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INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
Publications Department

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Characterization of weak type by the entropy distribution of r-nuclear operators

by

MARTIN DEFANT and MARIUS JUNGE (Kiel)

Abstract. The dual of a Banach space X is of weak type p if and only if the entropy numbers of an r-nuclear operator with values in a Banach space of weak type q belong to the Lorentz sequence space $\ell_{s,r}$ with $1/s+1/p+1/q=1+1/r$ $(0<r<1,\ 1\le p,q\le 2)$. It is enough to test this for $Y=X^*$. This extends results of Carl, König and Kühn.

Introduction. We show that the notion of weak type p spaces ($1\le p \le 2$) just characterizes the entropy distribution of (r,w)-nuclear operators. Our main results are contained in the following theorem (for precise definitions see below).

THEOREM. *For $1\le p\le 2$ the following are equivalent:*

1) *X^* is of weak type p.*

2) *For all $0<r<1$, $0<s,t\le\infty$, $1\le q\le 2$ with $1/s+1/p+1/q = 1+1/r$ and all Banach spaces Y of weak type q,*
$$\mathfrak{N}_{r,t}(X,Y)\subset\mathfrak{L}^e_{s,t}(X,Y).$$

3) *There are $0<r\le 1$ and $0<s,\ t\le\infty$ with $1/s+2/p=1+1/r$ and*
$$\mathfrak{N}_{r,t}(X,X^*)\subset\mathfrak{L}^e_{s,\infty}(X,X^*).$$

Results in this direction were shown by König [KÖN] and Carl [CA3]. Kühn [KÜH] proved that statement 2) of the above theorem holds provided that X^* is of type p and Y of type q. He also remarked that the parameters r, s, t, p, q are optimal.

The proof of our theorem divides into two cases, $1\le p<2$ and $p=2$. The first uses the generalized Carl–Maurey inequality (see Theorem 1.1), and then follows the proofs of Carl [CA3] and Kühn [KÜH] (see Chapter 1).

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Since there is a gap in the generalized Carl–Maurey inequality for $p=2$ we look in Chapter 2 for an alternative approach by estimating the Kolmogorov numbers in the right way by the Weyl numbers.

Preliminaries. In what follows c_0 always denotes a universal constant. We use standard Banach space notations. In particular, the unit ball of a Banach space X is denoted by B_X. For a closed subspace E of X we denote by

$$\iota_E : E \to X,\ x \mapsto x, \text{ the natural injection,}$$
$$Q_E : X \to X/E,\ x \mapsto x+E, \text{ the natural surjection.}$$

The Lorentz spaces $\ell_{r,w}$, ℓ_r^n, L_r, $0<r,w\le\infty$, $n\in\mathbb{N}$, are defined in the usual way. We also need the vector-valued generalizations $L_p(X)$.

Standard references on s-numbers and operator ideals are the monographs of Pietsch [PI1] and [PI2]. The ideals of all linear bounded and all finite rank operators are denoted by $\mathfrak{L}$ and $\mathfrak{F}$, respectively.

For every Banach ideal (A,α) the component $A^*(X,Y)$ of the conjugate ideal (A^*,α^*) is the class of all operators $T\in\mathfrak{L}(X,Y)$ such that

$$\alpha^*(T) := \sup\{|\operatorname{tr} TS| \mid S\in\mathfrak{F}(Y,X),\ \alpha(S)\le 1\}<\infty .$$

Next we recall the usual notation of some s-numbers of an operator $T\in\mathfrak{L}(X,Y)$:

$$a_n(T) := \inf\{\|T-S\| \mid S\in\mathfrak{L}(X,Y),\ \operatorname{rank}(S)<n\},$$
the nth *approximation number*,
$$c_n(T) := \inf\{\|T\iota_E\| \mid E\subset X,\ \operatorname{codim} E<n\}$$
the nth *Gelfand number*,
$$d_n(T) := \inf\{\|Q_F T\| \mid F\subset Y,\ \dim F<n\},$$
the nth *Kolmogorov number*,
$$x_n(T) := \sup\{a_n(Tu) \mid u\in\mathfrak{L}(\ell_2,X),\ \|u\|\le 1\},$$
the nth *Weyl number*.

