[3] M. Hernández and C. Houdré, *Disjointness results for some classes of stable processes*, this issue, 235–252.
[4] G. Maruyama, *Infinitely divisible processes*, Probab. Theory Appl. 15 (1970), 3–23.
[5] K. Urbanik, *Some prediction problems for strictly stationary processes*, in: Proc. 5th Berkeley Sympos. Math. Statist. Probab., Vol. 2, Part I, Univ. California Press, 1967, 235–258.

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Weak invertibility and strong spectrum

by

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Abstract. A notion of weak invertibility in a unital associative algebra $\mathcal{A}$ and a corresponding notion of strong spectrum of an element of $\mathcal{A}$ is defined. It is shown that many relationships between the Jacobson radical, the group of invertibles and the spectrum have analogues relating the strong radical, the set of weakly invertible elements and the strong spectrum. The nonunital case is also discussed. A characterization is given of all (submultiplicative) norms on $\mathcal{A}$ in which every modular maximal ideal $M \subseteq \mathcal{A}$ is closed.

1. Introduction. Let $\mathcal{A}$ be a unital associative algebra over the field of complex numbers and let G, $S = \mathcal{A} \setminus G$, and $\mathrm{Rad}(\mathcal{A})$ denote the group of invertibles, the set of singular elements of $\mathcal{A}$ and the Jacobson radical of $\mathcal{A}$ respectively. For an element $a \in \mathcal{A}$, let $\mathrm{Sp}(a)$ denote the spectrum of a in $\mathcal{A}$, that is, the set of scalars λ such that $\lambda - a \in S$ and let $\varrho(a) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}(a)\}$ be the spectral radius of a in the algebra $\mathcal{A}$.

For a subset $F \subseteq \mathcal{A}$, let $P(F)$ denote the perturbation class of F in $\mathcal{A}$, that is, the set of all elements $a \in \mathcal{A}$ such that $a + F \subseteq F$.

In reasonable algebras (such as for example all Banach algebras) the Jacobson radical admits several characterizations in terms of invertibility and spectrum [3, Theorem 2.5] and [4]:

(a) $\mathrm{Rad}(\mathcal{A})$ is the perturbation class of the group G of invertibles.
(b) $\mathrm{Rad}(\mathcal{A}) = \{r \in \mathcal{A} : \mathrm{Sp}(a + r) = \mathrm{Sp}(a)$, for all $a \in \mathcal{A}\}$.
(c) $\mathrm{Rad}(\mathcal{A})$ is the largest ideal in $\mathcal{A}$ on which the spectral radius is identically zero.

If the algebra $\mathcal{A}$ carries a submultiplicative norm, then all primitive ideals in $\mathcal{A}$, and hence the Jacobson radical of $\mathcal{A}$, are closed, whenever the group G of invertibles of $\mathcal{A}$ is open in this norm. We will henceforth assume all norms under consideration to be submultiplicative. Following [4] we call a norm on $\mathcal{A}$ *spectral* if the group of invertibles of $\mathcal{A}$ is open in the corresponding topology. The term *Q-norm* is also employed by several

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authors. Every complete norm is spectral and the spectral property is often a sufficient substitute for completeness of a norm. See [4].

Recall that the strong radical of $\mathcal{A}$ is the intersection of all maximal two-sided ideals in $\mathcal{A}$. It is the purpose of this paper to show that it is possible to define a notion of weak invertibility and according notions of strong spectrum and strong spectral radius in such a manner that many of the theorems concerning the Jacobson radical, primitive ideals, invertibles, spectrum, spectral radius and spectral norms remain true if these notions are replaced by the strong radical, maximal ideals, weakly invertible elements, strong spectrum, strong spectral radius and weakly spectral norms respectively.

For an element $a \in \mathcal{A}$, let $(a)_L$, $(a)_R$, (a) denote the left ideal, the right ideal and the two-sided ideal generated by a in $\mathcal{A}$ respectively. In particular,

$$(a) = \Big\{ \sum_{j=1}^{n} b_j a c_j : n \geq 1,\ b_1, c_1, \ldots, b_n, c_n \in \mathcal{A} \Big\}.$$

Clearly the element a is invertible in $\mathcal{A}$ if and only if $(a)_L = (a)_R = \mathcal{A}$. Let us now call a *weakly invertible* if the two-sided ideal generated by a in $\mathcal{A}$ is all of $\mathcal{A}$, and call a *strongly singular* otherwise. Let G_w and S_s denote the sets of weakly invertible and strongly singular elements of $\mathcal{A}$ respectively.

For an element $a \in \mathcal{A}$ define the *strong spectrum* $\mathrm{Sp}_s(a)$ and the *strong spectral radius* $\varrho_s(a)$ of a in $\mathcal{A}$ as

$$\mathrm{Sp}_s(a) = \{\lambda \in \mathbb{C} : \lambda - a \in S_s\} \quad \text{and}$$
$$\varrho_s(a) = \sup_{\lambda \in \mathrm{Sp}_s(a)} |\lambda| \quad (= -\infty \text{ if } \mathrm{Sp}_s(a) = \emptyset).$$

Let us call a norm on $\mathcal{A}$ *weakly spectral* if the set G_w of weakly invertible elements is open in the corresponding topology. The term "ideal" shall mean "two-sided ideal". Our first result is the analogue of the above characterization of the Jacobson radical.

THEOREM 1. *Let $\mathcal{A}$ be an associative algebra with identity over the field of complex numbers and R the strong radical of $\mathcal{A}$. Then*

(1) *R is contained in the perturbation class $P(G_w)$.*

(2) *R is the largest ideal in $\mathcal{A}$ on which the strong spectral radius is identically zero.*

With an additional assumption on $\mathcal{A}$ these results can be sharpened:

THEOREM 2. *Let $\mathcal{A}$, R be as in Theorem 1 and $r \in \mathcal{A}$. Suppose that every element in $\mathcal{A}$ can be written as a sum of invertibles. Then*

(1) *R is the perturbation class $P(G_w)$.*

(2) *We have $r \in R$ if and only if $\mathrm{Sp}_s(a + r) = \mathrm{Sp}_s(a)$, for all $a \in \mathcal{A}$.*

Concerning the notion of weakly spectral norm we have

THEOREM 3. *Suppose that $\mathcal{A}$ is a normed algebra with identity. Then the following are equivalent:*

(1) *The norm on $\mathcal{A}$ is weakly spectral.*

(2) *Every maximal ideal $M \subseteq \mathcal{A}$ is closed.*

(3) *We have $\varrho_s(a) \leq \|a\|$, for all $a \in \mathcal{A}$.*

(4) *$\|1 - a\| < 1$ implies $a \in G_w$, for all $a \in \mathcal{A}$.*

(5) *The identity element is in the interior of G_w.*

COROLLARY 1. *Let $\mathcal{A}$ be as in Theorem 3 and suppose that the norm on $\mathcal{A}$ is weakly spectral. Then the strong radical is closed in $\mathcal{A}$.* ■

COROLLARY 2. *Every spectral norm on $\mathcal{A}$ is weakly spectral.*

Proof. Consider any norm on $\mathcal{A}$ and endow $\mathcal{A}$ with the corresponding topology. If the norm is spectral, $1 \in \mathrm{int}(G)$. Since $G \subseteq G_w$ it follows that $1 \in \mathrm{int}(G_w)$ and thus, by Theorem 3, the norm is weakly spectral. ■

The following proposition gives a relation to completeness. It is the analogue of [4, 4.11].

