). If C, μ support Conjecture S for each closed, convex, symmetric $C \subset F$ then we will say that μ *supports Conjecture S*.

The main result of this paper is the following:

THEOREM 1. *If μ is a centered Gaussian measure on a separable Banach space F, K is a symmetric, convex, closed subset of F and there exists a sequence of vectors in F which is both a representing sequence for μ and 1-unconditional basis for K then K, μ support Conjecture S.*

In particular, if $F = H$ is a Hilbert space and μ is a centered Gaussian measure on H, then there exists a representing sequence $x_i, i = 1, 2, \ldots$, for μ which is an orthogonal sequence in H. Therefore the sequence x_i, $i = 1, 2, \ldots$, is a 1-unconditional basis in each $B_r \cap H_0$ where $B_r = \{x \in H : \|x\| \le r\}$ and H_0 is the closure of the linear space spanned by the sequence x_i, $i = 1, 2, \ldots$ This and Theorem 1 yield the following.

COROLLARY 1. *If μ is a centered Gaussian measure on a Hilbert space H then*

$$\mu(B_r) \ge \Phi\left(\frac{r}{s} \Phi^{-1}(B_s)\right) \quad \textit{for each } r \ge s > 0.$$

As an application of Corollary 1 we get

THEOREM 2. *If μ is a centered Gaussian measure on a Hilbert space H and $\int_H \|x\|^2 \mu(dx) = s^2$ then*

$$\mu(B_r) \ge \Phi\left(\frac{r}{s}\right) \quad \textit{for } \frac{r}{s} \ge \sqrt{\frac{22}{3}},$$

$$\mu(B_r) \le \Phi\left(\frac{r}{s}\right) \quad \textit{for } \frac{r}{s} \le \sqrt{\frac{4\sqrt{2}-5}{7}}.$$

Remark 1. A much stronger result than the first part of Theorem 1 was proved by N. K. Bakirov [1], by a different method.

Remark 2. If μ is a centered Gaussian measure on a Banach space F and K is a convex, symmetric, closed subset of F then K, μ support Conjecture S if and only if $\frac{1}{r}\Phi^{-1}(\mu(rK))$ is nondecreasing for $r > 0$. This condition is stronger than $\frac{1}{r}\Psi^{-1}(\mu(rK))$ being nondecreasing for $r > 0$. The last fact is known to be true and it is connected with Ehrhard's result (cf. Section III) from which it follows that the function $\Psi^{-1}(\mu(rK))$ is concave on $\mathbb{R}^1_+$. It was proved by T. Byczkowski [2] that $\Phi^{-1}(\mu(rK))$ need not be convex or concave in general.

Remark 3. Estimates from below of Gaussian measures of balls with centers not necessarily at 0 can be obtained by the following

THEOREM 3. *If μ is a centered Gaussian measure on F, $C \subset F$ is a symmetric Borel subset, $x \in F$ and $P = \{y \in F : |x'(y)| \le 1\}$ is a strip orthogonal to x, i.e. $|x'(x)| = \sup\{|y'(x)| : \int_F y'(y)^2 \mu(dy) \le \int_F x'(y)^2 \mu(dy)\}$, such that $\mu(C) = \mu(P)$ then $\mu(C + x) \ge \mu(P + x)$.*

If we put $r = |x'(x)|(\int_F x'(y)^2\,\mu(dy))^{-1/2}$ then the last inequality can be written in the form

$$\mu(C+x) \ge \Psi(r+\Phi^{-1}(\mu(C))) - \Psi(r-\Phi^{-1}(\mu(C)))\,.$$

The proof of Theorem 3 follows easily from the Cameron–Martin Theorem which gives

$$\mu(C+x) = e^{-r^2/2}\int_C e^{sx'(y)}\,\mu(dy) = e^{-r^2/2}\int_C \cosh(sx'(y))\,\mu(dy)$$
$$\ge e^{-r^2/2}\int_P \cosh(sx'(y))\,\mu(dy) = \mu(P+x)\,;$$

here we have set $s = |x'(x)|(\int_F x'(y)^2\,\mu(dy))^{-1}$ and the inequality is true since $\cosh(sx'(y)) \ge \cosh(sx'(z))$ for each $y \in C\setminus P$ and $z \in P$.

Using isoperimetric methods it is possible to prove that Conjecture S is true whenever $\dim F \le 3$ (cf. [6]).

II. Preliminary results. Let us observe that if x_i, $i = 1,2,\ldots$, is a representing sequence for a Gaussian measure μ on a separable Banach space F, and ξ_i, $i = 1,2,\ldots$, is the canonical Gaussian sequence then for each closed, convex, symmetric set $K \subset F$ we have

$$\lim_{n\to\infty} P\Big(\sum_{i=1}^{n} x_i\xi_i \in K\Big) = \mu(K)\,. \tag{3}$$

Indeed, since $S_n = \sum_{i=1}^n x_i\xi_i$ is a.s. convergent to $S = \sum_{i=1}^\infty x_i\xi_i$, it is convergent in law and therefore $\limsup_{n\to\infty} P(S_n \in K) \le P(S \in K) = \mu(K)$, since K is closed. On the other hand, for each n, $P(S_n \in K) \ge P(S \in K)$ by the Anderson Inequality. This proves (3).

