STUDIA MATHEMATICA 106 (3) (1993)

On the weak $(1,1)$ boundedness of a class of oscillatory singular integrals

by

YIBIAO PAN (Pittsburgh, Penn.)

Abstract. We prove the uniform weak $(1,1)$ boundedness of a class of oscillatory singular integrals under certain conditions on the phase functions. Our conditions allow the phase function to be completely flat. Examples of such phase functions include $\phi(x) = e^{-1/x^2}$ and $\phi(x) = xe^{-1/|x|}$. Some related counterexample is also discussed.

1. Introduction. Oscillatory singular integrals have arisen in many problems in Fourier analysis and related areas. Such operators have been studied by many authors ([1], [2], [4], [5], [8]–[14]). In this article we shall focus on the study of the weak $(1,1)$ boundedness of such operators.

For a given function f, we let λ_f be its distribution function, i.e. $\lambda_f(s) = m\{x : f(x) > s\}$. The function f is said to be in $L^{1,\infty}$ if $\|f\|_{1,\infty} = \sup_s s\lambda_f(s) < \infty$. An operator is said to be of weak type $(1,1)$ if it is bounded from L^1 to $L^{1,\infty}$.

Let $\lambda \in \mathbb{R}$, ϕ be a real-valued function and $\varphi \in C_0^\infty(\mathbb{R})$. Define the operator T_λ by

$$T_\lambda f(x) = \text{p.v.} \int_{\mathbb{R}} e^{i\lambda\phi(x-y)} \frac{\varphi(x-y)}{x-y} f(y)\, dy, \tag{1}$$

where $x \in \mathbb{R}$. We now state our main result.

THEOREM A. *Suppose $\phi \in C^3([0,d])$ for some $d > 0$ and satisfies* (i) *ϕ is even or odd*; (ii) *$\phi'''(t) \geq 0$, for $0 \leq t \leq d$. Then the operators T_λ are uniformly bounded from L^1 to $L^{1,\infty}$, i.e. there exists a constant C which is independent of λ such that*

$$m\{x : |T_\lambda f(x)| > \alpha\} \leq C\alpha^{-1}\|f\|_1 \tag{2}$$

for all $\alpha > 0$, $f \in L^1$.

Earlier results on the weak $(1,1)$ estimates for oscillatory singular integrals can be found in [1], [2] and [10], among others. Recall that in [12]

1991 *Mathematics Subject Classification*: Primary 42B20.

Ricci and Stein proved the L^p boundedness of oscillatory singular integrals with polynomial phases. Shortly thereafter, Chanillo and Christ proved the weak $(1,1)$ boundedness for operators with polynomial phases ([1]). In [10], the author obtained the uniform weak $(1,1)$ estimate for the operators T_λ with real-analytic phase functions. Actually, the estimates obtained in [10] are valid for all dimensions and the analyticity of the phase was used in an essential way only for dimensions greater than one. For $n=1$, the method in [10] can be used to prove the weak $(1,1)$ estimates for T_λ with smooth phases. Namely, we have

THEOREM B. *Let T_λ be given by* (1). *Suppose there is an integer $k \geq 2$, such that ϕ is in C^{2k-1} on* $\operatorname{supp}(\varphi)$ *and $\phi^{(k)}(0) \neq 0$. Then the operators T_λ are uniformly bounded from L^1 to $L^{1,\infty}$.*

Although Theorem B has not appeared before, its proof does not require new ideas. Therefore, we decide not to include a proof for Theorem B here. The reader may consult [8] and [10].

If the phase function ϕ satisfies $\phi^{(j)}(0)=0$ for $j=2,3,\ldots$, the operators T_λ may fail to be uniformly bounded from L^1 to $L^{1,\infty}$. An example will be given by using a function constructed by Nagel and Wainger in [7]. This shows that Theorem B becomes false if the condition $\phi^{(k)}(0) \neq 0$ (for some $k \geq 2$) is removed. The phase function in the example is also convex, which shows that condition (ii) in Theorem A cannot be replaced by $\phi'' \geq 0$.

On the other hand, Theorem A implies that the uniform boundedness of T_λ from L^1 to $L^{1,\infty}$ holds for many phase functions with vanishing derivatives at $x=0$. Examples of such phase functions include $\phi(x)=e^{-1/x^2}$ and $\phi(x)=xe^{-1/|x|}$.