The nth *entropy number*, nth *volume number* and nth *volume ratio number* are defined by

$$e_n(T) := \inf\Big\{\varepsilon>0 \;\Big|\; \exists (y_k)_{k=1}^{2^{n-1}}\subset Y : T(B_X)\subset \bigcup_{k=1}^{2^{n-1}} (y_k+\varepsilon B_Y)\Big\},$$

$$v_n(T) := \sup\Big\{\Big(\frac{\operatorname{vol}(T(B_E))}{\operatorname{vol}(B_F)}\Big)^{1/(\mu n)} \;\Big|\; E\subset X,\ T(E)\subset F\subset Y,\ \dim E=\dim F=n\Big\},$$

$$vr_n(T) := \sup\Big\{\Big(\frac{\operatorname{vol}(Q_F T(B_X))}{\operatorname{vol}(B_{X/F})}\Big)^{1/(\mu n)} \;\Big|\; F\subset Y,\ \operatorname{codim} F=n\Big\},$$

where μ is 1 or 2 according as the scalar field $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. Let us note that the entropy numbers are surjective and quasi-injective, i.e. for every metric surjection Q and metric injection J we have

$$e_n(T)\le e_n(TQ) \quad\text{and}\quad e_n(T)\le 2e_n(JT).$$

The relations between these different geometric numbers are as follows (see [DJ], [MA2]): for all $k,n\in\mathbb{N}$,

$$v_n(T)\le 2\cdot 2^{k/n}e_k(T),$$
$$vr_n(T)\le 2^{k/n}e_k(T),$$
$$\frac{1}{c_0}v_n(T)\le vr_n(T^*)\le c_0 v_n(T),$$
$$\frac{1}{c_0}vr_n(T)\le v_n(T^*)\le c_0 vr_n(T).$$

Let $\det:\mathbb{K}^{n^2}\to\mathbb{K}$ be the unique determinant. Then the nth *Grothendieck number* is defined by

$$\Gamma_n(T) := \sup\{|\det(\langle Tx_i,y_j^*\rangle)|^{1/n} \mid (x_k)_{k=1}^n\subset B_X,\ (y_k^*)_{k=1}^n\subset B_{Y^*}\}.$$

The Grothendieck numbers were investigated by Geiss. Let us mention that

$$\Gamma_n(T)=\Gamma_n(T^*),$$
$$\Gamma_n(T)=\sup\{\Gamma_n(Q_F T\iota_E)\mid E\subset X,\ \dim E=n,\ F\subset Y,\ \operatorname{codim} F=n\}.$$

Carl [CA4] proved that for $s\in\{c,d\}$,

$$\Big(\prod_{k=1}^n s_k(T)\Big)^{1/n}\le \Gamma_n(T).$$

In particular, if $T\in\mathfrak{L}(X,Y)$ and X or Y is a Hilbert space then

$$a_n(T)\le\Gamma_n(T). \tag{1}$$

If both are Hilbert spaces one has (see [GEI], [DJ])

$$v_n(T)=\Gamma_n(T)=\Big(\prod_{k=1}^n a_k(T)\Big)^{1/n}. \tag{2}$$

The following multiplication formula holds for $s\in\{v,vr,\Gamma\}$:

$$s_n(TS)\le s_n(T)s_n(S),$$

whereas for $s\in\{a,c,d,e,x\}$,

$$s_{n+m-1}(ST)\le s_n(S)s_m(T).$$

For $s \in \{a, c, d, e, v, vr, \Gamma, x\}$ and $0 < r, w \leq \infty$ the operator ideal $\mathfrak{L}^s_{r,w}$ is the class of all operators T such that

$$\ell^s_{r,w}(T) := \|(s_n(T))_{n\in\mathbb{N}}\|_{r,w} < \infty .$$

The multiplication formula yields immediately (see [PI2])

$$\mathfrak{L}^s_{r_1,w_1} \circ \mathfrak{L}^s_{r_2,w_2} \subset \mathfrak{L}^s_{r,w} \quad \text{with } 1/r = 1/r_1 + 1/r_2 \text{ and } 1/w = 1/w_1 + 1/w_2 .$$

For $2 \leq r < \infty$ an operator $T \in \mathfrak{L}(X, Y)$ is said to be *absolutely* $(r, 2)$-*summing* ($T \in \Pi_{r,2}(X, Y)$) if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$ and $(x_k)^n_{k=1} \subset X$,

$$\Big(\sum_{k=1}^n \|Tx_k\|^r\Big)^{1/r} \leq c \sup_{x^*\in B_{X^*}} \Big(\sum_{k=1}^n |\langle x_k, x^*\rangle|^2\Big)^{1/2} .$$

We set $\pi_{r,2}(T) := \inf c$, where the infimum is taken over all c satisfying the above inequality.