Given a convex, closed, symmetric set $K \subset F$ and a representing sequence x_i, $i = 1,2,\ldots$, for μ let $K_n \subset \mathbb{R}^n$ be defined by

$$K_n = \Big\{a = (\alpha_1,\ldots,\alpha_n) \in \mathbb{R}^n : \sum_{i=1}^{n}\alpha_i x_i \in K\Big\}\,.$$

Then K_n is a convex, closed and symmetric subset of $\mathbb{R}^n$ and we have $P(\sum_{i=1}^n x_i\xi_i \in K) = \gamma_n(K_n)$.

By (3) we have $\lim_{n\to\infty}\gamma_n(K_n) = \mu(K)$ and clearly $\lim_{n\to\infty}\gamma_n(\lambda K_n) = \mu(\lambda K)$.

This proves that Conjecture S is true if each γ_n supports it.

Moreover, if (x_i) is a 1-unconditional basis for K, then each K_n is symmetric with respect to each coordinate in $\mathbb{R}^n$, i.e. if $a = (\alpha_1,\ldots,\alpha_n) \in K_n$ then $(\varepsilon_1\alpha_1,\ldots,\varepsilon_n\alpha_n) \in K_n$ for each sequence of signs $\varepsilon_i = \pm 1$, $i = 1,\ldots,n$.

Therefore to prove Theorem 1 it is enough to prove

THEOREM 1'. *If K is a convex closed subset of $\mathbb{R}^n$ which is symmetric with respect to each coordinate then K,γ_n support Conjecture S.*

Let σ_n denote the normalized surface Lebesgue measure on $S_n = \{x \in \mathbb{R}^n : |x| = 1\}$. The following was proved by H. J. Landau and L. A. Shepp [4]:

PROPOSITION 1. *If $K \subset \mathbb{R}^n$ is a Borel set such that K,σ_n support Conjecture S then K,γ_n support Conjecture S.*

Proof. Let $t > 0$, $K' = tK$ and let P be a symmetric strip such that $\gamma_n(P) = \gamma_n(K')$. We have to prove that $\gamma_n(\lambda P) \le \gamma_n(\lambda K')$ for each $\lambda \ge 1$. Since K_n,σ_n support Conjecture S we deduce that there exists $c \in \mathbb{R}^1_+ \cup \{+\infty\}$ such that

$$\sigma_n(sK') \le \sigma_n(sP) \quad \text{for } 0 \le s < c\,,$$
$$\sigma_n(sK') \ge \sigma_n(sP) \quad \text{for } s > c\,.$$

Hence if $p : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ is such that

(4) for each $\lambda > 1$ the function $g(s) = p(s/\lambda)/p(s)$ is nondecreasing on $\mathbb{R}^1_+$,

then

$$\int_0^c p(s/\lambda)(\sigma_n(sK') - \sigma_n(sP))\,ds \ge \frac{p(c/\lambda)}{p(c)}\int_0^c p(s)(\sigma_n(sK') - \sigma_n(sP))\,ds$$

and

$$\int_c^\infty p(s/\lambda)(\sigma_n(sK') - \sigma_n(sP))\,ds \ge \frac{p(c/\lambda)}{p(c)}\int_c^\infty p(s)(\sigma_n(sK') - \sigma_n(sP))\,ds\,.$$

Therefore if

$$\int_0^\infty p(s)(\sigma_n(sK') - \sigma_n(sP))\,ds = 0$$

then

$$\int_0^\infty p(s/\lambda)(\sigma_n(sK') - \sigma_n(sP))\,ds \ge 0\,.$$

This proves Proposition 1, since for each Borel set $A \subset \mathbb{R}^n$ we have

$$\gamma_n(\lambda A) = \frac{1}{\lambda}\int_0^\infty p_n(s/\lambda)\sigma_n(sA)\,ds$$

where $p_n(s) = \frac{2\pi^{n/2}}{\Gamma(n/2)}s^{-n-1}e^{-s^{-2}/2}$ and p_n has the property (4).

Remark 4. It is clear from the above proof that Proposition 1 remains valid if we replace γ_n by any measure on $\mathbb{R}^n$ which has a density with respect to the Lebesgue measure of the form $p(|x|)$ where $p : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ has the property (4).

If μ is a probability measure on $\mathbb{R}^n$ and $C \subset \mathbb{R}^n$ is a Borel set we will denote by m_C the function on $\mathbb{R}^1_+$ given by $m_C(t) = \mu(tC)$ (it will always be clear which measure μ is taken into consideration).

If $f : \mathbb{R}^1_+ \to \mathbb{R}^1_+$ is a function then by definition

$$f'_+(s) = \limsup_{t\to s, t>s} \frac{f(t)-f(s)}{t-s}.$$

A set $C \subset \mathbb{R}^n$ is said to be a *star set* if $\lambda C \subset C$ for each $\lambda \in \mathbb{R}^1_+$, $\lambda < 1$.