PROPOSITION 1. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{B}$ the completion of $\mathcal{A}$. Then the norm on $\mathcal{A}$ is weakly spectral if and only if $G_w(\mathcal{A}) = G_w(\mathcal{B}) \cap \mathcal{A}$ (that is, if every element of $\mathcal{A}$ which is weakly invertible in $\mathcal{B}$ is weakly invertible in $\mathcal{A}$).*

Let us now investigate properties of the strong spectrum and the strong spectral radius. Denote by $\mathcal{M}(\mathcal{A})$ the space of maximal ideals in $\mathcal{A}$ and for each element $a \in \mathcal{A}$ define $\widehat{a} \in \prod_{M \in \mathcal{M}(\mathcal{A})} \mathcal{A}/M$ to be the map

$$\widehat{a} : \mathcal{M}(\mathcal{A}) \ni M \to a + M \in \mathcal{A}/M.$$

If $\mathcal{A}$ is a commutative algebra carrying a spectral norm, then [4] all the quotients $\mathcal{A}/M$ coincide with the scalars and $\widehat{a}$ is the usual Gelfand transform of the element a. It is well known that in this case $\mathrm{Sp}(a) = \mathrm{range}(\widehat{a})$, for each element $a \in \mathcal{A}$. In the general case the scalars are contained in each quotient $\mathcal{A}/M$, $M \in \mathcal{M}(\mathcal{A})$, and we have

THEOREM 4. *Let $\mathcal{A}$ be as in Theorem 1 and $a \in \mathcal{A}$. Then*

(1) *If a is in the center of $\mathcal{A}$, then a is weakly invertible if and only if it is invertible. Consequently, $\mathrm{Sp}_s(a) = \mathrm{Sp}(a)$ and $\varrho_s(a) = \varrho(a)$.*

(2) *We have $\mathrm{Sp}_s(a) = \mathrm{range}(\widehat{a}) \cap$ scalars.*

(3) *The strong spectrum $\mathrm{Sp}_s(a)$ is a subset of $\mathrm{Sp}(a)$. If $\mathcal{A}$ carries a weakly spectral norm $\|\ \|$, then the strong spectrum $\mathrm{Sp}_s(a)$ is compact and we have $\varrho_s(a) \leq \|a\|$.*

(4) *If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital homomorphism, then we have $\mathrm{Sp}_s(\phi(a)) \subseteq \mathrm{Sp}_s(a)$.*

(5) *If f is a rational function of one complex variable with all its poles contained in the complement of the spectrum of a, then $f(\mathrm{Sp}_s(a)) = \{f(\lambda) : \lambda \in \mathrm{Sp}_s(a)\} \subseteq \mathrm{Sp}_s(f(a))$.*

(6) *If $\mathcal{A}$ is a Banach algebra, then* (5) *holds for all functions f which are holomorphic on a neighborhood of the spectrum of a.*

(7) $\mathrm{Sp}_s(a) = \bigcup_M \mathrm{Sp}_s(Q_M(a))$, *where the union is taken over all maximal ideals $M \subsetneq \mathcal{A}$ and for each such M, $Q_M : \mathcal{A} \to \mathcal{A}/M$ denotes the quotient map.*

(8) *If $\mathcal{A}$ contains at most n distinct maximal ideals, then $\mathrm{Sp}_s(a)$ contains at most n points, for each element $a \in \mathcal{A}$. If $\mathcal{A}$ contains at least n distinct maximal ideals, then for all scalars $\lambda_1, \ldots, \lambda_n$ there exists an element $a \in \mathcal{A}$ such that $\mathrm{Sp}_s(a) = \{\lambda_1, \ldots, \lambda_n\}$.*

Remark. Not all properties of the spectrum carry over to the strong spectrum: The strong spectrum can be empty (see Corollary 4 below) and equality does not hold in general in the Spectral Mapping Theorem (5). For the spectrum it is well known that $\mathrm{Sp}(ab) \setminus \{0\} = \mathrm{Sp}(ba) \setminus \{0\}$. The strong spectrum does not have this property. In reasonable algebras [4] the spectral radius is subadditive on commuting elements. The strong spectral radius fails to be subadditive even on commuting selfadjoint elements of the algebra $\mathcal{B}(H)$ of all bounded operators on separable Hilbert space. Examples are given below.

However, the well-known upper semicontinuity of the spectrum and the spectral radius in an algebra carrying a spectral norm do hold for the strong spectrum. Indeed, as has been observed in [6], they hold in a much more general setting:

THEOREM 5. *Let $\mathcal{A}$ be as in Theorem 1 and assume that $\mathcal{A}$ carries a weakly spectral norm. Then the set E of all elements $a \in \mathcal{A}$ such that $\mathrm{Sp}_s(a) \neq \emptyset$ is closed in $\mathcal{A}$. The set-valued function $\mathcal{A} \ni a \to \mathrm{Sp}_s(a)$ is upper semicontinuous on E in the following sense: If U is an open subset of the plane, then the set $\{a \in E : \mathrm{Sp}_s(a) \subseteq U\}$ is open in E. Consequently, the strong spectral radius $\mathcal{A} \ni a \to \varrho_s(a)$ is also upper semicontinuous on E.*

Proof. See [6, Proposition 2]. ∎

Let now $\mathcal{A}$, $\mathcal{B}$ be normed algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map. Then [10] the subspace

$$S(\phi) = \{b \in \mathcal{B} : \inf_{a\in\mathcal{A}} (\|a\| + \|b - \phi(a)\|) = 0\}$$

of $\mathcal{B}$ is called the *separating space* of ϕ. If $\mathcal{A}$ and $\mathcal{B}$ are complete, then continuity of ϕ is equivalent to $S(\phi) = \{0\}$. If ϕ is a homomorphism with dense range, then $S(\phi) \subseteq \mathcal{B}$ is an ideal. With this notation one of the great unsolved problems of automatic continuity theory can be stated as follows:

(S) If $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ a homomorphism with dense range, does $b \in S(\phi)$ imply that $\varrho(b) = 0$?