Let $(g_k)_{k\in\mathbb{N}}$ be a sequence of independent, normalized gaussian variables on a measure space (Ω, μ). For $n \in \mathbb{N}$ and any operator $u \in \mathfrak{L}(\ell^n_2, X)$, we define

$$l(u) := \Big\| \sum_{k=1}^n g_k u(e_k) \Big\|_{L_2(X)} ,$$

where e_k $(k = 1, \ldots, n)$ are the unit vectors in ℓ^n_2. An operator $T \in \mathfrak{L}(X, Y)$ is said to be *absolutely* γ-*summing* ($T \in \ell(X, Y)$) if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$ and $u \in \mathfrak{L}(\ell^n_2, X)$,

$$l(Tu) \leq c\|u\| .$$

We define $l(T) := \inf c$, where the infimum is taken over all c satisfying the above inequality.

As is well-known $l^*(v) \leq l(v^*)$ holds for all $v \in \mathfrak{L}(X, \ell_2)$. But in general the converse is not true. A Banach space is K-*convex* if there is a constant $c \geq 0$ such that for all $v \in \ell^*(X, \ell_2)$,

$$l(v^*) \leq cl^*(v) .$$

In this case we write $K(X) := \inf c$, where the infimum is taken over all c satisfying the above inequality. Let us mention that X is K-convex if and only if X^* is K-convex, and in this case $K(X) = K(X^*)$. Pisier [PS] proved that for every n-dimensional Banach space E,

$$K(E) \leq c_0(1 + \ln n) . \tag{3}$$

To end the preliminaries, we recall estimates for entropy numbers. The first is due to Sudakov, Pajor, and Tomczak-Jaegermann (see [PS]). For $u \in \ell(\ell_2, X)$ we have

$$\max\{\ell^e_{2,\infty}(u), \ell^e_{2,\infty}(u^*)\} \leq c_0 l(u) . \tag{4}$$

The second estimate concerns the duality problem for entropy numbers and is due to Bourgain, Pajor, Szarek and Tomczak-Jaegermann [BPST]. Let X or Y be a K-convex Banach space. Then for every $0 < p < \infty$ there is a constant $c \geq 0$, depending on the K-convexity constant and p, such that for all compact operators $T \in \mathfrak{L}(X, Y)$ and $n \in \mathbb{N}$,

$$\Big(\sum_{k=1}^n e_k(T^*)^p\Big)^{1/p} \leq c\Big(\sum_{k=1}^n e_k(T)^p\Big)^{1/p} .$$

In particular (using Hardy's inequality), there are constants $c_{r,w} \geq 0$ such that

$$\ell^e_{r,w}(T^*) \leq c_{r,w}\ell^e_{r,w}(T) . \tag{5}$$

1. Consequences of the Carl–Maurey inequality. We start with the notion of type and weak type of operators. An operator $T \in \mathfrak{L}(X, Y)$ is of (*gaussian*) *type* p, $1 \leq p \leq 2$ ($T \in \mathfrak{T}_p(X, Y)$), if there is a constant $c \geq 0$ such that for all $n \in \mathbb{N}$ and $(x_k)^n_{k=1} \subset X$,

$$\Big\| \sum_{k=1}^n g_k T x_k \Big\|_{L_2(Y)} \leq c\Big(\sum_{k=1}^n \|x_k\|^p\Big)^{1/p} .$$

In this case $t_p(T) := \inf c$, where the infimum is taken over all c satisfying the above inequality. The translation into the language of operator ideals is due to Tomczak-Jaegermann (see [TOJ]):

$T \in \mathfrak{L}(X, Y)$ is of type p if there is a constant $c \geq 0$ such that for all $v \in \ell^*(Y, \ell_2)$,

$$\pi_{q,2}((vT)^*) \leq cl^*(v) , \quad \text{where } 2 \leq q \leq \infty \text{ with } 1/q + 1/p = 1 .$$

By the use of the well-known inequality $\ell^x_{q,\infty} \leq \pi_{q,2}$ this reformulation leads to Pisier's definition of weak type p (see also [MA1]). An operator $T \in \mathfrak{L}(X, Y)$ is of *weak type* p, $1 \leq p \leq 2$ ($T \in \omega\mathfrak{T}_p(X, Y)$), if there is a constant $c \geq 0$ such that for all $v \in \ell^*(Y, \ell_2)$,

$$\ell^a_{q,\infty}(vT) \leq cl^*(v) , \quad \text{where } 2 \leq q \leq \infty \text{ with } 1/q + 1/p = 1 .$$

In this case $\omega t_p(T) := \inf c$, where the infimum is taken over all c satisfying the above inequality.