2. Some reductions and lemmas. To prove Theorem A, first we make a few simple reductions. Since the singularities of the kernel of T_λ are along the line $\{x=y\}$, by using appropriate cut-off functions, we may assume that $\varphi(x) \equiv 1$ near the origin. Without loss of generality, we shall assume that ϕ is defined on $[-1,1]$ and T_λ is of the following form:

$$T_\lambda f(x) = \text{p.v.} \int\limits_{|x-y|\leq 1} e^{i\lambda\phi(x-y)} \frac{1}{x-y} f(y)\,dy. \tag{3}$$

In order to prove Theorem A, it suffices to show that T_λ are uniformly bounded from L^1 to $L^{1,\infty}$ provided that ϕ satisfies the following conditions:

(C1) ϕ is even or odd;
(C2) $\phi'''(t) \geq 0$, for $t \in [0,1]$;
(C3) $\phi(0)=\phi'(0)=\phi''(0)=0$.

Let us clarify condition (C3). One may assume that $\phi(0)=\phi'(0)=0$ because replacing the phase function $\phi(x-y)$ by $\phi(x-y)-(\phi(0)+\phi'(0)(x-y))$ does not change the operator norm. One may further assume that $\phi''(0)=0$ because, if $\phi''(0) \neq 0$, the uniform boundedness of T_λ from L^1 to $L^{1,\infty}$ would follow automatically from Theorem B.

We now recall a few known facts.

LEMMA 1 (van der Corput). *Suppose ψ is smooth and real-valued on $[a,b]$. If $|\psi'(x)| \geq \lambda$, and ψ' is monotone on $[a,b]$, then*

$$\Big| \int_a^b e^{i\psi(x)}\,dx \Big| \leq 4\lambda^{-1}. \tag{4}$$

LEMMA 2. *Assume that ϕ satisfies* (C1)–(C3). *Let T_λ be given as in* (3). *Then T_λ are uniformly bounded on $L^2(\mathbb{R})$, i.e. there exists a constant C which is independent of λ such that*

$$\|T_\lambda f\|_{L^2(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})} \tag{5}$$

for $f \in L^2(\mathbb{R})$, $\lambda \in \mathbb{R}$.

Proof. If $\phi''(t_0)=0$, for some $t_0>0$, then $\phi''(t) \equiv 0$ for $|t| \leq t_0$, which implies that ϕ is identically zero in $[-t_0,t_0]$. In this case, the uniform boundedness of $\|T_\lambda\|_{2,2}$ is well-known.

We now assume that $\phi''(t)>0$ for $t \in [0,1]$. Since $\phi'(t)/t$ is increasing in $[0,1]$, we have

$$\phi'(2t) > 2\phi'(t). \tag{6}$$

By Plancherel's Theorem, we find

$$\sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = \|H_\phi\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)}, \tag{7}$$

where H_ϕ is the Hilbert transform along the curve $t \to (t,\phi(t))$ in $\mathbb{R}^2$, i.e.

$$H_\phi g(x_1,x_2) = \int_{-1}^{1} g(x_1-t, x_2-\phi(t))\frac{dt}{t}. \tag{8}$$

By (C1), (6)–(8) and a result of Nagel, Vance, Wainger and Weinberg ([6], Theorems 1, 2 and Lemma 2), we find

$$\sup_{\lambda\in\mathbb{R}} \|T_\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} < \infty. \ \blacksquare \tag{9}$$

3. The proof of Theorem A. Let $f \in L^1(\mathbb{R})$. For $\alpha>0$, we make a Calderón–Zygmund decomposition $f = g + \sum_j b_j$, where

(i) $\|g\|_\infty < \alpha$, and $\|g\|_1 \leq \|f\|_1$;
(ii) b_j is supported on an interval I_j, with $|I_j| = 2^{n_j}$, and the I_j are pairwise disjoint;

(iii) $(1/|I_j|)\int_{I_j}|b_j(y)|dy \le C\alpha$, $\sum_j |I_j| \le C\alpha^{-1}\|f\|_1$;
(iv) $d(I_i, I_j) \le 2^{n_i}$ implies $|n_i - n_j| \le N_0$, for some fixed N_0.