As a consequence ϕ would be automatically continuous if $\mathcal{B}$ is semisimple. If the spectral radius is replaced by the strong spectral radius, then (S) is true. This is merely a slight generalization of [10, Theorem 6.18]. The result and its proof are included for the sake of completeness and the convenience of the reader.

THEOREM 6. *Let $\mathcal{A}$, $\mathcal{B}$ be unital normed algebras, R the strong radical of $\mathcal{B}$ and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map which satisfies $\phi(xy) - \phi(x)\phi(y) \in R$, for all $x, y \in \mathcal{A}$. Assume that the norms on $\mathcal{A}$ and $\mathcal{B}$ are spectral. Then $b \in S(\phi)$ implies $b \in R$ and hence $\varrho_s(b) = 0$.*

Remark. We do not know if this result remains true under the weaker assumption that the norms on $\mathcal{A}$ and $\mathcal{B}$ are weakly spectral. The proof of Theorem 6 uses Lemma 3, which relies on the fact that the spectral radius is subadditive on commuting elements. Here it is not possible to replace the spectral radius by the strong spectral radius, since the latter does not have this property.

Let us turn to the proofs of these results before we discuss algebras without identity.

2. Proofs. For the proofs of Theorems 1–6 assume that $\mathcal{A}$, $\mathcal{B}$ are associative algebras with identity. A scalar λ will be identified with the element $\lambda \cdot 1 \in \mathcal{A}$. If $M \subsetneq \mathcal{A}$ is an ideal, then $Q_M : \mathcal{A} \to \mathcal{A}/M$ denotes the quotient map. Note that an element $a \in \mathcal{A}$ is weakly invertible if and only if there exist elements $b_1, c_1, \ldots, b_n, c_n \in \mathcal{A}$ such that $\sum_{j=1}^n b_j a c_j = 1$.

LEMMA 1. *$G \subset G_w$ and G_w is invariant under multiplication from the left and from the right by invertibles.*

Proof. It is clear that $G \subsetneq G_w$. Let $g \in G$ be arbitrary. Suppose that $a \in G_w$. Choose elements $b_1, c_1, \ldots, b_n, c_n \in \mathcal{A}$ such that $\sum_{j=1}^n b_j a c_j = 1$. Then $1 = \sum_{j=1}^n (b_j g^{-1}) g a c_j = \sum_{j=1}^n b_j a g (g^{-1} c_j)$. This shows that $ga, ag \in G_w$. ∎

COROLLARY 3. *If every element of $\mathcal{A}$ is a sum of invertibles, then the permutation class $P(G_w)$ is a proper ideal in $\mathcal{A}$.*

Proof. According to [3, Lemma 2.3] or [2, 5.5.5, p. 96], Lemma 1 implies that $P(G_w)$ is an ideal in $\mathcal{A}$. The above references state their result for Banach algebras. However, the existence of a complete norm on $\mathcal{A}$ is used merely to imply that every element of $\mathcal{A}$ is a sum of invertibles. It is easily checked that $1 \notin P(G_w)$. ∎

LEMMA 2. *Let $a \in \mathcal{A}$ and λ be a scalar. Then the following are equivalent*:

(a) $\lambda \in \mathrm{Sp}_s(a)$.
(b) *There exists a maximal ideal $M \subseteq \mathcal{A}$ such that $Q_M(a) = \lambda$.*

Proof. Note that an element $a \in \mathcal{A}$ is strongly singular if and only if it belongs to some proper ideal in $\mathcal{A}$, which is true if and only if it belongs to some maximal ideal $M \subseteq \mathcal{A}$. Consequently,

$$\begin{aligned}\lambda \in \mathrm{Sp}_s(a) &\Leftrightarrow \lambda - a \in S_s \\ &\Leftrightarrow \lambda - a \in M, \text{ for some maximal ideal } M \subseteq \mathcal{A} \\ &\Leftrightarrow Q_M(a) = \lambda, \text{ for some maximal ideal } M \subseteq \mathcal{A}. \ \blacksquare\end{aligned}$$

COROLLARY 4. *Suppose that the algebra $\mathcal{A}$ is simple. Then for each element $a \in \mathcal{A}$, the strong spectrum $\mathrm{Sp}_s(a)$ is empty if a is not a scalar, and $\mathrm{Sp}_s(a) = \{\lambda\}$ if a is the scalar λ.*

Proof. This follows from Lemma 2, since $M = \{0\}$ is the only maximal ideal in $\mathcal{A}$. ■

The following lemma is well known. It is included for the convenience of the reader.

LEMMA 3. *Let $\mathcal{A}$, $\mathcal{B}$ be normed algebras, $1 \in \mathcal{B}$ and $\psi : \mathcal{A} \to \mathcal{B}$ a homomorphism with dense range. If the norms on $\mathcal{A}$ and $\mathcal{B}$ are spectral, then $1 \notin S(\psi)$.*

Proof (similar to [8, Theorem 2.5.2]). The spectral radius is subadditive on commuting elements of an algebra which admits a spectral norm [4]. Let $a \in \mathcal{A}$. Since the elements $\psi(a)$, $1-\psi(a)$ commute in $\mathcal{B}$, the homomorphism ψ contracts the spectral radius and the norms on $\mathcal{A}$, $\mathcal{B}$ are spectral, we have

$$1 = \varrho(\psi(a) + 1 - \psi(a)) \le \varrho(a) + \varrho(1 - \psi(a)) \le \|a\| + \|1 - \psi(a)\|. \ \blacksquare$$

LEMMA 4. *Let $a, b_1, c_1, \ldots, b_n, c_n \in \mathcal{A}$. Then $\sum_{j=1}^{n} b_j a c_j \in G_w$ implies $a \in G_w$.*

Proof. Suppose that $f = \sum_j b_j a c_j \in G_w$ and choose (finitely many) elements $u_i, v_i \in \mathcal{A}$ such that $\sum_i u_i f v_i = 1$. Then $1 = \sum_i u_i f v_i = \sum_{i,j} u_i b_j a c_j v_i$, and so $a \in G_w$. ■

Proof of Theorem 1. (1) Suppose that $r \in R \setminus P(G_w)$. Then $r + G_w \not\subseteq G_w$ and we can choose an element $a \in G_w$ such that $r + a \notin G_w$, that is, $r + a \in S_s$. Consequently, there exists a maximal ideal $M \subseteq \mathcal{A}$ such that $r + a \in M$. Since r also belongs to M we conclude that $a \in M$. But this contradicts $a \in G_w$.