It is obvious that every operator of type p is of weak type p. Conversely, every operator of weak type p is of type r for all $1 \leq r < p$.

This chapter relies on the Carl–Maurey inequality for weak type p operators (see [DJ]):

1.1. Theorem. *Let* $1 \leq p \leq 2$. *For every operator* $T \in \mathfrak{L}(X, Y)$ *we have the following implications*: (i)⇒(ii)⇒(iii) *for* $p < 2$, (i)⇒(iii) *for* $p = 2$ *and* (iii)⇒(i), *provided* Y *is* K-*convex.*

(i) *T is of weak type p.*

(ii) *There is a constant $c_1(T) \geq 0$ such that for all $k, n \in \mathbb{N}$, $1 \leq k \leq n$, and $S \in \mathfrak{L}(\ell_1^n, X)$,*

$$e_k(TS) \leq c_1(T)\left(\frac{1+\ln(n/k)}{k}\right)^{1-1/p}\|S\|.$$

(iii) *There is a constant $c_2(T) \geq 0$ such that for all $n \in \mathbb{N}$ and $S \in \mathfrak{L}(\ell_1^n, X)$,*

$$e_n(TS) \leq c_2(T) n^{1/p-1}\|S\|.$$

If $T = \mathrm{Id}_X$ is the identity on a Banach space X the implication (iii)⇒(i) *holds without the assumption of K-convexity.*

In the following we want to indicate some consequences of the Carl–Maurey inequality which were originally formulated with the assumption "type p" instead of "weak type p". In this sense the following theorem is due to Carl [CA3].

1.2. THEOREM. *Let $1 \leq p < 2$. For every operator $T \in \mathfrak{L}(X, Y)$ of weak type p and for all $S \in \mathfrak{L}(\ell_1, X)$, $\sigma \in \ell_{r,t}$ and $D_\sigma \in \mathfrak{L}(\ell_q, \ell_1)$,*

$$TSD_\sigma \in \mathfrak{L}^e_{s,t}(\ell_q, Y) \quad \text{with } 1/s + 1/p = 1/r + 1/q,$$

where $0 < r < \infty$, $0 < s, t \leq \infty$, $1 \leq q \leq \infty$ such that $1/r + 1/q > 1$.

Since the duality problem for entropy numbers is solved for Lorentz norms in K-convex spaces, we are able to extend a result of Kühn [KÜH] which describes the dual situation.

1.3. COROLLARY. *Let $1 \leq p < 2$. For every operator $T \in \mathfrak{L}(X, Y)$ whose dual T^* is of weak type p, and for all $R \in \mathfrak{L}(Y, \ell_\infty)$, $\sigma \in \ell_{r,t}$, $D_\sigma \in \mathfrak{L}(\ell_\infty, \ell_q)$,*

$$D_\sigma RT \in \mathfrak{L}^e_{s,t}(X, \ell_q) \quad \text{with } 1/s + 1/p + 1/q = 1 + 1/r,$$

where $1 \leq q \leq \infty$, $0 < r < q \leq \infty$, and $0 < s, t \leq \infty$.

Proof. Let first $1 < q < \infty$. For $R \in \mathfrak{L}(Y, \ell_\infty)$ we define

$$S := \sum_{i \in \mathbb{N}} e_i \otimes R^* e_i \in \mathfrak{L}(\ell_1, Y^*).$$

By Theorem 1.2,

$$T^* S D_\sigma \in \mathfrak{L}^e_{s,t}(\ell_{q'}, X^*) \quad \text{if } 1/s + 1/p = 1/r + 1/q'.$$

The K-convexity of ℓ_q together with (5) and $S^*_{|Y} = R$ implies that

$$\ell^e_{s,t}(D_\sigma RT) \leq \ell^e_{s,t}(D_\sigma S^* T^{**}) \leq c_{r,t,q}\ell^e_{s,t}(T^* S D_\sigma) < \infty.$$

If $q = 1$ or $q = \infty$ there exist $0 < r_1, r_2 < \infty$ and $1 < u < \infty$ with

$$1/r = 1/r_1 + 1/r_2, \quad 1/r_1 > 1/u \quad \text{and} \quad 1/r_2 + 1/u > 1/q.$$