This decomposition may be obtained via a Whitney decomposition of the set where the Hardy–Littlewood maximal function of f is greater than α ([1], [15]).

Let $\lambda > 0$ be sufficiently large and $\omega = \phi^{-1}(1/\lambda)$. Define S_λ^1 and S_λ^2 by

$$S_\lambda^1 f(x) = \int\limits_{\omega \le |x-y| \le 1} \frac{e^{i\lambda\phi(x-y)}}{x-y} f(y)\,dy\,, \tag{10}$$

$$S_\lambda^2 f(x) = T_\lambda f(x) - S_\lambda^1 f(x). \tag{11}$$

Let

$$H_a f(x) = \text{p.v.} \int\limits_{|x-y|\le a} \frac{f(y)}{x-y}\,dy\,. \tag{12}$$

Then we have

LEMMA 3. *Let M be the Hardy–Littlewood maximal operator. Then*

$$|S_\lambda^2 f(x)| \le 16Mf(x) + |H_{\omega/4} f(x)|\,.$$

P r o o f. By the definition, we have

$$\begin{aligned}
|S_\lambda^2 f(x)| &= \Bigg|\int\limits_{|x-y|\le\omega} \frac{e^{i\lambda\phi(x-y)}}{x-y} f(y)\,dy\Bigg| \\
&\le \int\limits_{|x-y|\le\omega/4} \Bigg|\frac{e^{i\lambda\phi(x-y)}-1}{x-y}\Bigg| |f(y)|\,dy \\
&\quad + |H_{\omega/4} f(x)| + \int\limits_{\omega/4\le|x-y|\le\omega} \Bigg|\frac{f(x)}{x-y}\Bigg|\,dy \\
&\le \lambda \int\limits_{|x-y|\le\omega/4} \Bigg|\frac{\phi(x-y)}{x-y}\Bigg| |f(y)|\,dy + |H_{\omega/4} f(x)| + 8Mf(x)\,.
\end{aligned} \tag{13}$$

Since ϕ is increasing on $[0,1]$, we find

$$\begin{aligned}
\int\limits_{|x-y|\le\omega/4} \Bigg|\frac{\phi(x-y)}{x-y}\Bigg| |f(y)|\,dy &\le \sum_{2^j\le\omega/2}\ \int\limits_{2^{j-1}\le|x-y|\le 2^j} \frac{|\phi(x-y)|}{|x-y|} |f(y)|\,dy \\
&\le \sum_{2^j\le\omega/2} \frac{\phi(2^j)}{2^{j-1}} \int\limits_{|x-y|\le 2^j} |f(y)|\,dy \\
&\le 8Mf(x) \sum_{2^j\le\omega/2}\ \int\limits_{2^j}^{2^{j+1}} \frac{\phi(t)}{t}\,dt \\
&\le 8Mf(x) \int\limits_0^\omega \phi'(t)dt \le 8\lambda^{-1}Mf(x)\,.
\end{aligned} \tag{14}$$

By (13) and (14), the lemma is proved. ∎

R e m a r k. By Lemmas 2 and 3, we conclude that both S_λ^1 and S_λ^2 are uniformly bounded on $L^2(\mathbb{R})$.

For each j, let $\omega_j = \max\{\omega, 2^{n_j}\}$, and

$$T_{\lambda,j}^0 f(x) = \int\limits_{\omega_j\le x-y\le 1} \frac{e^{i\lambda\phi(x-y)}}{x-y} f(y)\,dy\,, \tag{15}$$

$$T_{\lambda,j}^1 f(x) = \int\limits_{-1\le x-y\le -\omega_j} \frac{e^{i\lambda\phi(x-y)}}{x-y} f(y)\,dy\,. \tag{16}$$

We have

LEMMA 4. *There is a constant C which is independent of λ and α such that*

$$\Big\|\sum_j T_{\lambda,j}^t b_j\Big\|_2^2 \le C\alpha\|f\|_1 \tag{17}$$

for $t = 0, 1$.