LEMMA 1. *Assume that a probability measure μ on $\mathbb{R}^n$ has the following property: for each symmetric strip $P \subset \mathbb{R}^n$ the function $m_P(t)$ is differentiable and $m'_P(t) > 0$ at each point t such that $m_P(t) < 1$. Let $C \subset \mathbb{R}^n$ be a Borel star set. Then C, μ support Conjecture S if for each symmetric strip P and each $t > 0$ the condition $m_C(t) = m_P(t) < 1$ implies that $m'_{C+}(t) \geq m'_P(t)$.*

Proof. If Q is a symmetric strip such that $\mu(sC) = \mu(Q)$ then for the strip $P = \frac{1}{s}Q$ we have $m_C(s) = m_P(s)$ and therefore in order to prove that C, μ support Conjecture S it is enough to prove that for each $t > s > 0$ and each strip P the equality $m_C(s) = m_P(s)$ implies $m_C(t) \geq m_P(t)$. Assume that the implication at the end of Lemma 1 holds true. Then for each $t, \bar{t} \in \mathbb{R}^1_+$ and each symmetric strip P such that $m_C(t) = m_P(\bar{t})$ we have $m_C(t) = m_{\bar{P}}(t)$ where $\bar{P} = \frac{\bar{t}}{t}P$ and hence we obtain

$$m'_{C+}(t) \geq m'_{\bar{P}}(t) = \frac{\bar{t}}{t} m'_P(\bar{t})$$

or equivalently $tm'_{C+}(t) \geq \bar{t}m'_P(\bar{t})$. Therefore if we put $h(t) = m_P^{-1}(m_C(t))$ defined on the interval $\{t \in \mathbb{R}^1_+ : m_C(t) < 1\}$ then h is a nondecreasing function and $tm'_{C+}(t) \geq h(t)m'_P(h(t))$. Also, since $m_C(t) = m_P(h(t))$ we get $m'_{C+}(t) = m'_P(h(t))h'_+(t)$. Consequently, combining this with the previous inequality we obtain

$$th'_+(t) \geq h(t) \quad \text{whenever } m_C(t) < 1. \tag{5}$$

The condition (5) implies that $(\ln h)'_+(t) \geq 1/t$. If $m_C(s) = m_P(s)$ or, which is the same, if $h(s) = s$ then since h is nondecreasing we get $h(t) \geq t$ for $t > s$ whenever $m_C(t) < 1$. But $h(t) \geq t$ means that $m_C(t) \geq m_P(t)$. Thus we have proved that $m_C(s) = m_P(s)$ implies $m_C(t) \geq m_P(t)$ for $t > s > 0$, and the proof is complete.

We will apply the above lemma in three cases. The first is when μ is equal to γ_n. Since for a symmetric strip $P = \{x \in \mathbb{R}^n : |x'(x)| \leq 1\}$ we have $m_P(t) = \Phi(t/\|x'\|)$, the measure γ_n satisfies the assumptions of Lemma 1. Moreover, we have

$$m'_C(t) = t\Big(nm_C(t) - \int_{tC} x^2\,\gamma_n(dx)\Big).$$

Hence by Lemma 1 we obtain

COROLLARY 2. *The measure γ_n supports Conjecture S if for each convex, closed, symmetric $C \subset \mathbb{R}^n$ and each symmetric strip $P \subset \mathbb{R}^n$ with $\gamma_n(C) = \gamma_n(P)$ we have*

$$\int_C x^2\,\gamma_n(dx) \leq \int_P x^2\,\gamma_n(dx).$$

Remark 5. It was observed by T. Żak that if the diameter of C is less than $2\sqrt{n-1}$ then C satisfies the inequality of Corollary 2. This is clear since $\int_C x^2\,\gamma_n(dx) \leq (n-1)\gamma_n(C)$ and $\int_P x^2\,\gamma_n(dx) \geq (n-1)\gamma_n(P)$.

Another case in which we will apply Lemma 1 is when $\mu = \sigma_2$ and $F = \mathbb{R}^2$. In this case

$$m_P(t) = \begin{cases} \dfrac{2}{\pi}\arcsin\dfrac{t}{\|x'\|} & \text{if } t \leq \|x'\|, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$m'_P(t) = \frac{2}{\pi t}\tan\left(\frac{\pi}{2}m_P(t)\right) \quad \text{whenever } m_P(t) < 1.$$

So we get

COROLLARY 3. *Let $C \subset \mathbb{R}^2$ be a Borel star set. Then C, σ_2 support Conjecture S if*

$$tm'_{C+}(t) \geq \frac{2}{\pi}\tan\left(\frac{\pi}{2}m_P(t)\right) \quad \textit{whenever } m_C(t) < 1.$$

Similarly for σ_3 we have

$$m_P(t) = \begin{cases} t/\|x'\| & \text{if } t \leq \|x'\|, \\ 1 & \text{otherwise,} \end{cases}$$

and Lemma 1 yields

COROLLARY 4. *Let $C \subset \mathbb{R}^3$ be a Borel star set. Then C, σ_3 support Conjecture S if*

$$tm'_{C+}(t) \geq m_C(t) \quad \textit{whenever } m_C(t) < 1.$$

III. Proof of Theorem 1. In fact, we will prove Theorem 1′, which is enough to prove Theorem 1, as shown in Section II. The proof of Theorem 1′ is by induction on n and it is based on the result of A. Ehrhard [3] which states that if K_1, K_2 are convex subsets of $\mathbb{R}^n$ and $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1$ then

$$\Psi^{-1}(\gamma_n(\lambda_1 K_1 + \lambda_2 K_2)) \ge \lambda_1 \Psi^{-1}(\gamma_n(K_1)) + \lambda_2 \Psi^{-1}(\gamma_n(K_2)).$$