Since $\ell_{r,t} = \ell_{r_1,t} \circ \ell_{r_2,\infty}$ (see [PI2]), for each $\sigma \in \ell_{r,t}$ we can find two sequences $\mu \in \ell_{r_1,t}$, $\tau \in \ell_{r_2,\infty}$ with $\sigma = \tau\mu$. By a result of Carl [CA2] we have

$$D_\mu \in \mathfrak{L}^e_{s_2,\infty}(\ell_u, \ell_q) \quad \text{with } 1/s_2 = 1/r + 1/u - 1/q.$$

Hence by the first part applied to $D_\tau \in \mathfrak{L}(\ell_\infty, \ell_u)$,

$$D_\sigma RT = D_\mu D_\tau RT \in \mathfrak{L}^e_{s_2,t} \circ \mathfrak{L}^e_{s_1,\infty} \subset \mathfrak{L}^e_{s,t}$$

with $1/s = 1/s_1 + 1/s_2 = 1 + 1/r_1 - 1/u - 1/p + 1/r_2 + 1/u - 1/q = 1 + 1/r - 1/p - 1/q$. ∎

A combination of Theorem 1.2 and Corollary 1.3 yields a result for (r, w)-nuclear operators which was proved for "type p" by Kühn [KÜH]. Let $0 < r < 1$ and $0 < w \leq \infty$. An operator $T \in \mathfrak{L}(X, Y)$ is called *(r, w)-nuclear* ($T \in \mathfrak{N}_{r,w}(X, Y)$) if there are $R \in \mathfrak{L}(X, \ell_\infty)$, $S \in \mathfrak{L}(\ell_1, Y)$ and $\sigma \in \ell_{r,w}$ such that

$$T = SD_\sigma R.$$

In this case $N_{r,w}(T) := \inf \|R\|\|\sigma\|_{r,w}\|S\|$, where the infimum is taken over all representations as above.

1.4. COROLLARY. *Let $1 \leq p, q < 2$, $0 < r < 1$ and $0 < s, w \leq \infty$. If X^* is of weak type p and Y of weak type q then*

$$\mathfrak{N}_{r,w}(X, Y) \subset \mathfrak{L}^e_{s,w}(X, Y) \quad \text{with } 1/s + 1/p + 1/q = 1/r + 1.$$

Proof. First we show that for $0 < r < 1$,

$$\mathfrak{N}_r(X, Y) \subset \mathfrak{L}^e_{s,\infty}(X, Y) \quad \text{with } 1/s + 1/p + 1/q = 1/r + 1.$$

For this let $T = SD_\sigma R$, $R \in \mathfrak{L}(X, \ell_\infty)$, $S \in \mathfrak{L}(\ell_1, Y)$, and $\sigma \in \ell_r$, $\sigma \geq 0$. We set $\tau = \sigma^{1/2}$ and consider $D_\tau \in \mathfrak{L}(\ell_\infty, \ell_2)$, $D_\tau \in \mathfrak{L}(\ell_2, \ell_1)$, respectively. Then we have

$$\|\sigma\|_r = \|\tau\|^2_{2r} \quad \text{and} \quad 1/2 + 1/(2r) > 1, \ 2r < 2.$$

Hence by Theorem 1.2 and Corollary 1.3,

$$SD_\tau \in \mathfrak{L}^e_{s_1,2r}(\ell_2, Y) \subset \mathfrak{L}^e_{s_1,\infty}(\ell_2, Y) \quad \text{with } 1/s_1 + 1/q = 1/(2r) + 1/2,$$

$$D_\tau R \in \mathfrak{L}^e_{s_2,2r}(X, \ell_2) \subset \mathfrak{L}^e_{s_2,\infty}(X, \ell_2) \quad \text{with } 1/s_2 + 1/p + 1/2 = 1/(2r) + 1.$$

By the multiplicativity of the entropy numbers we get

$$T = SD_\sigma R = SD_\tau D_\tau R \in \mathfrak{L}^e_{s,\infty}(X, Y) \quad \text{with } 1/s = 1/s_1 + 1/s_2 = 1 + 1/r - 1/p - 1/q.$$

Finally, the assertion follows from the real interpolation method, since for $0 < r_1 < r < r_2 < 1$ there is a $0 < \theta < 1$ such that (see e.g. [PI2])

$$\ell_{r,w} = (\ell_{r_1}, \ell_{r_2})_{\theta,w} \quad \text{with } 1/r = (1-\theta)/r_1 + \theta/r_2 \quad \text{and}$$

$$(\mathfrak{L}^e_{s_1,\infty}, \mathfrak{L}^e_{s_2,\infty})_{\theta,w} \subset \mathfrak{L}^e_{s,w} \quad \text{with } 1/s = (1-\theta)/s_1 + \theta/s_2. \ ∎$$