(2) If $r \in R$, then $Q_M(r) = 0$, for all maximal ideals $M \subseteq \mathcal{A}$, and consequently $\mathrm{Sp}_s(r) = \{0\}$, thus $\varrho_s(a) = 0$, by Lemma 2. Conversely, suppose that $J \subseteq \mathcal{A}$ is an ideal such that $\varrho_s(a) = 0$, for all $a \in J$. We wish to show that $J \subseteq R$. Assume the contrary. Then there exists a maximal ideal $M \subseteq \mathcal{A}$ such that $J \not\subseteq M$. By maximality of M we have $M + J = \mathcal{A}$. Choose $m \in M$ and $a \in J$ such that $m + a = 1$. Then $a = 1 - m$ and consequently $Q_M(a) = 1$. This shows $1 \in \mathrm{Sp}_s(a)$ and contradicts $\varrho_s(a) = 0$. ■

Proof of Theorem 2. (1) According to (1) of Theorem 1 we must only show that $P(G_w) \subseteq R$. Let $M \subseteq \mathcal{A}$ be a maximal ideal. Suppose $P(G_w) \not\subseteq M$. Since every element of $\mathcal{A}$ is a sum of invertibles, $P(G_w)$ is an ideal in $\mathcal{A}$ by Corollary 3. Thus the maximality of M implies that $P(G_w) + M = \mathcal{A}$. In particular, there exist elements $b \in P(G_w)$ and $m \in M$ such that $-b + m = 1$. Since $1 \in G_w$ it follows that $m = 1 + b \in G_w$. But this contradicts the fact that m belongs to the proper ideal M in $\mathcal{A}$. Thus we must have $P(G_w) \subseteq M$, for each maximal ideal $M \subseteq \mathcal{A}$, and consequently $P(G_w) \subseteq R$.

(2) Note that $a \in G_w$ if and only if $0 \notin \mathrm{Sp}_s(a)$. Hence $\mathrm{Sp}_s(a+r) = \mathrm{Sp}_s(a)$, for all $a \in \mathcal{A}$, implies $a + r \in G_w \Leftrightarrow a \in G_w$, for all $a \in \mathcal{A}$, and in particular therefore $r + G_w \subseteq G_w$, that is, $r \in P(G_w) = R$. Conversely, if $r \in R = P(G_w)$, $a \in \mathcal{A}$ and λ is a scalar, then

$$\lambda \notin \mathrm{Sp}_s(a) \Rightarrow \lambda - a \in G_w \Rightarrow \lambda - (a + r) \in G_w \Rightarrow \lambda \notin \mathrm{Sp}_s(a + r).$$

This shows $\mathrm{Sp}_s(a + r) \subseteq \mathrm{Sp}_s(a)$. To obtain the reverse inclusion replace a with $a - r$ and subsequently r with $-r$. ■

Proof of Theorem 3. (1)⇒(2). Assume that the norm on $\mathcal{A}$ is weakly spectral and let $M \subseteq \mathcal{A}$ be a maximal ideal. Then M is either closed or dense in $\mathcal{A}$. Since G_w is open, we can choose $\varepsilon > 0$ such that $\|1 - g\| < \varepsilon$ implies $g \in G_w$, for all $g \in \mathcal{A}$. Then $m \in M$ implies $m \in S_s$ and so $\|1 - m\| \ge \varepsilon$. This shows that M is not dense.

(2)⇒(3). Let $a \in \mathcal{A}$ and suppose that $\lambda \in \mathrm{Sp}_s(a)$. Then there exists a maximal ideal $M \subseteq \mathcal{A}$ such that $Q_M(a) = \lambda$. By assumption M is closed (and proper). Consequently, the quotient norm $\|\ \|_M$, induced on the quotient $\mathcal{A}/M$ by the norm on $\mathcal{A}$, is not identically zero (and is also submultiplicative). Therefore we have $\|1\|_M \ge 1$ and so $\|\alpha\|_M \ge |\alpha|$, for all scalars α. We conclude that

$$\|a\| \ge \|Q_M(a)\|_M = \|\lambda\|_M \ge |\lambda|.$$

(3)⇒(4). Assume $\varrho_s(b) \le \|b\|$, for all $b \in \mathcal{A}$, and let $a \in \mathcal{A}$ be such that $\|1 - a\| < 1$. Then $\varrho_s(1 - a) < 1$ and consequently $1 \notin \mathrm{Sp}_s(1 - a)$. Thus $a = 1 - (1 - a) \in G_w$.

(4)⇒(5). Clear.

(5)⇒(1). Assume that $1 \in \mathrm{int}(G_w)$ and choose $\varepsilon > 0$ such that $\|1 - f\| < \varepsilon$ implies $f \in G_w$, for all $f \in \mathcal{A}$. We must show that G_w is open. Let $a \in G_w$ and choose (finitely many) elements $b_j, c_j \in \mathcal{A}$ such that $\sum_j b_j a c_j = 1$.

Suppose that $\|d-a\| < \varepsilon/\sum \|b_j\|\|c_j\|$. It will suffice to show that $d \in G_w$. Set $f = \sum_j b_j d c_j$. Then

$$\|1-f\| = \Big\|\sum b_j(a-d)c_j\Big\| \le \|d-a\|\sum \|b_j\|\|c_j\| < \varepsilon\,.$$

Consequently, $f \in G_w$. This implies $d \in G_w$, by Lemma 4. ∎

Proof of Proposition 1. The complete norm on $\mathcal{B}$ is spectral and so $G_w(\mathcal{B})$ is open in $\mathcal{B}$. Thus $G_w(\mathcal{A}) = G_w(\mathcal{B}) \cap \mathcal{A}$ implies that $G_w(\mathcal{A})$ is open in $\mathcal{A}$, or equivalently, that the norm on $\mathcal{A}$ is weakly spectral.

Assume now conversely that the norm on $\mathcal{A}$ is weakly spectral and choose $\varepsilon > 0$ such that $\|1-f\| < \varepsilon$ implies $f \in G_w(\mathcal{A})$, for all $f \in \mathcal{A}$. Note that clearly $G_w(\mathcal{A}) \subseteq G_w(\mathcal{B}) \cap \mathcal{A}$ and let $a \in G_w(\mathcal{B}) \cap \mathcal{A}$ be arbitrary. Choose (finitely many) elements $x_j, y_j \in \mathcal{B}$ such that $\sum x_j a y_j = 1$. By density of $\mathcal{A}$ in $\mathcal{B}$ we can now choose elements $b_j, c_j \in \mathcal{A}$ such that $\|1 - \sum b_j a c_j\| < \varepsilon$. But then $f = \sum b_j a c_j \in G_w(\mathcal{A})$, and so $a \in G_w(\mathcal{A})$, by Lemma 4. ∎

Proof of Theorem 4. (1) If a is in the center of $\mathcal{A}$, then $(a)_L = (a)_R = (a)$.

(2) This follows at once from Lemma 2.

(3) We have $S_s \subseteq S$, which implies that $\mathrm{Sp}_s(a) \subseteq \mathrm{Sp}(a)$ for all $a \in \mathcal{A}$. Suppose now that $\|\ \|$ is a weakly spectral norm on $\mathcal{A}$. Then the set S_s is closed in the corresponding topology and this implies that the strong spectrum $\mathrm{Sp}_s(a)$ is a closed subset of the plane for each $a \in \mathcal{A}$. The rest follows from Theorem 3.