It follows easily by this result that if K is a convex set in $\mathbb{R}^{n+1}$ and if we define

$$K_x = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : (x_1, \ldots, x_n, x) \in K\} \quad \text{for } x \in \mathbb{R}^1$$

then the function h defined by $h(x) = \Psi^{-1}(\gamma_n(K_x))$ is concave on $\mathbb{R}^1$. Recall that a function $h : \mathbb{R}^+ \to \mathbb{R} \cup \{\pm\infty\}$ is concave if for each $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1$, $x_1, x_2 \in \mathbb{R}^1$ we have $h(\lambda_1 x_1 + \lambda_2 x_2) \ge \lambda_1 h(x_1) + \lambda_2 h(x_2)$ and where $-\infty + \infty = +\infty$.

Now the induction step is proved as follows: if $K \subset \mathbb{R}^{n+1}$ is closed, convex, and symmetric with respect to each coordinate then the same is true for K_x for each $x \in \mathbb{R}^1$ and by the inductive assumption, for each $\lambda \ge 1$, $\gamma_n(\lambda K_{x/\lambda}) \ge \Phi(\lambda(\Phi^{-1}(\gamma_n(K_{x/\lambda}))))$. This gives

$$\begin{aligned}\gamma_{n+1}(\lambda K) &= \int_{\mathbb{R}} \gamma_n((\lambda K)_x)\,\gamma_1(dx) = \int_{\mathbb{R}} \gamma_n(\lambda K_{x/\lambda})\,\gamma_1(dx)\\ &\ge \int_{\mathbb{R}} \Phi(\lambda\Phi^{-1}(\gamma_n(K_{x/\lambda})))\,\gamma_1(dx) = \gamma_2(\lambda B_g)\end{aligned}$$

where $g : \mathbb{R}^1 \to \mathbb{R}^+ \cup \{+\infty\}$ is defined by $g(x) = \Phi^{-1}(\gamma_n(K_x))$, and $B_g = \{(x, y) \in \mathbb{R}^2 : |y| \le g(|x|)\}$. Therefore the proof of the inductive step, and the proof of Theorem 1′ for $n = 2$ as well, will be completed if we show that B_g, γ_2 support Conjecture S. If $\gamma_n(K_x) = 1$ for some $x \in \mathbb{R}^1$ then K is a symmetric strip and there is nothing to prove. So we can assume that g has finite values. The function $\Psi^{-1}\Phi g$ is even on $\mathbb{R}^1$ and concave by the Ehrhard result. Thus in view of Proposition 1 and Corollary 3 it is enough to prove the following two lemmas:

LEMMA 2. *Let $G = \Psi^{-1}\Phi$. Then the function $H(x) = xG'(x)$ is increasing on $\mathbb{R}^1_+ \setminus \{0\}$.*

LEMMA 3. *Let $G : \mathbb{R}^1_+ \to \mathbb{R}^1 \cup \{-\infty\}$ be an increasing function such that G is differentiable on $\mathbb{R}^1_+ \setminus \{0\}$ and $H(x) = xG'(x)$ is increasing on this set. If $g : \mathbb{R}^1 \to \mathbb{R}^1_+$ is an even function such that Gg is concave on $\mathbb{R}^1$ then*

$$tm'_{C+}(t) \ge \frac{2}{\pi} \tan\left(\frac{\pi}{2} m_C(t)\right) \quad \textit{whenever } m_C(t) < 1$$

where m is defined for $\mu = \sigma_2$ and $C = B_g = \{(x, y) \in \mathbb{R}^2 : |y| \le g(|x|)\}$.

Proof of Lemma 3. If G satisfies the assumptions of Lemma 3 then so does the function $\overline{G}(x) = G(x/t)$ for each $t > 0$. Moreover, $tm'_{C+}(t) = m'_{tC+}(1)$ and $tC = B_{\overline{g}}$ where $\overline{g}(x) = tg(x/t)$ for $x \in \mathbb{R}^1$, and clearly $\overline{G}\,\overline{g}$ is concave if so is Gg. Therefore it is enough to prove Lemma 3 for $t = 1$.

The set $S_2 \setminus C$ is at most a countable union of disjoint arcs. First consider the case when one of these arcs, including its endpoints, is contained in $\mathbb{R}^2_{++} = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$. Let I be such an arc and let $(x_1, y_1), (x_2, y_2)$ be the endpoints of I. Then $y_i = g(x_i)$ for $i = 1, 2$ and we can assume that $x_1 < x_2$. Let $\phi_1 = \arctan(x_1/y_1)$ and $\phi_2 = \arctan(y_2/x_2)$. Since $\cot x$ is decreasing for $x \in (0, \pi/2)$ and since C is symmetric with respect to each coordinate we obtain

$$\tan\left(\frac{\pi}{2}\sigma_2(C)\right) = \cot\left(\frac{\pi}{2}\sigma_2(S_2 \setminus C)\right) \le \cot(2\pi\sigma_2(I)) = \tan(\phi_1 + \phi_2). \tag{6}$$