P r o o f. We shall prove (17) for $t = 0$. The proof for $t = 1$ is similar. Let $L_{\lambda,i,j}(x,y)$ be the kernel of $(T_{\lambda,i}^0)^* T_{\lambda,j}^0$. Then, we have

$$L_{\lambda,i,j}(x,y) = \int\limits_{\omega_i\le z-x\le 1,\ \omega_j\le z-y\le 1} e^{i\lambda(\phi(z-y)-\phi(z-x))} \frac{dz}{(z-y)(z-x)}\,.$$

Suppose $\omega_i \ge \omega_j$. We shall need the following two inequalities:

$$|L_{\lambda,i,j}(x,y)| \le C\omega_i^{-1}(1 + \ln(\omega_i/\omega_j))\,, \tag{18}$$

$$|L_{\lambda,i,j}(x,y)| \le C(\lambda\phi''(\omega)\omega)^{-1}|x-y|^{-2}\,. \tag{19}$$

(18) can be proved by taking the absolute value of the integrand and the argument is straightforward. To prove (19), we let $A = \max\{\omega_i + x, \omega_j + y\}$, $B = \min\{1 + x, 1 + y\}$, and

$$J_r = \int\limits_A^r e^{i\lambda(\phi(z-y)-\phi(z-x))}\,dz\,,$$

for $A \le r \le B$. We find

$$\left|\frac{\partial}{\partial z}(\phi(z-y)-\phi(z-x))\right| \ge \phi''(\omega)|x-y|,$$

$$\frac{\partial^2}{\partial z^2}(\phi(z-y)-\phi(z-x)) = \phi'''(\xi)(x-y),$$

for some $\xi = \xi(x,y,z)$ which lies between $z-x$ and $z-y$. By van der Corput's lemma, we have

$$|J_r| \le C(\lambda\phi''(\omega)|x-y|)^{-1}. \tag{20}$$

By (20) and integration by parts, we find

$$|L_{\lambda,i,j}(x,y)| = \left|\int_A^B \frac{d}{dz}\Big(\int_A^z e^{i\lambda(\phi(t-y)-\phi(t-x))}\,dt\Big)\frac{dz}{(z-y)(z-x)}\right| \le C(\lambda\phi''(\omega)\omega)^{-1}|x-y|^{-2},$$

which proves (19). Now we prove that, for fixed i, $x \in I_i$, the following holds:

$$\sum_{j:\omega_j\le\omega_i}\left|\int_{I_j} L_{\lambda,i,j}(x,y)b_j(y)\,dy\right| \le C\alpha, \tag{21}$$

where the constant C is independent of i, α and x. To prove (21), we break the summation into two parts

$$\sum_{j:\omega_j\le\omega_i} = \sum_{I_j\in S_i} + \sum_{I_j\in F_i}, \tag{22}$$

where $S_i = \{I_j \mid \operatorname{dist}(I_j,I_i)\le\omega_i,\ \omega_j\le\omega_i\}$ and $F_i = \{I_j \mid \operatorname{dist}(I_j,I_i)>\omega_i,\ \omega_j\le\omega_i\}$.

If $\omega_i = 2^{n_i}$ and $I_j\in S_i$, then $1\le|\omega_i/\omega_j|\le 2^{N_0}$, $|S_i|\le 2^{N_0}$. By (18), we get

$$\left|\sum_{I_j\in S_i}\int L_{\lambda,i,j}(x,y)b_j(y)\,dy\right| \le C\alpha. \tag{23}$$

If $\omega_i=\omega$ and $I_j\in S_i$, then $\omega_j=\omega_i$. Hence

$$\left|\sum_{I_j\in S_i}\int L_{\lambda,i,j}(x,y)b_j(y)\,dy\right| \le C\omega^{-1}\alpha\sum_{I_j\in S_i}|I_j| \le C\alpha. \tag{24}$$

To treat the other part, we use (19). For $x\in I_i$, $I_j\in F_i$, and $y\in I_j$, we have $|x-y|\le C\min_{z\in I_j}|x-z|$ and

$$\left|\int L_{\lambda,i,j}(x,y)b_j(y)\,dy\right| \le C(\lambda\phi''(\omega)\omega)^{-1}\int_{I_j}\frac{|b_j(y)|}{|x-y|^2}\,dy \le C\alpha(\lambda\phi''(\omega)\omega)^{-1}\int_{I_j}|x-y|^{-2}\,dy.$$

