References

[1] C. Apostol, *The reduced minimum modulus*, Michigan Math. J. 32 (1985), 279–294.
[2] K. G. Brock, *A note on commutativity of a linear operator and its Moore–Penrose inverse*, Numer. Funct. Anal. Optim. 11 (1990), 673–678.
[3] S. R. Caradus, *Generalized Inverses and Operator Theory*, Queen's Papers in Pure and Appl. Math. 50, Queen's Univ., Kingston, Ont., 1978.
[4] C. W. Groetsch, *Representations of the generalized inverse*, J. Math. Anal. Appl. 49 (1975), 154–157.
[5] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York 1967.
[6] R. E. Harte, *Invertibility and Singularity*, Dekker, New York 1988.
[7] R. E. Harte and M. Mbekhta, *On generalized inverses in C^*-algebras*, Studia Math. 103 (1992), 71–77.
[8] R. E. Harte and M. Ó Searcóid, *Positive elements and the B^* condition*, Math. Z. 193 (1986), 1–9.
[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer, New York 1966.
[10] M. Mbekhta, *Conorme et inverse généralisé dans les C^*-algèbres*, Canad. Math. Bull. 35 (1992), 515–522.
[11] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc. 51 (1955), 406–413.
[12] A. Pietsch, *Operator Ideals*, North-Holland, 1980.

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On continuity properties of functions with conditions on the mean oscillation

by

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Abstract. In this paper we study distribution and continuity properties of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

Since the initial works by F. John and L. Nirenberg and J. Moser in 1961, the study of regularity of functions with properties on their mean oscillation over balls was developed by S. Campanato, G. Meyers, S. Spanne and A. P. Calderón. Extensions from the euclidean setting to spaces of homogeneous type were considered by N. Burger, R. Macías and C. Segovia and one of the authors.

In 1967, J. Moser in his paper on Harnack's inequality for parabolic equations introduces a BMO type condition with a time lag. In 1985, E. Fabes and N. Garofalo, applying an extension of Calderón's method as stated by U. Neri obtained a John–Nirenberg type lemma for this parabolic case. In 1988 one of us proved an extension of these results to the setting of spaces of homogeneous type that can be applied to degenerate parabolic equations. Related results come from the analysis of one-sided maximal functions and weights; in a recent paper F. Martín-Reyes and A. de la Torre prove a John–Nirenberg type lemma for one-sided BMO functions.

In this paper we study distribution and continuity property of functions satisfying a vanishing mean oscillation property with a lag mapping on a space of homogeneous type.

1. Main results. Let X be a set. A symmetric function $d : X \times X \to \mathbb{R}^+ \cup \{0\}$ is a *quasi-distance* on X if $d(x,y) = 0$ iff $x = y$ and there exists a constant K such that $d(x,z) \le K[d(x,y) + d(y,z)]$ for $x, y, z \in X$. The ball with center $x \in X$ and radius $r > 0$ is the set $B(x,r) = \{y \in X : d(x,y) < r\}$. We shall say that a positive measure μ defined on a σ-algebra contain-

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ing the balls satisfies the *doubling condition* if there is a positive constant A such that

$$0<\mu(B(x,2r))\le A\mu(B(x,r))<\infty \tag{1.1}$$

for every $x\in X$ and every $r>0$. If d is a quasi-distance on X and μ satisfies the doubling condition, then we say, following [MS], that (X,d,μ) is a *space of homogeneous type*. Given a ball B on (X,d) we shall usually write $x(B)$ and $r(B)$ for the center and the radius of B.

Given a space of homogeneous type (X,d,μ) we say that a function $T:X\times\mathbb{R}^+\to X\times\mathbb{R}^+$, $T(x,r)=(\xi,\varrho)$, $\xi=\xi(x,r)$, $\varrho=\varrho(x,r)$, is a *lag mapping* if there exist three constants K_ν, $\nu=1,2,3$, such that

$$d(x,\xi)\le K_1 r\,, \tag{1.2}$$

$$K_2\varrho\le r\le K_3\varrho\,, \tag{1.3}$$

for every $x\in X$ and $r>0$. Given a d-ball $B=B(x,r)$ we shall usually write TB for $B(\xi,\varrho)$.

Let $\phi:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing function satisfying the Δ_2 condition: there exists $C>0$ such that $\phi(2r)\le C\phi(r)$ for every positive r. We say that a locally integrable function $f:X\to\mathbb{R}$ is of *ϕ-bounded mean oscillation* with respect to the lag mapping T if there exists a real function $C(x,r)$ on $X\times\mathbb{R}^+$ such that

$$m_B((f-C_B)^+)\le D\phi(r)\,, \tag{1.4}$$

$$m_{TB}((C_B-f)^+)\le D\phi(r)\,, \tag{1.5}$$

where D is a constant, $B=B(x,r)$, $C_B=C(x,r)$ and $m_B(g)=\frac{1}{\mu(B)}\int g\,d\mu$. Let $\mathrm{BMO}(\phi,T)$ be the class of such functions.