Let P_1 (resp. P_2) be a symmetric strip in $\mathbb{R}^2$ whose boundary is tangent at the point (x_1, y_1) (resp. (x_2, y_2)) to the graph of g restricted to the interval $[x_1, x_2]$. Since Gg is a concave function and since G is differentiable such strips do exist. We compute easily that

$$\frac{\pi}{2}\sigma_2(P_1) = \phi_1 + \psi_1 \quad \text{where } \cot\psi_1 = -g'_+(x_1) \quad \text{and} \tag{7}$$

$$\frac{\pi}{2}\sigma_2(P_2) = \phi_2 + \psi_2 \quad \text{where } \tan\psi_2 = -g'_-(x_2); \tag{8}$$

here

$$g'_-(x_2) = \liminf_{x \to x_2, x < x_2} \frac{g(x) - g(x_2)}{x - x_2}.$$

For $i = 1, 2$ and each neighborhood V_i of the point (x_i, y_i) we obtain, by the tangency of the boundary of P_i and the graph of g,

$$\begin{aligned}\lim_{t \to 1, t > 1} \frac{\sigma_2(tC \cap I \cap V_i)}{t - 1} &= \lim_{t \to 1, t > 1} \frac{\sigma_2(tP_i \cap I \cap V_i)}{t - 1}\\ &= \frac{1}{2\pi}\tan\left(\frac{\pi}{2}\sigma_2(P_i)\right) = \frac{1}{2\pi}\tan(\phi_i + \psi_i).\end{aligned}$$

Hence for V_1, V_2 disjoint we obtain

$$\begin{aligned}m'_{C+}(1) &= \limsup_{t \to 1, t > 1} \frac{\sigma_2(tC) - \sigma_2(C)}{t - 1} \ge \limsup_{t \to 1, t > 1} 4\frac{\sigma_2(tC \cap I)}{t - 1}\\ &= 4\limsup_{t \to 1, t > 1}\left(\frac{\sigma_2(tC \cap I \cap V_1)}{t - 1} + \frac{\sigma_2(tC \cap I \cap V_2)}{t - 1}\right)\\ &= \frac{2}{\pi}(\tan(\phi_1 + \psi_1) + \tan(\phi_2 + \psi_2)).\end{aligned}$$

Hence using (6) we see that to prove $m'_{C+}(1) \geq \frac{2}{\pi}\tan(\frac{\pi}{2}m_C(1))$ it is enough to show that

$$\tan(\phi_1+\psi_1)+\tan(\phi_2+\psi_2) \geq \tan(\phi_1+\phi_2).$$

This will be proved if we show that either $\psi_1 \geq \phi_2$ or $\psi_2 \geq \phi_1$, which is equivalent to: $\tan\psi_1 \geq \tan\phi_2$ or $\tan\psi_2 \geq \tan\phi_1$, which by (7) and (8) is equivalent to:

$$-g'_+(x_1) \leq \frac{x_2}{y_2} \quad \text{or} \quad -g'_-(x_2) \geq \frac{x_1}{y_1}. \tag{9}$$

Since Gg is concave on the interval $[x_1,x_2]$ we have

$$g'_+(x_1)G'(y_1) \geq \frac{G(y_2)-G(y_1)}{x_2-x_1} \geq g'_-(x_2)G'(y_2).$$

Therefore to prove (9) it suffices to show that either

$$\frac{1}{G'(y_1)}\frac{G(y_1)-G(y_2)}{x_2-x_1} \leq \frac{x_2}{y_2}$$

or

$$\frac{1}{G'(y_2)}\frac{G(y_1)-G(y_2)}{x_2-x_1} \geq \frac{x_1}{y_1},$$

which is the same as proving that one of the following inequalities holds:

$$y_2G'(y_2)x_1 \leq \frac{y_1y_2(G(y_1)-G(y_2))}{x_2-x_1} \leq y_1G'(y_1)x_2.$$

This is indeed true because $x_1<x_2$ and $y_1G'(y_1) > y_2G'(y_2)$, since $yG'(y)$ is increasing and $y_1>y_2$.

Now consider the case when $S_2\setminus C$ has no component contained in $\mathbb{R}^2_{++}$. Then if $\sigma_2(C)<1$ we can find $(x_0,y_0)\in\mathbb{R}^2_{++}$ such that either $S_2\setminus C$ contains the arc joining $(-x_0,y_0)$ and (x_0,y_0), except possibly its midpoint $(0,1)$, or $S_2\setminus C$ contains the arc joining (x_0,y_0) and $(x_0,-y_0)$, except possibly its midpoint $(1,0)$.

Let P be the symmetric strip $\{(x,y)\in\mathbb{R}^2 : |y|\leq y_0\}$ in the first case and $\{(x,y)\in\mathbb{R}^2 : |x|\leq x_0\}$ in the second case. Since g is nonincreasing we have in both cases

$$m'_{C+}(1) \geq m'_P(1) = \frac{2}{\pi}\tan\left(\frac{\pi}{2}\sigma_2(P)\right) \geq \frac{2}{\pi}\tan\left(\frac{\pi}{2}\sigma_2(C)\right).$$

This completes the proof of Lemma 3.