Hence

$$\sum_{I_j\in F_i}\left|\int L_{\lambda,i,j}(x,y)b_j(y)\,dy\right| \le C\alpha(\lambda\phi''(\omega)\omega)^{-1}\int_{|x-y|>\omega}|x-y|^{-2}\,dy = C\alpha(\lambda\phi''(\omega)\omega^2)^{-1} \le C\alpha(\lambda\phi(\omega))^{-1} = C\alpha. \tag{25}$$

(21) now follows from (22)–(25). By (21), we observe that

$$\Big\|\sum_j T^0_{\lambda,j}b_j\Big\|_2^2 \le 2\sum_{\omega_j\le\omega_i}|\langle T^0_{\lambda,i}b_i, T^0_{\lambda,j}b_j\rangle| \le C\alpha\sum_i\int|b_i(x)|\,dx \le C\alpha^2\sum_i|I_i| \le C\alpha\|f\|_1, \tag{26}$$

which concludes the proof of the lemma. ∎

We are now ready to prove Theorem A.

Proof Theorem A. For $\alpha>0$, we have

$$m\{x:|T_\lambda f(x)|>\alpha\} \le m\{x:|S^1_\lambda f(x)|>\alpha/2\} + m\{x:|S^2_\lambda f(x)|>\alpha/2\}.$$

By Lemma 3 and the well-known boundedness properties of the operators M and H_a, we can find a constant C such that

$$m\{x:|S^2_\lambda f(x)|>\alpha/2\} \le C\alpha^{-1}\|f\|_1, \tag{27}$$

where C is independent of λ. To obtain a similar estimate for $S^1_\lambda f$, we write $f = g+\sum_j b_j$. Let I_j^* be the 4-fold dilates of the I_j and $\Omega = (\bigcup_j I_j^*)^c$. Then we have

$$m\{x:|S^1_\lambda g(x)|>\alpha/4\} \le C\alpha^{-2}\|S^1_\lambda g\|_2^2 \le C\alpha^{-2}\|g\|_2^2 \le C\alpha^{-1}\|f\|_1, \tag{28}$$

where we used the uniform L^2 boundedness of S^1_λ. To finish the proof, it suffices to show that

$$m\Big\{x\in\Omega:\Big|S^1_\lambda\Big(\sum_j b_j\Big)(x)\Big|>\alpha/4\Big\} \le C\alpha^{-1}\|f\|_1. \tag{29}$$

Since $S^1_\lambda(\sum_j b_j)(x) = \sum_j(T^0_{\lambda,j}b_j)(x) + \sum_j(T^1_{\lambda,j}b_j)(x)$, for $x\in\Omega$. By Lemma 4, we find

$$m\Big\{x\in\Omega:\Big|S^1_\lambda\Big(\sum_j b_j\Big)(x)\Big|>\alpha/4\Big\} \le C\alpha^{-2}\Big\|\sum_j T^0_{\lambda,j}b_j + \sum_j T^1_{\lambda,j}b_j\Big\|_2^2 \le C\alpha^{-1}\|f\|_1.$$

The proof is now complete. ∎

Remark. The idea of using an $L^1\to L^2$ argument to prove weak type bounds for oscillatory integrals was used in [1] and [3]. See also [10].

4. An example

THEOREM C. *There exists an odd, C^∞ function $\phi(t)$, defined for $-1 \le t \le 1$, with $\phi''(t) \ge 0$ for $t > 0$, such that T_λ are not uniformly bounded from L^1 to $L^{1,\infty}$.*

To prove Theorem C, we shall use a function which was constructed in [7] by Nagel and Wainger. Let $\{r_n\}$ be a sequence of positive numbers such that $r_1 \le 1$ and $r_{n+1} \le r_n/4(n+1)$. Let $B_n = (r_n, nr_n)$ and $B'_n = ((n+1)r_{n+1}, r_n)$. Let $h(t)$ be a C^∞ function on $[-1,1]$ such that (i) h is odd; (ii) $h(t) > 0$ for $t \in \bigcup_{n=1}^\infty B'_n$; (iii) $h(t) = 0$ for $t \in \bigcup_{n=1}^\infty B_n$. Define the function ϕ by

$$\phi(t) = \int_0^t \int_0^s h(\tau)\, d\tau\, ds\,. \tag{30}$$

Clearly, $\phi \in C^\infty$. Since $\phi''(t) = 0$ for $t \in B_n$, there exist constants σ_n and ζ_n such that $\phi(t) = \sigma_n t + \zeta_n$, for $t \in B_n$. The following estimate can be found in [7].