We start with some simple results concerning $\mathrm{BMO}(\phi,T)$ functions.

(1.6) Lemma. *Let $f\in\mathrm{BMO}(\phi,T)$. Then for every ball B we have*

$$m_B((f-m_{TB}(f))^+)\le 2D\phi(r)\,, \tag{1.7}$$

$$m_{TB}((m_B(f)-f)^+)\le 2D\phi(r)\,. \tag{1.8}$$

Proof. Let us prove (1.7):

$$\begin{aligned}m_B((f-m_{TB}(f))^+)&\le m_B((f-C_B)^+)+(C_B-m_{TB}(f))^+\\&\le m_B((f-C_B)^+)+m_{TB}((C_B-f)^+)\le 2D\phi(r)\,.\ \blacksquare\end{aligned}$$

The following lemma provides the key for the study of the distribution of functions in $\mathrm{BMO}(\phi,T)$.

(1.9) Lemma. *There exist constants M_0 and N depending only on K, A and K_ν, $\nu=1,2,3$, such that given a fixed ball $B_0=B(x_0,r_0)$ in (X,d,μ), given $M\ge M_0$ and given $f\in\mathrm{BMO}(\phi,T)$ such that $m_{T\widetilde B_0}(f)=0$ with $\widetilde B_0=B(x_0,Nr_0)$, there exists a constant C_1 depending on M, K, A, K_ν, $\nu=1,2,3$, D and C such that*

$$m_{TB(x,r)}(f)\le C_1\lambda_i\,, \tag{1.10}$$

for every $x\in B_0$, every $i\in\mathbb{N}$ and every $r\in[r_0/M^{i+1},r_0/M^i)$ where $\lambda_i=\sum_{k=0}^{i-1}\phi(r_0/M^k)$.

Proof. Let $x\in B_0=B(x_0,r_0)$. For $N\ge K(K_2^{-1}+K(1+K_1))$ we have $B(x_0,Nr)\supset TB(x,r_0)$; in fact, if $y\in TB(x,r_0)$ then

$$\begin{aligned}d(y,x_0)&\le K[d(y,x(TB(x,r_0)))+K[d(x(TB(x,r_0)),x)+d(x,x_0)]]\\&<K[r(TB(x,r_0))+K[K_1r_0+r_0]]\le K[K_2^{-1}+K(K_1+1)]r_0\,.\end{aligned}$$

Taking now $M\ge K[K_2^{-1}+K_1]$ we clearly have $B(x,r)\supset TB(x,r/M)$ for any ball $B(x,r)$. Let $x\in B_0$, $r\in[r_0/M^{i+1},r_0/M^i)$ and $B=B(x,r)$, and construct a finite sequence of balls satisfying

$$\begin{aligned}B_1&=B(x,r_0/M^{i-1})\supset TB\,,\\B_2&=B(x,r_0/M^{i-2})\supset TB_1\,,\\&\cdots\cdots\cdots\cdots\\B_i&=B(x,r_0)\supset TB_{i-1}\end{aligned}$$

and

$$\widetilde B_0=B(x_0,Nr_0)\supset TB_i\,.$$

Pick $f\in\mathrm{BMO}(\phi,T)$ such that $m_{T\widetilde B_0}(f)=0$. Then from Lemma (1.6) we get

$$\begin{aligned}m_{TB}(f)&=[m_{TB}(f)-m_{TB_1}(f)]+\sum_{j=1}^{i-1}[m_{TB_j}(f)-m_{TB_{j+1}}(f)]\\&\quad+[m_{TB_i}(f)-m_{T\widetilde B_0}(f)]\\&\le\frac{1}{\mu(TB)}\int_{TB}(f-m_{TB_1}(f))^+d\mu\\&\quad+\sum_{j=1}^{i-1}\frac{1}{\mu(TB_j)}\int_{TB_j}(f-m_{TB_{j+1}}(f))^+d\mu\\&\quad+\frac{1}{\mu(TB_i)}\int_{TB_i}(f-m_{T\widetilde B_0}(f))^+d\mu\\&\le\frac{\mu(B_1)}{\mu(TB_1)}\cdot\frac{1}{\mu(B_1)}\int_{B_1}(f-m_{TB_1}(f))^+d\mu\end{aligned}$$

$$+\sum_{j=1}^{i-1}\frac{\mu(B_{j+1})}{\mu(TB_j)}\cdot\frac{1}{\mu(B_{j+1})}\int_{B_{j+1}}(f-m_{TB_{j+1}}(f))^+d\mu$$

$$+\frac{\mu(\widetilde{B}_0)}{\mu(TB_i)}\cdot\frac{1}{\mu(\widetilde{B}_0)}\int_{\widetilde{B}_0}(f-m_{\widetilde{B}_0}(f))^+d\mu$$

$$\le C_2\Big\{\sum_{j=0}^{i-1}\phi(r(B_{j+1}))+\phi(Nr_0)\Big\}$$

$$\le C_1\Big\{\sum_{j=0}^{i-1}\phi\Big(\frac{r_0}{M^{i-j-1}}\Big)+\phi(r_0)\Big\}=C_1\lambda_i\,.$$