(4) Clearly, $\sum_{j=1}^n b_j d c_j = 1$ implies $\sum_{j=1}^n \phi(b_j)\phi(d)\phi(c_j) = 1$. Consequently, $d \in G_w$ implies $\phi(d) \in G_w$, that is, $\phi(d) \in S_s$ implies $d \in S_s$. It follows that $\lambda \in \mathrm{Sp}_s(\phi(a))$ implies $\lambda \in \mathrm{Sp}_s(a)$.

(5) Suppose that $f = f(\lambda)$ is a rational function with all its poles contained in the complement of the spectrum of $a \in \mathcal{A}$. Then the element $f(a) \in \mathcal{A}$ is well defined. Suppose that $\lambda \in \mathrm{Sp}_s(a)$ and choose a maximal ideal $M \subseteq \mathcal{A}$ such that $Q_M(a) = \lambda$. Then $f(\lambda) = f(Q_M(a)) = Q_M(f(a))$, which shows that $f(\lambda) \in \mathrm{Sp}_s(f(a))$.

(6) The argument of (5) remains valid.

(7) If $M \subseteq \mathcal{A}$ is a maximal ideal then the quotient $\mathcal{A}/M$ is simple. The result now follows from Lemma 2 and Corollary 4.

(8) The first part follows immediately from Lemma 2, the second follows from the Chinese Remainder Theorem. ∎

Proof of Theorem 6 (similar to [10, 6.18]). Let $\phi : \mathcal{A} \to \mathcal{B}$ be as in Theorem 6, $b \in S(\phi)$ and $M \subseteq \mathcal{A}$ any maximal ideal. Then M is closed in $\mathcal{B}$ and so the norm on $\mathcal{B}$ induces the quotient norm on $\mathcal{B}/M$ which is again spectral [4]. The composition $Q_M\phi : \mathcal{A} \to \mathcal{B}/M$ is a homomorphism with dense range and $Q_M(b) \in S(Q_M\phi)$. But $S(Q_M\phi) = \{0\}$, since the quotient $\mathcal{B}/M$ is simple and $1 \notin S(Q_M\phi)$ (Lemma 3). Thus $Q_M(b) = 0$, that is, $b \in \mathcal{M}$. This shows that $b \in R$. ∎

3. Not necessarily unital algebras. Let now $\mathcal{A}$ be any associative algebra. Then, to obtain a definition of spectrum, one introduces the circle operation $a \circ b = a + b - ab$, with respect to which $\mathcal{A}$ is a semigroup with identity 0. One calls an element $a \in \mathcal{A}$ *quasiregular* if it is invertible in this semigroup, or equivalently, if

$$a \in (1-a)\mathcal{A} \cap \mathcal{A}(1-a)\,. \tag{1}$$

Note that the occurrence of the identity is purely formal: $(1-a)b$ stands for the element $b - ab$, for all $b \in \mathcal{A}$. With this understanding we could reinterpret (1) as saying that

$$a \in (1-a)_R \cap (1-a)_L\,, \tag{2}$$

where $(1-a)_R = (1-a)\mathcal{A}$ and $(1-a)_L = \mathcal{A}(1-a)$ are the "right and left ideals generated by $1-a$ in $\mathcal{A}$". Let Q_r denote the set of quasiregular elements in $\mathcal{A}$. If $\mathcal{A}$ does have an identity, then it is easily checked that

$$a \in Q_r \Leftrightarrow 1 - a \in G, \quad \text{for all } a \in \mathcal{A}\,. \tag{3}$$

Consequently, for a nonzero scalar λ we have

$$\lambda \in \mathrm{Sp}(a) \Leftrightarrow \lambda - a \notin G \Leftrightarrow 1 - a/\lambda \notin G \Leftrightarrow a/\lambda \notin Q_r\,.$$

Moreover, $0 \in \mathrm{Sp}(a) \Leftrightarrow a \in G$. Thus, without assuming an identity, one defines the spectrum $\mathrm{Sp}(a)$ as

$$\mathrm{Sp}(a) = \{\lambda \ne 0 : a/\lambda \notin Q_r\} \cup Z(a)\,,$$

where $Z(a) = \emptyset$ if $\mathcal{A}$ has an identity and $a \in G$, and $Z(a) = \{0\}$ otherwise. Moreover, the notion of quasiregular element leads to the following well-known characterization of the Jacobson radical [4]:

The Jacobson radical is the largest ideal in $\mathcal{A}$ consisting entirely of quasi-regular elements.

A corresponding characterization of the strong radical is also well known [5, Section 4.5]: For an element $a \in \mathcal{A}$ define the ideal $I(a) \subseteq \mathcal{A}$ as

$$I(a) = \Big\{ b(1-a) + (1-a)c + \sum_{j=1}^{n} b_j(1-a)c_j : \\ n \ge 1,\ b, c, b_1, c_1, \ldots, b_n, c_n \in \mathcal{A} \Big\}.$$

Again the occurrence of the identity is purely formal and we may think of $I(a)$ as "the *two-sided* ideal generated by $1-a$ in $\mathcal{A}$". This is similar to the definition of the ideal $G(a)$ in [1]. Proceeding as in the unital case we now call the element $a \in \mathcal{A}$ *weakly quasiregular* if $a \in I(a)$, or equivalently, if there exist (finitely many) elements $b, c, b_j, c_j \in \mathcal{A}$ such that

$$a = b(1-a) + (1-a)c + \sum b_j(1-a)c_j .$$

In [5] the term *G-regular* is used instead. Clearly every quasiregular element is weakly quasiregular. We have [5, Theorem 4.5.6]:

The strong radical is the largest ideal consisting entirely of weakly quasiregular elements.

A similar characterization is given in [1, Theorem 7].

Now let WQ_r denote the set of all weakly quasiregular elements $a \in \mathcal{A}$. Note that $I(a) \subseteq \mathcal{A}$ is a modular ideal with (two-sided) modular identity a. Thus $a \in I(a) \Leftrightarrow I(a) = \mathcal{A}$. If the algebra $\mathcal{A}$ does have an identity, then $I(a)$ is in fact the (two-sided) ideal $((1-a))$ generated by the element $1-a$ in $\mathcal{A}$ and consequently

$$a \in WQ_r \Leftrightarrow ((1-a)) = \mathcal{A} \Leftrightarrow 1-a \in G_w , \tag{4}$$

analogously to (3). We now define the strong spectrum $\mathrm{Sp}_s(a)$ and the strong spectral radius $\varrho_s(a)$ of an element $a \in \mathcal{A}$ as

$$\mathrm{Sp}_s(a) = \{\lambda \neq 0 : a/\lambda \notin WQ_r\} \cup Z(a) ,$$
$$\varrho_s(a) = \sup_{\lambda \in \mathrm{Sp}_s(a)} |\lambda| ,$$

where $Z(a) = \{0\}$ if there exists a modular maximal ideal $M \subseteq \mathcal{A}$ such that $Q_M(a) = 0$, and $Z(a) = \emptyset$ otherwise. Here $\varrho_s(a) = -\infty$ if $\mathrm{Sp}_s(a) = \emptyset$, and $\varrho_s(a) = +\infty$ if $\mathrm{Sp}_s(a)$ is an unbounded subset of the plane. It follows from (4) that the new definition of the strong spectrum agrees with the old one if the algebra $\mathcal{A}$ has an identity.