Proof of Lemma 2. By the formula for the derivative of the inverse function we have $G'(x) = 2e^{G(x)^2/2-x^2/2}$, which implies that $G''(x) = (G'(x)G(x)-x)G'(x)$ and

$$\begin{aligned} H'(x) &= xG''(x)+G'(x) = G'(x)(G'(x)G(x)-x^2+1)\\ &= G'(x)(2e^{G(x)^2/2-x^2/2}G(x)x-x^2+1). \end{aligned}$$

Since $G'(x)>0$ Lemma 2 will be proved if we show that

$$G(x)e^{G(x)^2/2} > \frac{1}{2}e^{x^2/2}\left(x-\frac{1}{x}\right) \quad \text{for } x>0. \tag{10}$$

Let $f(x)=xe^{x^2/2}$ for $x\in\mathbb{R}^1$, $g(x)=\frac{1}{2}e^{x^2/2}(x-1/x)$ for $x>0$ and $h(x)=f^{-1}(g(x))$ for $x>0$. Since f is strictly increasing on $\mathbb{R}^1$ the inequality (10) is equivalent to $\Phi(x)>\Psi(h(x))$ for $x>0$, i.e. $F(x)=\Phi(x)-\Psi(h(x))>0$ for $x>0$. Since $\lim_{x\to0,x>0}F(x)=\lim_{x\to+\infty}F(x)=0$ it is enough to show that $F'(x)>0$ for $0<x<c$ and $F'(x)<0$ for $x>c$ for some $c\in\mathbb{R}^1_+$. Since

$$F'(x) = \frac{1}{\sqrt{2\pi}}(2e^{-x^2/2}-h'(x)e^{-h(x)^2/2}) \quad \text{and} \quad h'(x) = \frac{\frac{1}{2}e^{x^2/2}\left(x^2+\frac{1}{x^2}\right)}{(1+h(x)^2)e^{h(x)^2/2}}$$

the inequality $F'(x)>0$ holds if and only if

$$(h(x)e^{h(x)^2/2})^2+e^{h(x)^2} > \frac{1}{4}e^{x^2}\left(x^2+\frac{1}{x^2}\right).$$

Since $h(x)e^{h(x)^2/2}=f(h(x))=g(x)$ this gives that $F'(x)>0$ if and only if

$$e^{h(x)^2} > \frac{1}{4}e^{x^2}\left(x^2+\frac{1}{x^2}\right)-g(x)^2 = e^{x^2-\ln 2}.$$

The last inequality is obviously satisfied if $x\in(0,\sqrt{\ln 2})$. For $x>\sqrt{\ln 2}$ it is satisfied if and only if $|g(x)|>f(\sqrt{x^2-\ln 2})$ or equivalently

$$\frac{1}{\sqrt{2}}\left|x-\frac{1}{x}\right| > \sqrt{x^2-\ln 2},$$

which, in turn, is equivalent to

$$x < \sqrt{\ln\frac{2}{e}+\sqrt{\left(\ln\frac{2}{e}\right)^2+1}} =: c.$$

So, finally, we conclude that $F'(x)>0$ if and only if $x<c$, which ends the proof.

IV. Proof of Theorem 2. Let μ be a centered Gaussian measure on a Hilbert space H. Then as explained in the introduction there exists a sequence of nonnegative numbers α_i, $i=1,2,\ldots$, an orthonormal sequence e_i, $i=1,2,\ldots$, and a canonical Gaussian sequence ξ_i, $i=1,2,\ldots$, such that $\sum_{i=1}^\infty\sqrt{\alpha_i}\xi_ie_i$ is a.s. convergent and μ is the distribution of the sum. Hence, if we set $Z=\sum_{i=1}^\infty\alpha_i\xi_i^2$ then $\mu(B_r)=P(Z\leq r^2)$ for each $r\in\mathbb{R}^1_+$. Without loss of generality we may assume that $\int_H\|x\|^2\,\mu(dx)=\sum_{i=1}^\infty\alpha_i=1$. It follows by Theorem 1 that if $\mu(B_c)=\Phi(c)$ for some $c\in\mathbb{R}^1_+$ then

$\mu(B_r) \le \Phi(r)$ for $r < c$ and $\mu(B_r) \ge \Phi(r)$ for $r > c$. This implies that $Ef(Z) > Ef(\xi_1^2)$ whenever $f : \mathbb{R}^1_+ \to \mathbb{R}^1$ is a differentiable function such that the two expectations exist and such that

$$f'(x) > 0 \quad \text{for } x < c^2 \quad \text{and} \quad f'(x) < 0 \quad \text{for } x > c^2 . \tag{11}$$

Indeed,

$$Ef(Z) - Ef(\xi_1^2) = \int_0^\infty f'(t)(P(Z > t) - P(\xi_1^2 > t))\,dt$$
$$= 2\int_0^\infty t f'(t^2)(\Phi(t) - \mu(B_t))\,dt > 0$$

because the last integrand is nonnegative on $\mathbb{R}^1_+$ and if the integral is 0 then $\mu(B_r) = \Phi(r)$ for all $r \in \mathbb{R}^1_+$ and the theorem is true. Hence, Theorem 2 will be proved if we show that for each $c^2 \in (0, (4\sqrt{2}-5)/7] \cup [22/3, \infty)$ we can find a differentiable function $f : \mathbb{R}^1_+ \to \mathbb{R}^1$ such that (11) is satisfied and such that

$$Ef(Z) \le Ef(\xi_1^2) \tag{12}$$

for each sequence of nonnegative numbers α_i, $i = 1, 2, \ldots$, with $\sum_{i=1}^\infty \alpha_i = 1$.