(1.11) LEMMA (Covering Lemma). *Let (X,d,μ) be a space of homogeneous type. Let $\mathcal{B}=\{B_\alpha=B(x_\alpha,r_\alpha):\alpha\in\Gamma\}$ be a given family of balls in X such that $\bigcup_{\alpha\in\Gamma}B_\alpha$ is bounded. Then there exists a sequence of disjoint balls $\{B_i\}\subset\mathcal{B}$ such that for every $\alpha\in\Gamma$ there exists i satisfying $r_\alpha\le 2r_i$ and $B_\alpha\subset B(x_i,5K^2r_i)$.*

For a proof of this lemma see [CW].

The following differentiation theorem will be useful in the proof of the main result.

(1.12) THEOREM. *Let (X,d,μ) be a space of homogeneous type and let T be a lag mapping on X. The maximal operator*

$$M_Tf(x)=\sup_{r>0}\frac{1}{\mu(TB(x,r))}\int_{TB(x,r)}|f|\,d\mu$$

is of weak type $(1,1)$, i.e.

$$\mu(\{x\in X:M_Tf(x)>\lambda\})\le\frac{C}{\lambda}\|f\|_1\,.$$

If μ is a regular measure, then for $f\in L^1(\mu)$,

$$f(x)=\lim_{r\to0}\frac{1}{\mu(TB(x,r))}\int_{TB(x,r)}f\,d\mu$$

almost everywhere.

Let $f\in\mathrm{BMO}(\phi,T)$ and $B_0=B(x_0,r_0)$ be such that $m_{T\widetilde{B}_0}(f)=0$. Let $t>0$ and $j\in\mathbb{N}$. Consider the set

$$\Omega^j=\{x\in B_0:\ \text{there exists } r\in(0,r_0/M)\text{ such that } m_{TB(x,r)}(f)>t\lambda_j\}$$

and, given $x\in\Omega^j$,

$$R^j(x)=\{r\in(0,r_0/M):m_{TB(x,r)}(f)>t\lambda_j\}\,.$$

(1.13) LEMMA. *$R^j(x)\subset(0,r_0/M^{j+1})$ provided that $t>C_1$.*

Proof. Let $r_0/M>r\ge r_0/M^{j+1}$ and $1\le h\le j$ such that

$$r_0/M^{j+1}\le r_0/M^{h+1}\le r<r_0/M^h\le r_0/M\,.$$

From Lemma (1.9) we have

$$m_{TB(x,r)}(f)\le C_1\lambda_h\le t\lambda_h\le t\lambda_j$$

so that $r\notin R^j(x)$. ■

(1.14) LEMMA. *Let n be a given positive integer. For $k=1,\dots,n$ there is a function r^k defined on Ω^k such that, for $t>C_1$ and $x\in\Omega^k$,*

$$r^k(x)\in R^k(x)\,,\tag{1.15}$$

$$0<r^k(x)<r_0/M^{k+1}\,,\tag{1.16}$$

$$m_{TB(x,r^k(x))}(f)>t\lambda_k\ge m_{TB(x,Mr^k(x))}(f)\,,\tag{1.17}$$

$$r^{k-1}(x)\ge r^k(x)\,.\tag{1.18}$$

Proof. Given $x\in\Omega^n$ pick $r^n(x)\in R^n(x)$ in such a way that $Mr^n(x)\notin R^n(x)$. Inequality (1.16) follows from Lemma (1.13). Inequalities (1.17) with $k=n$ follow from the definition of $R^k(x)$. Assume that r^k is defined. Let us define r^{k-1}. Let $x\in\Omega^{k-1}$. If $x\in\Omega^{k-1}-\Omega^k$ define r^{k-1} in the same way as we defined r^n. If, otherwise, $x\in\Omega^k$, then pick $r^{k-1}(x)\in R^{k-1}(x)$ in such a way that $r^{k-1}(x)\ge r^k(x)$ and $Mr^k(x)\notin R^{k-1}(x)$. ■

Given $k=1,\dots,n$, set

$$\mathcal{B}^k=\{TB(x,r^k(x)):x\in\Omega^k\}\,.$$

(1.19) LEMMA. *There exists $M\ge M_0$ and for each $k=1,\dots,n$ and $t>0$ there exists a sequence $\{x_i^k:i\in\mathbb{N}\}$ of points in Ω^k such that the following properties hold:*

$$TB(x_i^k,r^k(x_i^k))\cap TB(x_j^k,r^k(x_j^k))=\emptyset,\quad i\ne j\,;\tag{1.20}$$