2. A composition formula for different s-numbers. Up to now we do not know an analog to the Carl–Maurey inequality for $p=2$ (in the weak case). Nevertheless, by using a composition formula for s-numbers, most of the results of the previous chapter can be carried over to this case. We start with

2.1. PROPOSITION. *Let $1\le p\le 2$ and $0<r,s\le\infty$ with $1/s+1/p=1/r+1/2$. Then for all operators $T\in\mathfrak{L}(X,Y)$ of weak type p and $u\in\mathfrak{L}(\ell_2,X)$ whose dual is in $\mathfrak{L}^x_{s,\infty}(X^*,\ell_2)$,*

$$\ell^v_{s,\infty}(Tu)\le c_0(2e)^{1/r}\omega t_p(T)\ell^x_{s,\infty}(u^*)\,.$$

Proof. Let $n\in\mathbb{N}$ and $H\subset\ell_2$ with $\dim H=n$. Then for $E:=\operatorname{Im}Tu\iota_H\subset Y$ we have $\dim E\le n$. By a lemma of Lewis (see [PS]), there are $w\in\mathfrak{L}(\ell_2^n,E)$ and $v\in\mathfrak{L}(Y,\ell_2^n)$ such that

$$\mathrm{Id}_E=wv\iota_E\quad\text{and}\quad l(w)=l^*(v)\le\sqrt{n}\,.$$

Combining this with (2) and (4) we get

$$\begin{aligned}
v_n(Tu\iota_H)&=v_n(wvTu\iota_H)\le v_n(w)v_n(vTu\iota_H)\\
&\le 4e_n(w)\Big(\prod_{k=1}^n a_k(vTu\iota_H)\Big)^{1/n}\\
&\le 4c_0n^{-1/2}l(w)e^{1/r+1/p'}n^{-(1/r+1/p')}\sup_{k\in\mathbb{N}}k^{1/r+1/p'}x_k((vT)^*(u\iota_H)^*)\\
&\le 8c_0e(2e)^{1/r}n^{-(1/r+1/p')}\ell^a_{p',\infty}(vT)\ell^x_{r,\infty}(u^*)\\
&\le 8c_0e(2e)^{1/r}n^{-(1/r+1/p')}\omega t_p(T)l^*(v)\ell^x_{r,\infty}(u^*)\\
&\le 8c_0e(2e)^{1/r}n^{-(1/r+1/2-1/p)}\omega t_p(T)\ell^x_{r,\infty}(u^*)\,.\ \blacksquare
\end{aligned}$$

Now we can prove the following composition formula:

2.2. PROPOSITION. *Let $1\le p\le 2$, $0<r<p$ and $0<s\le\infty$ with $1/s+1/p=1/r+1/2$. For every K-convex Banach space Y and any operator $T\in\mathfrak{L}(X,Y)$ of weak type p, and for all $u\in\mathfrak{L}(\ell_2,X)$ whose dual is in $\mathfrak{L}^x_{r,\infty}(X^*,\ell_2)$,*

$$\ell^d_{s,\infty}(Tu)\le cK(Y)^{2/s}\omega t_p(T)\ell^x_{s,\infty}(u^*)\,,$$

where $c\ge 0$ is an absolute constant depending only on p and r.

Proof. We will use a result of Pajor and Tomczak-Jaegermann [PTJ] which states that for all $0<s<2$ there is a constant $c_s\ge 0$ such that for all $v\in\mathfrak{L}(Y^*,\ell_2)$,

$$\ell^c_{s,\infty}(v)\le c_sK(Y)^{2/s}\ell^{vr}_{s,\infty}(v)\,.$$

Therefore the assertion follows by duality from Proposition 2.1 (see also [PI2]):

$$\begin{aligned}
\ell^d_{s,\infty}(Tu)&\le\ell^c_{s,\infty}((Tu)^*)\le c_sK(Y)^{2/s}\ell^{vr}_{s,\infty}((Tu)^*)\\
&\le c_0c_sK(Y)^{2/s}\ell^v_{s,\infty}(Tu)\\
&\le c_0^2(2e)^{1/s}c_sK(Y)^{2/s}\omega t_p(T)\ell^x_{r,\infty}(u^*)\,.\ \blacksquare
\end{aligned}$$

Remark. (i) If T is the identity operator on some Banach space X we can formally drop the assumption of K-convexity.