First consider the case of $c^2 \in [22/3, \infty)$. Let f be defined by $f(t) = c^2t^2/2 - t^3/3$. Then f satisfies (11) and for each sequence of nonnegative numbers α_i, $i = 1, 2, \ldots$, with $\sum_{i=1}^\infty \alpha_i = 1$ easy computations yield

$$Ef(\xi_1^2) - Ef(Z) = c^2\Big(1 - \sum_{i=1}^\infty \alpha_i^2\Big) + 2\sum_{i=1}^\infty \alpha_i^2 + \frac{8}{3}\sum_{i=1}^\infty \alpha_i^3 - \frac{14}{3}$$
$$\ge -\frac{16}{3}\sum_{i=1}^\infty \alpha_i^2 + \frac{8}{3}\Big(\sum_{i=1}^\infty \alpha_i^2\Big)^2 + \frac{8}{3}$$
$$= \frac{8}{3}\Big(\sum_{i=1}^\infty \alpha_i^2 - 1\Big)^2 \ge 0 .$$

The first inequality above is a consequence of $c^2 > 22/3$ and of the inequality $\sum_{i=1}^\infty \alpha_i^3 \ge (\sum_{i=1}^\infty \alpha_i^2)^2$. This ends the proof of the first case.

The case of $c^2 < (4\sqrt{2}-5)/7$ is more complicated. Let f be of the form $f(t) = Ae^{-at} - Be^{-bt}$ where A, B, a, b are positive numbers such that $b > a$ and

$$c^2 = \frac{1}{b-a}\ln\frac{Bb}{Aa} .$$

We check easily that f satisfies (11). We will prove that there are A, B, a, b as above such that the condition (12) is satisfied, which will complete the proof. Simple computations give

$$Ef(Z) = A\prod_{i=1}^\infty (1 + 2a\alpha_i)^{-1/2} - B\prod_{i=1}^\infty (1 + 2b\alpha_i)^{-1/2} .$$

For fixed n let $G_n : \mathbb{R}^n_+ \to \mathbb{R}^n$ be the function given by

$$G_n(x_1, \ldots, x_n) = A\prod_{i=1}^n (1 + 2ax_i)^{-1/2} - B\prod_{i=1}^n (1 + 2bx_i)^{-1/2} .$$

Let $(y_1, \ldots, y_n) \in \mathbb{R}^n_+$ be a point at which G_n attains its maximum on the compact set $\{(x_1, \ldots, x_n) \in \mathbb{R}^n_+ : x_1 + \ldots + x_n = 1\}$. Hence, if for $i, j \le n$ we define g by

$$g(t) = G_n(y_1, \ldots, y_i - t, \ldots, y_j + t, \ldots, y_n)$$

then $g'(0) = 0$ if $y_i > y_j > 0$, $g'(0) \ge 0$ if $y_i > y_j = 0$ and

$$g'(0) = \frac{2Aa^2(y_j - y_i)}{(1 + 2ay_i)(1 + 2ay_j)}\prod_{k=1}^n (1 + 2ay_k)^{-1/2}$$
$$- \frac{2Bb^2(y_j - y_i)}{(1 + 2by_i)(1 + 2by_j)}\prod_{k=1}^n (1 + 2by_k)^{-1/2} .$$

It follows that if $y_i > y_j > 0$ (resp. $y_i > y_j = 0$) then

$$\frac{Aa^2}{Bb^2}\prod_{k=1}^n ((1 + 2by_k)/(1 + 2ay_k))^{1/2} = \ (\text{resp. } \ge)\ \frac{1 + 2ay_i}{1 + 2by_i}\,\frac{1 + 2ay_j}{1 + 2by_j} .$$

Since the left side of the above does not depend on i, j and since we have $(1 + 2ay_i)/(1 + 2by_i) < 1$ for $y_i > 0$ we deduce easily that there are at most two different values in the sequence $y_1, \ldots, y_n$. Hence, letting n tend to infinity and in view of the fact that $\lim_{k\to\infty}(1 + 2ay/k)^{-k/2} = e^{-ay}$ where the convergence is uniform for $y \in [0, 1]$ we derive that f satisfies (12) if and only if

$$Ae^{a(x-1)}\Big(1 + \frac{2ax}{k}\Big)^{-k/2} - Be^{b(x-1)}\Big(1 + \frac{2bx}{k}\Big)^{-k/2} \le Ef(\xi_1^2) \tag{13}$$

for each $x \in [0, 1]$ and each $k \ge 1$. Denote the left side of the above inequality by $H(x, k)$. Then the right side is equal to $H(1, 1)$ and (13) reads $H(x, k) \le H(1, 1)$ for $x \in [0, 1]$ and $k \ge 1$. Assume that $H : [0, 1] \times [1, \infty) \to \mathbb{R}^1$ attains its maximum at a point $(y, l) \in (1, 0] \times (1, \infty)$. Then