(1.21) *For every $x\in\Omega^k$ there exists $i\in\mathbb{N}$ such that $\varrho(x,r^k(x))\le 2\varrho(x_i^k,r^k(x_i^k))$ and*

$$TB(x,r^k(x))\subset B(\xi(x_i^k,r^k(x_i^k)),5K^2\varrho(x_i^k,r^k(x_i^k)))\,;$$

$$\Omega^k\subset\bigcup_{i=1}^{\infty}B(\xi(x_i^k,r^k(x_i^k)),P\varrho(x_i^k,r^k(x_i^k)))\tag{1.22}$$

where P depends only on the constants K, K_ν; $\nu=1,2,3$;

$$r^k(x_i^k)<r_0/M^{k+1},\quad\text{for all } k,i\,;\tag{1.23}$$

$$m_{TB(x_i^k,r^k(x_i^k))}(f)>t\lambda_k\ge m_{TB(x_i^k,Mr^k(x_i^k))}(f)\,;\tag{1.24}$$

(1.25) *Given $j \in \mathbb{N}$, there exists $i \in \mathbb{N}$ such that*

$$TB(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, Mr^k(x_i^k)) \subset B(\xi(x_i^k, r^k(x_i^k)), S\varrho(x_i^k, r^k(x_i^k)))$$

where S depends only on K and the K_ν's;

(1.26) *Given $i \in \mathbb{N}$, set*

$$J_i = \{j \in \mathbb{N} : TB(x_j^{k+1}, r^{k+1}(x_j^{k+1})) \subset B(x_i^k, Mr^k(x_i^k)) \text{ but } TB(x_j^{k+1}, r^{k+1}(x^{k+1} - j)) \not\subset B(x_\ell^k, Mr^k(x_\ell^k)) \text{ for } \ell = 1, \ldots, i-1\}.$$

Then $J_i \cap J_h = \emptyset$ if $i \neq h$ and $\mathbb{N} = \bigcup_i J_i$.

Proof. Applying the covering Lemma (1.11) to the family $\mathcal{B}^k$ we obtain a sequence $\{x_i^k : i \in \mathbb{N}\}$ satisfying (1.20) to (1.24). In order to prove (1.25) observe that given $j \in \mathbb{N}$ the point x_j^{k+1} belongs to Ω^{k+1} which is contained in Ω^k, therefore $B(x_j^{k+1}, r^k(x_j^{k+1})) \in \mathcal{B}^k$, and thus from (1.21) there exists $i \in \mathbb{N}$ such that $\varrho(x_j^{k+1}, r^k(x_j^{k+1})) \leq 2\varrho(x_i^k, r^k(x_i^k))$ and

$$TB(x_j^{k+1}, r^k(x_j^{k+1})) \subset B(\xi(x_i^k, r^k(x_i^k)), 5K^2\varrho(x_i^k, r^k(x_i^k))).$$

For this $i \in \mathbb{N}$ the first inclusion in (1.25) follows readily with an appropriate choice of M. The second inclusion is a consequence of the properties of lag mappings. Finally, (1.26) follows from (1.25). ∎

The following lemma provides an estimate for the size of the distribution function of f^+.

(1.27) Lemma. *Let (X, d, μ) be a space of homogeneous type with μ regular. Let T be a lag mapping on (X, d). Let ϕ be an increasing function satisfying the Δ_2 condition. Let $f \in \mathrm{BMO}(\phi, T)$, $B_0 = B(x_0, r_0)$, $\widetilde{B}_0 = B(x_0, Nr_0)$ and $m_{T\widetilde{B}_0}(f) = 0$. Then there exist constants t and $\widetilde{C}$ such that*

$$\mu(\{x \in B_0 : f^+(x) > t\lambda_n\}) \leq \frac{\widetilde{C}}{2^n}\mu(B_0)$$

with $\lambda_n = \sum_{k=0}^{n-1} \phi(r_0/M^k)$ and $n \in \mathbb{N}$.