The element a is a modular identity for the ideal $I(a)$ and, if $M \subseteq \mathcal{A}$ is any ideal, then a is a modular identity for M if and only if $I(a) \subseteq M$. Consequently, the following statements are all equivalent: $a \notin WQ_r$, $I(a) \neq \mathcal{A}$, a is a modular identity for some proper ideal in $\mathcal{A}$, a is a modular identity for some maximal ideal $M \subseteq \mathcal{A}$. Recall also that a is a modular identity for the ideal $M \subseteq \mathcal{A}$ if and only if the quotient $\mathcal{A}/M$ has an identity and $Q_M(a) = 1$.

Thus our definition of the strong spectrum shows that the analogue of Lemma 2 holds:

LEMMA 2′. *Let $a \in \mathcal{A}$ and λ be any scalar. Then $\lambda \in \mathrm{Sp}_s(a)$ if and only if there exists a modular maximal ideal $M \subseteq \mathcal{A}$ such that $Q_M(a) = \lambda \in \mathcal{A}/M$.* ∎

As in the unital case we have

THEOREM 2′. *The strong radical of $\mathcal{A}$ is the largest ideal in $\mathcal{A}$ on which the strong spectral radius is identically zero.* ∎

Theorem 5 also generalizes to the general case with minor changes. We omit the details. Recall [4] that a norm on $\mathcal{A}$ is called *spectral* if the group Q_r of quasiregular elements is open. (In the presence of an identity we have $Q_r = 1 - G$, and so this new definition of spectral norm agrees with the one given above [4].)

Let us now call a norm on $\mathcal{A}$ *weakly spectral* if the set WQ_r of weakly quasiregular elements of $\mathcal{A}$ is open in the corresponding topology. To obtain the analogue of Theorem 3, it is most convenient to relate the strong spectrum $\mathrm{Sp}_s(a, \mathcal{A})$ of an element a in the algebra $\mathcal{A}$ to the strong spectrum $\mathrm{Sp}_s(a, \mathcal{A}^1)$ of a in the unitization $\mathcal{A}^1$ of $\mathcal{A}$.

Note first that for every modular maximal ideal $M \subseteq \mathcal{A}$ there exists a maximal ideal $M_1 \subseteq \mathcal{A}^1$ with $M_1 \cap \mathcal{A} = M$. (Let $Q : \mathcal{A}^1 \to \mathcal{A}/M$ be the extension of $Q_M : \mathcal{A} \to \mathcal{A}/M$ defined by $Q(a + \lambda) = Q_M(a) + \lambda$, for all $a \in \mathcal{A}$ and all scalars λ, and set $M_1 = \ker(Q)$.) Similarly, if $M_1 \subseteq \mathcal{A}^1$ is a maximal ideal, then $M_1 = \mathcal{A}$ or $M = M_1 \cap \mathcal{A}$ is a modular maximal ideal in $\mathcal{A}$. Now Lemma 2′ shows that

$$\mathrm{Sp}_s(a, \mathcal{A}) \setminus \{0\} = \mathrm{Sp}_s(a, \mathcal{A}^1) \setminus \{0\}, \quad \text{and thus}$$
$$\varrho_s(a, \mathcal{A}) = \varrho_s(a, \mathcal{A}^1), \quad \text{for all } a \in \mathcal{A}.$$

We shall also require the following lemmas:

LEMMA 5. *We have $WQ_r(\mathcal{A}) = WQ_r(\mathcal{A}^1) \cap \mathcal{A}$.*

Proof. Clearly $WQ_r(\mathcal{A}) \subseteq WQ_r(\mathcal{A}^1) \cap \mathcal{A}$. Conversely, let $a \in WQ_r(\mathcal{A}^1) \cap \mathcal{A}$. Then there exist (finitely many) elements $b+\lambda, c+\mu, b_j+\lambda_j, c_j+\mu_j \in \mathcal{A}^1$ such that

$$\begin{aligned} a &= (b+\lambda)(1-a) + (1-a)(c+\mu) + \sum (b_j+\lambda_j)(1-a)(c_j+\mu_j) \\ &= b(1-a) + (1-a)c + \sum b_j(1-a)c_j + \sum \lambda_j(1-a)c_j \\ &\quad + \sum \mu_j b_j(1-a) + \Big[\lambda + \mu + \sum \lambda_j\mu_j\Big](1-a) . \end{aligned}$$

Since $a \in \mathcal{A}$, the scalar part $\lambda + \mu + \sum \lambda_j\mu_j$ on the right must be zero and we obtain

$$a = \Big[b + \sum \mu_j b_j\Big](1-a) + (1-a)\Big[c + \sum \lambda_j c_j\Big] + \sum b_j(1-a)c_j .$$

This shows that $a \in WQ_r(\mathcal{A})$. ∎

LEMMA 6. *Suppose that $\mathcal{A}$ is a normed algebra. Let $0 < \varepsilon \leq 1/2$. If $\|a\| < 2\varepsilon$ implies $a \in WQ_r(\mathcal{A})$, for all $a \in \mathcal{A}$, then $\|a\| + |\alpha| < \varepsilon$ implies $a + \alpha \in WQ_r(\mathcal{A}^1)$, for all elements $a \in \mathcal{A}$ and all scalars α.*

Proof. Assume that $b \in WQ_r(\mathcal{A})$, for all elements $b \in \mathcal{A}$ with $\|b\| < 2\varepsilon$. Let $a \in \mathcal{A}$ and α be any scalar such that $\|a\| + |\alpha| < \varepsilon$. Then $|\alpha| < \varepsilon \leq 1/2$ and so $\|a/(1-\alpha)\| \leq 2\|a\| < 2\varepsilon$. Thus $a/(1-\alpha) \in WQ_r(\mathcal{A})$ and consequently there exist (finitely many) elements $b', c', b'_j, c_j \in \mathcal{A}$ such that

$$\frac{a}{1-\alpha} = b'\left(1-\frac{a}{1-\alpha}\right) + \left(1-\frac{a}{1-\alpha}\right)c' + \sum b'_j\left(1-\frac{a}{1-\alpha}\right)c_j .$$