$$\frac{\partial H}{\partial x}(y, l) = \frac{2Aa^2y}{l} e^{a(y-1)} \left(1 + \frac{2ay}{l}\right)^{-l/2-1} - \frac{2Bb^2y}{l} e^{b(y-1)} \left(1 + \frac{2by}{l}\right)^{-l/2-1} \geq 0,$$

$$\begin{aligned}\frac{\partial H}{\partial k}(y, l) = {} & \frac{A}{2} e^{a(y-1)} \left(1 + \frac{2ay}{l}\right)^{-l/2-1} \\ & \times \left(\frac{2ay}{l} - \left(1 + \frac{2ay}{l}\right) \ln \left(1 + \frac{2ay}{l}\right)\right) \\ & - \frac{B}{2} e^{b(y-1)} \left(1 + \frac{2by}{l}\right)^{-l/2-1} \\ & \times \left(\frac{2by}{l} - \left(1 + \frac{2by}{l}\right) \ln \left(1 + \frac{2by}{l}\right)\right) = 0.\end{aligned}$$

Combining the above equality and inequality we obtain $g(u) \leq g(v)$ where $u = 2ay/l$, $v = 2by/l$ and $g(x) = x^{-2}((1 + x) \ln(1 + x) - x)$. But this is a contradiction since $u < v$ and it is easy to check that g is decreasing on $\mathbb{R}^1_+$. Also, since $\lim_{k \to \infty} H(x, k) = H(0, 1)$ and the convergence is uniform for $x \in [0, 1]$, we see that $H(x, k) \leq H(1, 1)$ for all $(x, k) \in [0, 1] \times [0, \infty)$ if $H(x, 1) \leq H(1, 1)$ for all $x \in [0, 1]$. Write $h(x)$ for $H(x, 1)$. So, (13) is equivalent to $h(x) \leq h(1)$ for $x \in [0, 1]$. Simple computations show that $h'(x) > 0$ if and only if

$$\frac{Aa^2}{Bb^2} e^{b-a} \geq e^{(b-a)x} \left(\frac{1 + 2ax}{1 + 2bx}\right)^{3/2}$$

and that the right side, say $q(x)$, is decreasing on $[0, d]$ and increasing on $[d, \infty)$ for some $d \in \mathbb{R}^1_+$. Hence, if

$$\frac{Aa^2}{Bb^2} e^{b-a} \geq q(1)$$

then $h'(x)$ is negative on $[0, \overline{d}]$ and positive on $[\overline{d}, 1]$ for some $\overline{d} \in \mathbb{R}^1$, which implies that $h(x) \leq h(1)$ for $x \in [0, 1]$ if $h(0) \leq h(1)$. Thus (13) is satisfied if

$$\frac{Aa^2}{Bb^2} e^{b-a} \geq q(1) \quad \text{and} \quad h(0) \leq h(1).$$

The last two conditions can be written in the form

$$\frac{1}{b-a} \ln \frac{Bb}{Aa} \leq \frac{g_1(b) - g_1(a)}{b - a}$$

and

$$\frac{1}{b-a} \ln \frac{Bb}{Aa} \leq \frac{g_2(b) - g_2(a)}{b - a}$$

where $g_1(x) = \frac{3}{2} \ln(1 + 2x) - \ln x$ and $g_2(x) = \ln x - \ln((1 + 2x)^{-1/2} - e^{-x})$. Since

$$c^2 = \frac{1}{b-a} \ln \frac{Bb}{Aa}$$

it is clear that if we put $r = \max_{s \in \mathbb{R}_+} \min\{g_1'(s), g_2'(s)\}$ then for each $c^2 < r$ we can find A, B, a, b with the required properties. If we put $s = (2 + \sqrt{2})/2$ then $g_1'(s), g_2'(s) \geq (4\sqrt{2} - 5)/7$ so that $r \geq (4\sqrt{2} - 5)/7$, which concludes the proof.

Remark 6. We conjecture that the second inequality of Theorem 2 holds true for all $r/s \leq 1$.

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References

[1] N. K. Bakirov, *Extremal distributions of quadratic forms of gaussian variables*, Teor. Veroyatnost. i Primenen. 34 (1989), 241–250 (in Russian).

[2] T. Byczkowski, *Remarks on Gaussian isoperimetry*, preprint, Wrocław University of Technology, 1991.

[3] A. Ehrhard, *Symétrisation dans l'espace de Gauss*, Math. Scand. 53 (1983), 281–381.

[4] H. J. Landau and L. A. Shepp, *On the supremum of a Gaussian process*, Sankhyā Ser. A 32 (1970), 369–378.

[5] M. Ledoux and M. Talagrand, *Probability in Banach Spaces*, Springer, 1991.

[6] S. Kwapień and J. Sawa, *On minimal volume of the convex hull of a set with fixed area on the sphere*, preprint, Warsaw University, to appear.

[7] S. J. Szarek, *Condition numbers of random matrices*, J. Complexity 7 (1991), 131–149.

[8] N. N. Vakhania, V. I. Tarieladze and S. A. Chobanyan, *Probability Distributions on Banach Spaces*, Reidel, Dordrecht 1987.

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