Proof. Applying the first inequality in (1.24) for $k+1$, (1.26), the second inequality in (1.24), (1.20), the fact that $f \in \mathrm{BMO}(\phi, T)$, (1.23) and the second inclusion of (1.25) we get

$$t\lambda_{k+1} \sum_{j \in \mathbb{N}} \mu(TB(x_j^{k+1}, r^{k+1}(x_j^{k+1}))) \leq \sum_{j \in \mathbb{N}} \int_{TB(x_j^{k+1}, r^{k+1}(x_j^{k+1}))} f\, d\mu$$

$$= \sum_{i \in \mathbb{N}} \sum_{j \in J_i} \int_{TB(x_j^{k+1}, r^{k+1}(x_j^{k+1}))} [f - m_{TB(x_i^k, Mr^k(x_i^k))}(f)]^+\, d\mu + t\lambda_k \sum_{j \in \mathbb{N}} \mu(TB(x_j^{k+1}, r^{k+1}(x_j^{k+1})))$$

$$\leq \sum_{i \in \mathbb{N}} \int_{B(x_i^k, Mr^k(x_i^k))} [f - m_{TB(x_i^k, Nr^k(x_i^k))}(f)]^+\, d\mu + t\lambda_k \sum_{j \in \mathbb{N}} \mu(TB(x_j^{k+1}, r^{k+1}(x_j^{k+1})))$$

$$\leq C_2\phi(r_0/M_k) \sum_{i \in \mathbb{N}} \mu(B(x_i^k, Mr^k(x_i^k))) + t\lambda_k \sum_{j \in \mathbb{N}} \mu(TB(x_j^{k+1}, r^{k+1}(x_j^{k+1})))$$

$$\leq C_3\phi(r_0/M^k) \sum_{i \in \mathbb{N}} \mu(TB(x_i^k, r^k(x_i^k))) + t\lambda_k \sum_{j \in \mathbb{N}} \mu(TB(x_j^{k+1}, r^{k+1}(x_j^{k+1}))).$$

Set $\Sigma_k = \sum_j \mu(TB(x_j^k, r^k(x_j^k)))$. Then

$$t(\lambda_{k+1} - \lambda_k)\Sigma_{k+1} \leq C_3\phi(r_0/M^k)\Sigma_k,$$

and so

$$t\Sigma_{k+1} \leq C_3\Sigma_k.$$

Taking $t \geq 2C_3$ we get $\Sigma_{k+1} \leq \frac{1}{2}\Sigma_k$ and, by iteration,

$$\Sigma_n \leq \frac{1}{2^{n-1}}\Sigma_1 \leq \frac{C_4}{2^n}\mu(B_0).$$

From (1.22) we have

$$\mu(\Omega^n) \leq \sum_i \mu(B(\xi(x_i^n, r^n(x_i^n)), P\varrho(x_i^n, r^n(x_i^n)))) \leq C_5\Sigma_n \leq \frac{C_6}{2^n}\mu(B_0).$$

In order to finish the proof of the lemma we only need to show that for almost every $x \notin \Omega^n$ we have $f(x) \leq t\lambda_n$. If $x \notin \Omega^n$, then for every $r \in (0, r_0/M)$ we have $m_{TB(x,r)}(f) \leq t\lambda_n$ and from Lemma (1.12) we get the desired result. ∎

Given a measurable function $g : X \to \mathbb{R}^+ \cup \{0\}$ the distribution function of g is given by $\eta(\lambda) = \mu(\{x \in X : g(x) > \lambda\})$, $\lambda \geq 0$. The function η is decreasing and continuous to the right. The function

$$\psi(s)=\sup\{\lambda:\eta(\lambda)>s\}$$

is the nonincreasing rearrangement of g in the sense that ψ and g are equidistributed even when g is defined in an abstract measure space and ψ is defined on $\mathbb{R}^+\cup\{0\}$.

(1.28) THEOREM. *Let (X,d,μ) be a space of homogeneous type with μ regular. Let T be a lag mapping on (X,d). Let ϕ be an increasing function satisfying the Δ_2 condition. Let $f\in \mathrm{BMO}(\phi,T)$. Let ψ_B be the nonincreasing rearrangement of $[f-m_{T\widetilde{B}}(f)]^+$ on $B=B(x,r)$ with $\widetilde{B}=B(x,Nr)$. Then there exist constants α, β and γ such that*

$$\psi_B(\tau)\le\beta\int_{(r/M)[\tau/\gamma\mu(B)]^\alpha}^{r}\frac{\phi(\xi)}{\xi}\,d\xi\quad \textit{for } 0<\tau<\gamma\mu(B). \tag{1.29}$$

Proof. Let $B=B(x,r)$ be a given ball in (X,d,μ). Since $f\in \mathrm{BMO}(\phi,T)$ if and only if $f-\mathrm{const}\in\mathrm{BMO}(\phi,t)$ with the same bounds, we can assume that $m_{T\widetilde{B}}(f)=0$ with $\widetilde{B}=B(x,Nr)$ and so we can apply Lemma (1.27). Given $0<s<1$, take $n\in\mathbb{N}$ such that $1/2^n<s\le 1/2^{n-1}$. Since $\mu(\{x\in B: f^+>t\lambda_n\})\le(\widetilde{C}/2^n)\mu(B)$, for the rearrangement of $f^+=[f-m_{T\widetilde{B}}(f)]^+$ on B we have