(ii) Actually, we can in general prove the proposition without the assumption of K-convexity.

With the composition formula of Proposition 2.2 we can extend the results of the previous chapter:

2.3. COROLLARY. *Let Y be a K-convex Banach space and $T\in\mathfrak{L}(X,Y)$ an operator of weak type 2. Then for all $S\in\mathfrak{L}(\ell_1,X)$, $\sigma\in\ell_{r,t}$ and $D_\sigma\in\mathfrak{L}(\ell_q,\ell_1)$,*

$$TSD_\sigma\in\mathfrak{L}^e_{s,t}(\ell_q,Y)\qquad\text{with } 1/s+1/2=1/r+1/q\,,$$

where $1\le q\le\infty$, $0<r<\min\{2,q'\}$ and $0<s,t\le\infty$.

Proof. By interpolation (compare the proof of Corollary 1.4), it is sufficient to prove that for $S\in\mathfrak{L}(\ell_1,X)$, $\sigma\in\ell_r$ and $D_\sigma\in\mathfrak{L}(\ell_q,\ell_1)$,

$$TSD_\sigma\in\mathfrak{L}^e_{s,\infty}(\ell_q,Y)\qquad\text{with } 1/s+1/2=1/r+1/q\,.$$

For $q=2$ the assertion follows from Proposition 2.2 since by [CA1], [PI2],

$$\begin{aligned}
\ell^e_{s,\infty}(TSD_\sigma)&\le d_s\ell^d_{s,\infty}(TSD_\sigma)\le d_sc_sK(Y)^{2/s}\omega t_2(T)\ell^x_{s,\infty}((SD_\sigma)^*)\\
&\le d_sc_sK(Y)^{2/s}\omega t_2(T)\|S\|c_s\|\sigma\|_{s,\infty}\\
&\le c_s^2d_sK(Y)^{2/s}\omega t_2(T)\|\sigma\|_r\|S\|\,.
\end{aligned}$$

For $q\ne 2$ we decompose $\sigma\in\ell_r$ as $\sigma=\tau\mu$ with $\tau\in\ell_{r_1}$, $\mu\in\ell_{r_2}$ such that $r_1<2$, $1/r_2>\max\{0,1/2-1/q\}$ and $1/r=1/r_1+1/r_2$. By [CA2], for $1/s_2=1/r_2+1/q-1/2$ we have

$$D_\mu\in\mathfrak{L}^e_{s_2,\infty}(\ell_q,\ell_2)\,.$$

Hence by the first step and the multiplicativity of the entropy numbers,

$$TSD_\sigma=TSD_\tau D_\mu\in\mathfrak{L}^e_{r_1,\infty}\circ\mathfrak{L}^e_{s_2,\infty}\subset\mathfrak{L}^e_{s,\infty}\quad\text{with } 1/s=1/r+1/q-1/2\,.\ \blacksquare$$

Remark. It is clear that a corresponding dual formulation for operators whose dual is of weak type 2 follows in the same way from Corollary 2.3 as in the previous chapter Corollary 1.3 follows from Theorem 1.2.

3. Characterization of weak type p. First we will show that the conditions of Theorem 1.2 and Corollaries 1.3 and 2.3 are even necessary:

3.1. PROPOSITION. *Let $1 \le p \le 2$, $0 < r < \infty$, $0 < s,t \le \infty$ and $1 \le q \le \infty$ with $1/s + 1/p = 1/r + 1/q$ and Y a K-convex Banach space. Assume that for an operator $T \in \mathfrak{L}(X,Y)$ and for all $\sigma \in \ell_{r,t}$, $D_\sigma \in \mathfrak{L}(\ell_q, \ell_1)$ and $S \in \mathfrak{L}(\ell_1, X)$,*

$$TSD_\sigma \in \mathfrak{L}^e_{s,\infty}(\ell_q, Y)\,.$$

Then T is of weak type p.