Set $b = b'/(1-\alpha)$, $c = c'/(1-\alpha)$, $b_j = b'_j/(1-\alpha)$ and $\lambda = \alpha/(1-\alpha)$. Then

$$\begin{aligned}\frac{a+\alpha}{1-\alpha} &= \frac{a}{(1-\alpha)^2} + \lambda\left(1-\frac{a}{1-\alpha}\right)\\ &= (b+\lambda)\left(1-\frac{a}{1-\alpha}\right) + \left(1-\frac{a}{1-\alpha}\right)c + \sum b_j\left(1-\frac{a}{1-\alpha}\right)c_j .\end{aligned}$$

Multiply with $1-\alpha$ to obtain

$$a+\alpha = (b+\lambda)(1-(a+\alpha)) + (1-(a+\alpha))c + \sum b_j(1-(a+\alpha))c_j .$$

This shows that $a+\alpha \in WQ_r(\mathcal{A}^1)$. ∎

COROLLARY 5. *Let $\mathcal{A}$ be a normed algebra and extend the norm on $\mathcal{A}$ to the unitization $\mathcal{A}^1$ as follows: $\|a+\alpha\| = \|a\| + |\alpha|$, for all elements $a \in \mathcal{A}$ and all scalars α. Then the norm on $\mathcal{A}$ is weakly spectral if and only if its extension to $\mathcal{A}^1$ is weakly spectral on $\mathcal{A}^1$.*

Proof. Assume first that the extended norm is weakly spectral on $\mathcal{A}^1$. Then the set $WQ_r(\mathcal{A}^1)$ is open in $\mathcal{A}^1$. By Lemma 5, $WQ_r(\mathcal{A}) = WQ_r(\mathcal{A}^1) \cap \mathcal{A}$. Consequently, the set $WQ_r(\mathcal{A})$ is open in $\mathcal{A}$. This means that the norm on $\mathcal{A}$ is weakly spectral.

Assume now conversely that the norm on $\mathcal{A}$ is weakly spectral. Since $0 \in WQ_r(\mathcal{A})$ we can choose $\varepsilon > 0$ such that $\varepsilon \le 1/2$ and $\|a\| < 2\varepsilon$ implies that $a \in WQ_r(\mathcal{A})$, for all elements $a \in \mathcal{A}$. Then, according to Lemma 6, $\|a\| + |\alpha| < \varepsilon$ implies $a + \alpha \in WQ_r(\mathcal{A}^1)$, for all $a \in \mathcal{A}$ and all scalars α. This shows that $0 \in \operatorname{int}(WQ_r(\mathcal{A}^1))$. But in the unitization $\mathcal{A}^1$ we have $WQ_r(\mathcal{A}^1) = 1 - G_w(\mathcal{A}^1)$, according to (4). This shows that $1 \in \operatorname{int}(G_w(\mathcal{A}^1))$. Now Theorem 3 shows that the norm on $\mathcal{A}^1$ is weakly spectral. ∎

THEOREM 3′. *The following conditions are equivalent for a normed algebra $\mathcal{A}$:*

(1) *The norm on $\mathcal{A}$ is weakly spectral.*
(2) *Every modular maximal ideal $M \subseteq \mathcal{A}$ is closed.*
(3) *We have $\varrho_s(a) \le \|a\|$, for all $a \in \mathcal{A}$.*
(4) *$\|a\| < 1$ implies $a \in WQ_r$, for all $a \in \mathcal{A}$.*
(5) *The zero element is in the interior of WQ_r.*

Proof. We show (1)⇒(2)⇒(3) ⇒(4)⇒(5)⇒(1). The implications (2)⇒(3)⇒(4) ⇒(5) are shown as in the proof of Theorem 3.

(1)⇒(2). Suppose that the norm on $\mathcal{A}$ is weakly spectral and extend it to the unitization $\mathcal{A}^1$ as above. Now let $M \subseteq \mathcal{A}^1$ be any modular maximal ideal. Choose a maximal ideal $M_1 \subseteq \mathcal{A}^1$ such that $M = M_1 \cap \mathcal{A}$. By Corollary 5 and Theorem 3, M_1 is closed in $\mathcal{A}^1$. Consequently, M is closed in $\mathcal{A}$.

(5)⇒(1). Suppose that $0 \in \operatorname{int}(WQ_r)$. Extend the norm on $\mathcal{A}$ to $\mathcal{A}^1$ as above. Then Lemma 5 shows that $0 \in \operatorname{int}(WQ_r(\mathcal{A}^1)) = \operatorname{int}(1 - G_w(\mathcal{A}^1))$. Consequently, $1 \in \operatorname{int}(G_w(\mathcal{A}^1))$. Now Theorem 3 shows that the norm on $\mathcal{A}^1$ is weakly spectral. It follows from Corollary 5 that the norm on $\mathcal{A}$ is weakly spectral. ∎

COROLLARY 2′. *Let $\mathcal{A}$ be a normed algebra. If the norm on $\mathcal{A}$ is spectral, then it is weakly spectral.*

Proof. Suppose that the norm on $\mathcal{A}$ is spectral. Then $0 \in \operatorname{int}(Q_r)$. Since $Q_r \subseteq WQ_r$, it follows that $0 \in \operatorname{int}(WQ_r)$ and hence the norm on $\mathcal{A}$ is weakly spectral. ∎

PROPOSITION 1′. *Let $\mathcal{A}$ be a normed algebra and $\mathcal{B}$ the completion of $\mathcal{A}$. Then the norm on $\mathcal{A}$ is weakly spectral if and only if $WQ_r(\mathcal{A}) = WQ_r(\mathcal{B}) \cap \mathcal{A}$.*

This can be reduced to the unital case by using Corollary 5 and (4).

4. Examples. Suppose that the algebra $\mathcal{A}$ has a unique largest proper ideal $\mathcal{K}$, that is, a proper ideal $\mathcal{K}$ which contains all other proper ideals.

From Lemma 2 it follows that $\mathrm{Sp}_s(a)$ is empty if $Q_{\mathcal{K}}(a)$ is not a scalar, and $\mathrm{Sp}_s(a) = \{\lambda\}$ if $Q_{\mathcal{K}}(a)$ is the scalar λ, for each element $a \in \mathcal{A}$. Moreover, the set S_s of strongly singular elements, which in general is the union of all proper ideals in $\mathcal{A}$, coincides with the ideal $\mathcal{K}$. Consequently, a norm on $\mathcal{A}$ is weakly spectral if and only if the ideal $\mathcal{K}$ is closed in this norm.

In particular, a simple algebra $\mathcal{A}$ is of this form with $\mathcal{K} = \{0\}$. In this case every norm on $\mathcal{A}$ is (trivially) weakly spectral.