$$\psi_B(sC\mu(B))\le t\lambda_n\le C'\sum_{k=1}^{n}\phi\left(\frac{r}{M_r}\right)\le\beta\int_{(r/M)s^\alpha}^{r}\frac{\phi(\xi)}{\xi}\,d\xi,$$

from which the theorem follows. ∎

(1.30) THEOREM. *Let (X,d,μ), T, ϕ and f be as in Theorem (1.28). Assume that T is one-to-one and onto $X\times\mathbb{R}^+$. Let Ψ_B be the nonincreasing rearrangement of $[m_{T^{-1}\widetilde{B}}(f)-f]^+$ on B. Then Ψ_B satisfies an estimate like* (1.29):

$$\Psi_B(\tau)\le\beta\int_{(r/M)[\tau/\gamma\mu(B)]^\alpha}^{r}\frac{\phi(\xi)}{\xi}\,d\xi. \tag{1.31}$$

Proof. First observe that the inverse T^{-1} of T is also a lag mapping and that $-f\in\mathrm{BMO}(\phi,T^{-1})$. In fact, taking $\widetilde{C}_B=-C_{T^{-1}B}$, from (1.4) and (1.5) we get

$$m_B((-f-\widetilde{C}_B)^+)=m_{T(T^{-1}B)}((C_{T^{-1}B}-f)^+)\le D\phi(r),$$
$$m_{T^{-1}B}((\widetilde{C}_B-(-f))^+)=m_{T^{-1}B}((f-C_{T^{-1}B})^+)\le D\phi(r),$$

so that Theorem (1.28) can be applied to $-f$ and T^{-1}. The function $[-f-m_{T^{-1}\widetilde{B}}(-f)]^+$ coincides with $[m_{T^{-1}\widetilde{B}}(f)-f]^+$ and thus its distribution

function on $B=B(x,r)$, $\Psi_B(\tau)$, satisfies

$$\Psi_B(\tau)\le\beta\int_{(r/M)[\tau/\gamma\mu(B)]^\alpha}^{r}\frac{\phi(\xi)}{\xi}\,d\xi. \ \blacksquare$$

Given a ball $B=B(x,r)$ we are using the notation $\widetilde{B}$ for $B(x,Nr)$. Let us write B' for $B(x,r/N)$. Given a lag mapping T on (X,d) such that T^{-1} exists, for $x\in X$ we define

$$S^+(x)=\{y\in X:\ \text{there exists a ball } B \text{ with } x\in B \text{ and } y\in(T^2\widetilde{B})'\},$$
$$S^-(x)=\{y\in X:\ \text{there exists a ball } B \text{ with } x\in B \text{ and } y\in(T^{-2}\widetilde{B})'\}.$$

Notice that $y\in S^+(x)$ if and only if $x\in S^-(y)$. Let, then, S be the set of points $(x,y)\in X\times X$ such that $y\in S^+(x)$ (or $x\in S^-(y)$). For $(x,y)\in S$ define $\Delta(x,y)=\inf\{r>0: r=r(B) \text{ with } x\in B \text{ and } y\in(T^2\widetilde{B})'\}$.

(1.32) COROLLARY. *Let (X,d,μ), T, ϕ and f be as in Theorem (1.30). Assume that $\int_0^1(\phi(\xi)/\xi)\,d\xi<\infty$. Then*

$$[f(x)-f(y)]^+\le C\int_0^{\Delta(x,y)}\frac{\phi(\xi)}{\xi}\,d\xi,\qquad (x,y)\in S. \tag{1.33}$$

In the special case $\phi(\xi)=\xi^\beta$, $0<\beta$, we have $[f(x)-f(y)]^+\le C\Delta(x,y)^\beta$, $(x,y)\in S$. If, moreover, $S=X\times X$, then $|f(x)-f(y)|\le C\Delta(x,y)^\beta$ for every x and every y in X.

Proof. Let $(x,y)\in S$. Let B be a ball such that $x\in B$ and $y\in(T^2\widetilde{B})'$. Then

$$\begin{aligned}[f(x)-f(y)]^+&\le[f(x)-m_{T\widetilde{B}}(f)]+[m_{T^{-1}(T^2\widetilde{B})}(f)-f(y)]^+\\&\le\psi_B(0)+\psi_{(T^2\widetilde{B})'}(0)\le C\int_0^{r(B)}\frac{\phi(\xi)}{\xi}\,d\xi.\end{aligned}$$

Taking the infimum over the family of balls B such that $x\in B$ and $y\in(T^2\widetilde{B})'$ we get the desired result. ∎

If in some subset A of S we have a bound of the form

$$\Delta(x,y)\le Cd(x,y),\qquad (x,y)\in A,$$

then the result in Corollary (1.32) applies to provide Hölder type estimates

$$[f(x)-f(y)]^+\le C\int_0^{d(x,y)}\frac{\phi(\xi)}{\xi}\,d\xi,\qquad (x,y)\in A, \tag{1.34}$$

$$[f(x)-f(y)]^+\le Cd(x,y)^\beta,\qquad (x,y)\in A,\ \text{when } \phi(\xi)=\xi^\beta, \tag{1.35}$$

$$(1.36)\qquad |f(x)-f(y)|\le Cd(x,y)^{\beta}\,,\qquad x,y\in X\,,$$

when $\phi(\xi)=\xi^{\beta}$ and $A=X\times X$.