PROPOSITION 1 ([7], p. 248). *There exists a sequence of positive numbers $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and*

$$\left| \int_0^1 \sin(\lambda_n \phi(t) - \sigma_n \lambda_n t)\, \frac{dt}{t} \right| > (\ln n)/2\,,$$

for $n = 1, 2, \ldots$

Proof of Theorem C. Let ϕ be given as in (30). For $\beta \in \mathbb{R}$, let $f_\beta(x) = \chi_{[-2,2]}(x) e^{i\beta x}$. Then we have, for $x \in [-1,1]$,

$$\begin{aligned} T_\lambda f_\beta(x) &= \int_{|x-y|\le 1} \frac{e^{i\lambda\phi(x-y)}}{x-y} f_\beta(y)\, dy \\ &= \int_{-1}^{1} e^{i\lambda\phi(t)} e^{i\beta(x-t)}\, \frac{dt}{t} \\ &= 2ie^{i\beta x} \int_0^1 \sin(\lambda\phi(t) - \beta t)\, \frac{dt}{t}\,. \end{aligned}$$

Let $\beta_n = \lambda_n \sigma_n$. By Proposition 1, we find

$$m\{x : |T_{\lambda_n} f_{\beta_n}(x)| > \ln n\} \ge m([-1,1]) = 2\,,$$

while $\|f_{\beta_n}\|_1 = 4$. Therefore, (2) cannot hold uniformly in λ. ∎

Remark. It is easy to see that the function ϕ defined in (30) satisfies $\phi^{(k)}(0) = 0$, for $k \ge 0$.

References

[1] S. Chanillo and M. Christ, *Weak (1,1) bounds for oscillatory singular integrals*, Duke Math. J. 55 (1987), 141–155.

[2] S. Chanillo, D. Kurtz and G. Sampson, *Weighted weak (1,1) and weighted L^p estimates for oscillating kernels*, Trans. Amer. Math. Soc. 295 (1986), 127–145.

[3] C. Fefferman, *Inequalities for strongly singular convolution operators*, Acta Math. 124 (1970), 9–36.

[4] Y. Hu, *Oscillatory singular integrals on weighted Hardy spaces*, Studia Math. 102 (1992), 145–156.

[5] Y. Hu and Y. Pan, *Boundedness of oscillatory singular integrals on Hardy spaces*, Ark. Mat. 30 (1992), 311–320.

[6] A. Nagel, J. Vance, S. Wainger, and D. Weinberg, *Hilbert transforms for convex curves*, Duke Math. J. 50 (1983), 735–744.

[7] A. Nagel and S. Wainger, *Hilbert transforms associated with plane curves*, Trans. Amer. Math. Soc. 223 (1976), 235–252.

[8] Y. Pan, *Uniform estimates for oscillatory integral operators*, J. Funct. Anal. 100 (1991), 207–220.

[9] —, *Hardy spaces and oscillatory singular integrals*, Rev. Mat. Iberoamericana 7 (1991), 55–64.

[10] —, *Weak (1,1) estimate for oscillatory singular integrals with real-analytic phases*, Proc. Amer. Math. Soc., to appear.

[11] D. H. Phong and E. M. Stein, *Hilbert integrals, singular integrals and Radon transforms I*, Acta Math. 157 (1986), 99–157.

[12] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals, I*, J. Funct. Anal. 73 (1987), 179–194.

[13] P. Sjölin, *Convolution with oscillating kernels on H^p spaces*, J. London Math. Soc. 23 (1981), 442–454 .

[14] E. M. Stein, *Oscillatory integrals in Fourier analysis*, in: Beijing Lectures in Harmonic Analysis, Princeton Univ. Press, Princeton 1986, 307–355.

[15] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton 1970.

[16] A. Zygmund, *Trigonometric Series*, Cambridge Univ. Press, Cambridge 1959.

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH, PENNSYLVANIA 15260
U.S.A.

Received February 18, 1993 (3042)