These extreme cases provide easy examples to complement Theorem 1. To be specific, let $H = l_2$ be separable Hilbert space, B the algebra of all bounded linear operators on H, $K \subseteq B$ the ideal of compact operators, $C = B/K$ the Calkin algebra and $Q : B \to C$ the quotient map. K is well known to be the unique largest ideal in B. Consequently, the algebra C is simple.

EXAMPLE 1. Choose a selfadjoint element $a \in B$ such that $Q_K(a) \in C$ is not a scalar. Then neither is the element $Q_K(1-a)$. Consequently, $\mathrm{Sp}_s(a) = \mathrm{Sp}_s(1-a) = \emptyset$. But $\mathrm{Sp}_s(a+1-a) = \{1\}$. This shows that the strong spectral radius is not subadditive on the commuting selfadjoint elements $a, 1-a \in B$.

EXAMPLE 2. We show that the equality $\mathrm{Sp}_s(f(a)) = f(\mathrm{Sp}_s(a))$ fails for certain elements of the algebra C, even for the polynomial $f(\lambda) = \lambda^2$.

Let $t \in B$ be the bounded linear operator which permutes the coordinates as follows:

$$t : l_2 \ni (x_1, x_2, \ldots) \to (x_2, x_1, x_4, x_3, \ldots) \in l_2 ,$$

and set $a = Q(t) \in C$. We have $t^2 = 1 \in B$ and consequently $a^2 = 1 \in C$. Thus $\mathrm{Sp}_s(f(a)) = \mathrm{Sp}_s(a^2) = \{1\}$. It will now suffice to show that the element a is not a scalar in C. Then, by simplicity of C, $\mathrm{Sp}_s(a) = \emptyset$ and so $f(\mathrm{Sp}_s(a)) = \emptyset$. Let λ be a scalar. We have to show that the operator $t - \lambda$ is not compact. Note that $t - \lambda$ is the operator

$$l_2 \ni (x_1, x_2, \ldots) \to (x_2 - \lambda x_1, x_1 - \lambda x_2, x_4 - \lambda x_3, x_3 - \lambda x_4, \ldots) \in l_2$$

and let $p : (x_1, x_2, \ldots) \to (x_1, 0, x_3, 0, \ldots)$ be the orthogonal projection onto $\overline{\mathrm{span}}\{e_{2k+1}\}$, where (e_k) denotes the standard unit vector basis of $H = l_2$. Then $p(t - \lambda)$ is the linear operator

$$l_2 \ni (x_1, x_2, \ldots) \to (x_2 - \lambda x_1, 0, x_4 - \lambda x_3, 0, \ldots) \in l_2 .$$

Thus $p(t-\lambda)$ is an isometry on the infinite-dimensional subspace $\overline{\mathrm{span}}\{e_{2k}\}$. Consequently, the operator $p(t - \lambda)$ is not compact and hence neither is the operator $t-\lambda$. Notice also that the algebra C is a C^*-algebra and the element $a \in C$ is selfadjoint.

EXAMPLE 3. We show that the equality $\mathrm{Sp}_s(ab) = \mathrm{Sp}_s(ba)$ fails for certain elements $a, b \in C$. Let $s, t \in B$ be the following bounded linear operators on $H = l_2$:

$$s : l_2 \ni (x_k) \to (x_{2k}) \in l_2 \quad \text{and}$$
$$t : l_2 \ni (x_k) \to (0, x_1, 0, x_2, \ldots) \in l_2 ,$$

and set $a = Q(s)$ and $b = Q(t)$. Then $st = 1 \in B$ and consequently $ab = 1 \in C$, but the operator ts has infinite-dimensional kernel, and so the element $ba = Q(ts) \in C$ is not invertible, in particular $ba \neq 1$. The simplicity of the algebra C now implies that $\mathrm{Sp}_s(ab) = \{1\}$ but $\mathrm{Sp}_s(ba) \neq \{1\}$. In fact, it is easy to see that the operator ts is not compact and so $ba \neq 0$. Thus ba is not a scalar in C and so the strong spectrum $\mathrm{Sp}_s(ba)$ is empty.

EXAMPLE 4. Recently [9] L. B. Schweitzer has constructed a simple unital Banach algebra $\mathcal{A}$ with involution which carries a nonspectral B^*-norm. Since all norms on $\mathcal{A}$ are weakly spectral, this provides an example of a norm which is weakly spectral but not spectral.

Remark. Another reason to consider the notion of a weakly spectral norm is the following: A classical result by C. E. Rickart [7] shows that on a commutative, semisimple, completely regular Banach algebra all norms are spectral. The question was left unresolved in the noncommutative case. The basic building blocks (via the subdirect product representation) of noncommutative, completely regular Banach algebras are the simple, unital Banach algebras, and these in turn are completely regular (by default). Consequently, Schweitzer's example gives a negative solution. The trouble seems to arise from the fact that invertibility (and hence the spectrum) are defined in terms of *one-sided ideals* (of which there are many, even in a simple noncommutative Banach algebra), whereas complete regularity is defined in terms of *two-sided ideals* (and hence true by default for a simple Banach algebra).

Our notions of weak regularity, strong spectrum and weakly spectral norm are defined in terms of two-sided ideals. In fact, all norms on a simple unital algebra are (trivially) weakly spectral. We are therefore led to the following question:

Is every norm on a strongly semisimple, completely regular Banach algebra weakly spectral?

In the commutative case we have $G_w = G$ and consequently the notions of spectrum and strong spectrum and of spectral norm and weakly spectral norm coincide.

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References

[1] B. Brown and M. H. McCoy, *Radicals and subdirect sums*, Amer. J. Math. 69 (1947), 46–58.
[2] S. R. Caradus, W. E. Pfaffenberger and B. Yood, *Calkin Algebras and Algebras of Operators on Banach Spaces*, Marcel Dekker, New York 1974.
[3] A. Lebow and M. Schechter, *Semigroups of operators and measures of noncompactness*, J. Funct. Anal. 7 (1971), 1–26.
[4] T. W. Palmer, *Spectral algebras, subalgebras and pseudonorms*, Rocky Mountain J. Math. 22 (1992), 293–328.
[5] —, *Banach Algebras and the General Theory of *-Algebras*, Cambridge University Press, 1993, to appear.
[6] T. J. Ransford, *Generalized spectra and analytic multivalued functions*, J. London Math. Soc. (2) 29 (1984), 306–322.
[7] C. E. Rickart, *On spectral permanence for certain Banach algebras*, Proc. Amer. Math. Soc. 4 (1953), 191–196.
[8] —, *General Theory of Banach Algebras*, Van Nostrand, New York 1960.
[9] L. B. Schweitzer, *A nonspectral dense Banach subalgebra of the irrotational rotation algebra*, preprint.
[10] A. M. Sinclair, *Automatic Continuity of Linear Operators*, London Math. Soc. Lecture Note Ser. 21, Cambridge Univ. Press, Cambridge 1976.

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