Remark. It is clear from Hölder's inequality that if instead of (1.4) and (1.5) we have

$$m_B(((f-C_B)^+)^p)^{1/p}\le D\phi(r)$$

and

$$m_B(((C_B-f)^+)^p)^{1/p}\le D\phi(r)\,,$$

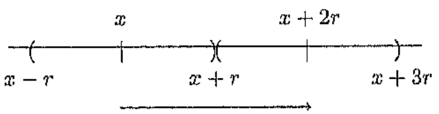
$p>1$, we have the same results. It is also true that we can get similar results when $0<p<1$ or even when instead of $g(t)=(t^+)^p$ we take some function satisfying $h(t)=0$ if $t<0$, $h(t)$ increasing for $t>0$, $h(t+s)\le h(t)+h(s)$ and $e^{-\varepsilon h(t)}$ integrable on $(0,\infty)$ for every $\varepsilon>0$ (see [A2] for the BMO case).

2. Examples and applications

(2.1) When $T:X\times\mathbb{R}^+\to X\times\mathbb{R}^+$ is the identity we get the "elliptic" results contained in [A2]. See also [MS].

(2.2) When $\phi\equiv 1$ we get the "elliptic and parabolic" versions of the John–Nirenberg theorem [A1].

(2.3) $X=\mathbb{R}$, d the usual distance in $\mathbb{R}$ and μ the Lebesgue measure. (X,d,μ) is a space of homogeneous type. Let $T:\mathbb{R}\times\mathbb{R}^+\cup\{0\}\to\mathbb{R}\times\mathbb{R}^+\cup\{0\}$ be given by $T(x,r)=(x+2r,r)$, so that, for $B=B(x,r)$, $TB=B(x+2r,r)$.



It is clear that, given $x\in\mathbb{R}$, we have $S^+(x)=\{y:y>x\}$, $S^-(x)=\{y:y<x\}$ and $S=\{(x,y):y>x\}$. On the other hand, for $(x,y)\in S$ we have $\Delta(x,y)=(y-x)/2<d(x,y)$. Then the results of Section 1 can be applied. If $\phi\equiv 1$, we have a John–Nirenberg type lemma which was also proved by F. Martín-Reyes and A. de la Torre. Let ϕ be any function satisfying the Δ_2 condition. Then the family of increasing functions is contained in $\mathrm{BMO}(\phi,T)$. In fact, for $I=(a,b)$ and $C_I=f(b)$ we have

$$m_I((f-C_I)^+)=m_{TI}((C_I-f)^+)=0\,.$$

When $\phi(s)=s^{\beta}$, $\beta>0$, we have

$$[f(x)-f(y)]^+\le Cd(x,y)^{\beta}\qquad\text{for } x<y\,,$$

in other words,

$$[f(x)-f(y)]^+\le C(y-x)^{\beta}\,,\qquad y>x\,,$$

which is a one-sided Hölder type condition for f.

(2.4) *The spaces* BMO *associated with parabolic diferential equations.* Let (X,d,μ) be a space of homogeneous type, let $\mathcal{B}$ be the class of all d-balls on (X,d) and let h be a function from $\mathcal{B}$ to $\mathbb{R}^+\cup\{0\}$. A function $v:X\times\mathbb{R}\to\mathbb{R}$ belongs to $\mathrm{BMOP}(\varphi)$ if and only if there is a constant C and, for every $B\in\mathcal{B}$, a function $V(t)=V_B(t)$ of class $C^1(\mathbb{R})$ such that

$$h(B)\frac{dV}{dt}+\frac{1}{\mu(B)}\int_B|v(x,t)-V(t)|^2\,d\mu(x)\le C\varphi(r)\,.$$

Given a ball $B\in\mathcal{B}$ and a real number t, put $t_1=t-\frac12 h(B)/\varphi(r)^{1/2}$, $t_2=t+\frac12 h(B)/\varphi(r)^{1/2}$, $t_3=t-\frac32 h(B)/\varphi(r)^{1/2}$, $R^+=B\times(t_1,t_2)$ and $R^-=B\times(t_3,t_1)$. We have the following.

Lemma. *Let $v\in\mathrm{BMOP}(\varphi)$. Then there exists C such that for every ball $B\in\mathcal{B}$ there is a $U\in\mathbb{R}$ satisfying*

$$\frac{1}{(\mu\times\lambda)(R^+)}\iint_{R^+}\sqrt{(v-U)^+}\,d\mu\,d\lambda\le C\varphi(r)^{1/2}\,,$$

$$\frac{1}{(\mu\times\lambda)(R^+)}\iint_{R^+}\sqrt{(U-v)^+}\,d\mu\,d\lambda\le C\varphi(r)^{1/2}\,.$$

Proof. Let $x_0\in X$, $r>0$ and $t_0\in\mathbb{R}$ be given. Write $B_0=B(x_0,r)$. Since $h(B_0)\frac{dV}{dt}\le C\varphi(t)$, for $t\in(t_1,t_2)$ we have

$$V-V_1=V(t)-V(t_1)\le\frac{C\varphi(r)}{h(B_0)}(t-t_1)\le C\varphi(r)^{1/2}\,.$$

For $s>3C\varphi(r)^{1/2}$ define

$$B_s(t)=\{x\in B_0:v(x,t)-V_1>s\}\,.$$

If $x\in B_s(t)$ we have $v(x,t)-V>s+V_1-V\ge s-C\varphi(r)^{1/2}>0$, so that, from the definition of $\mathrm{BMOP}(\varphi)$ it follows that

$$\frac{\mu(B_s(t))}{\mu(B_0)}\le\frac{C\varphi(r)}{(s-V_1-V)^2}-h(B_0)\frac{d}{dt}\left(\frac{1}{V-V_1-s}\right).$$

Integrating from t_1 to t_2 we get

$$\frac{1}{\mu(B_0)}\int_{t_1}^{t_2}\mu(B_s(t))\,dt\le 2h(B_0)\frac{1}{s-C\varphi(r)^{1/2}}\,.$$

We can now compute $\iint_{R^+}\{(v-V_1)^+\}^{1/2}\,d\mu\,d\lambda$ by using the distribution function:

$$\int_0^\infty s^{-1/2}(\mu\times\lambda)(\{(x,t)\in R^+ : (v-V_1)^+ > s\})$$
$$\le C\varphi(r)^{1/4}(\mu\times\lambda)(R^+ + C\mu(B_0)h(B_0))\int_{3C\varphi(r)^{1/2}}^{\infty}\frac{ds}{s^{1-\alpha}(s-C\varphi(r)^{1/2})}$$
$$\le C(\mu\times\lambda)(R^+)\varphi(r)^{1/2}.$$

The second inequality follows in the same way. ∎

For $X=\mathbb{R}^n$, d the euclidean distance, μ the Lebesgue measure, $h(B)=r(B)^2$ and $\varphi\equiv 1$, we have the result of J. Moser [M2], i.e. a John–Nirenberg type lemma which allows one to prove A_2 type conditions for small powers of e^{-v}.

For $X=\mathbb{R}^n$, d the euclidean distance, $d\mu(x)=w(x)dx$ with $w\in A_{1+2/n}$ and $h(B)=r(B)^2|B|/\mu(B)$ we are in the setting of degenerate parabolic equations first considered in [CS].

References

[A1] H. Aimar, *Rearrangement and continuity properties of* BMO(ϕ) *functions on spaces of homogeneous type*, Ann. Scuola Norm. Sup. Pisa, to appear.

[A2] —, *Elliptic and parabolic* BMO *and Harnack's inequality*, Trans. Amer. Math. Soc. 306 (1988), 265–276.

[B] N. Burger, *Espace des fonctions à moyenne bornée sur un espace de nature homogène*, C. R. Acad. Sci. Paris Sér. A 286 (1978), 139–142.

[C] S. Campanato, *Proprietà di hölderianità di alcune classi di funzioni*, Ann. Scuola Norm. Sup. Pisa 17 (1963), 175–188.

[CS] F. Chiarenza and R. Serapioni, *A Harnack inequality for degenerate parabolic equations*, Comm. Partial Differential Equations 9 (1984), 719–749.

[CW] R. Coifman et R. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. 242, Springer, Berlin 1972.

[FG] E. Fabes and N. Garofalo, *Parabolic* BMO *and Harnack's inequality*, Proc. Amer. Math. Soc. 95 (1985), 63–69.

[JN] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 14 (1961), 415–426.

[Me] G. Meyers, *Mean oscillation over cubes and Hölder continuity*, Proc. Amer. Math. Soc. 15 (1964), 717–724.

[M1] J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. 14 (1961), 577–591.

[M2] —, *A Harnack inequality for parabolic differential equations*, ibid. 17 (1964), 101–134; Correction, ibid. 20 (1967), 232–236.

[MS] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Adv. in Math. 33 (1979), 257–270.

[MT] F. Martín-Reyes and A. de la Torre, preprint.

[N] U. Neri, *Some properties of functions with bounded mean oscillation*, Studia Math. 61 (1977), 63–75.

[S] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa 19 (1965), 593–608.

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