Proof. We will prove the existence of a constant $c \ge 0$ such that for all $n \in \mathbb{N}$ and $S \in \mathfrak{L}(\ell_1^n, X)$,

$$e_n(TS) \le cn^{1/p-1}\|S\|\,.$$

Then the assertion follows from (iii)⇒(i) of Theorem 1.1. By the closed graph theorem the assumption yields a constant $c \ge 0$ such that for all $S \in \mathfrak{L}(\ell_1, X)$, $\sigma \in \ell_{r,t}$ and $D_\sigma \in \mathfrak{L}(\ell_q, \ell_1)$,

$$\ell^e_{s,\infty}(TSD_\sigma) \le c\|S\|\|\sigma\|_{r,t}\,.$$

Now let $n \in \mathbb{N}$, $S \in \mathfrak{L}(\ell_1^n, X)$ and define $\sigma = (\sigma_j)_{j\in\mathbb{N}}$ by $\sigma_j := 1$ for $j = 1, \ldots, n$ and $\sigma_j := 0$ for $j > n$. Hence

$$\|\sigma\|_{r,t} \le c_{r,t} n^{1/r}\,.$$

On the other hand, it is known (see [CA2]) that for the formal identity $\iota_n : \ell_1^n \to \ell_q^n$ and $m := [(n+1)/2]$ we have

$$e_m(\iota_n) \le c_0 n^{1/q-1}\,.$$

Denoting by $P_n : \ell_1 \to \ell_1^n$ the natural projection and $I_n : \ell_q^n \to \ell_q$ the natural injection, we can conclude:

$$\begin{aligned} e_n(TS) &= e_n(TSP_nD_\sigma I_n\iota_n) \le e_m(TSP_nD_\sigma)e_m(I_n\iota_n)\\ &\le m^{-1/s}\ell^e_{s,\infty}(TSP_nD_\sigma)c_0n^{1/q-1}\\ &\le 2^{1/s}c_0n^{-1/s+1/q-1}c\|SP_n\|\|\sigma\|_{r,t} \le 2^{1/s}c_0c_{r,t}cn^{1/p-1}\|S\|\,. \ \blacksquare \end{aligned}$$

Remark. If T is the identity of some Banach space X we can drop the assumption of K-convexity, since the condition

$$e_n(TS) \le cn^{1/p-1}\|S\| \quad \text{for all } n \in \mathbb{N},\ S \in \mathfrak{L}(\ell_1^n, X)$$

just characterizes weak type p spaces (see Theorem 1.1).

For the proof of our main theorem we will need the following

3.2. LEMMA. *Let H, K be Hilbert spaces and $T \in \mathfrak{L}(H,K)$. Then for all $n \in \mathbb{N}$,*

$$(e_n(T))^2 \le 144e_{2n}(T^*T) \le 144(e_n(T))^2\,.$$

Proof. Note that $a_i(T^*T) = (a_i(T))^2$ for $i \in \mathbb{N}$. Hence it follows from [GKS] with $\mu = 1$ or 2 according as $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ that

$$\begin{aligned} (e_n(T))^2 &\le \Big(12 \sup_{k=1,\ldots,n} 2^{-n/(\mu k)}\Big(\prod_{i=1}^k a_i(T)\Big)^{1/k}\Big)^2\\ &= 144 \sup_{k=1,\ldots,n} 2^{-2n/(\mu k)}\Big(\prod_{i=1}^k a_i(T^*T)\Big)^{1/k}\\ &\le 144e_{2n}(T^*T) \le 144e_n(T^*)e_n(T) \le 144(e_n(T))^2\,. \ \blacksquare \end{aligned}$$

We finish this chapter with the proof of our main theorem.

Proof (main theorem).

1)⇒2). If $1 \le p, q < 2$ this implication is Corollary 1.4. If either $p = 2$ or $q = 2$ the proof of Corollary 1.4 still works, provided one uses Corollary 2.3 and its dual formulation instead of Theorem 1.2 and Corollary 1.3.

2)⇒1). We will apply Proposition 3.1. Let $1 < q \le 2$, $S \in \mathfrak{L}(\ell_1, X^*)$ and $\sigma \in \ell_{r,t}$. Then we set

$$R := \sum_{i\in\mathbb{N}} Se_i \otimes e_i \in \mathfrak{L}(X, \ell_\infty) \quad \text{and}$$

$$I := \sum_{i\in\mathbb{N}} e_i \otimes e_i \in \mathfrak{L}(\ell_1, \ell_q), \quad \text{the formal identity}\,.$$

Since $T = ID_\sigma R \in \mathfrak{N}_{r,w}(X, \ell_q)$ and ℓ_q is of type q our assumption yields

$$T \in \mathfrak{L}^e_{s,\infty}(X, \ell_q) \quad \text{with } 1/s + 1/p + 1/q = 1 + 1/r\,.$$

By the K-convexity of ℓ_q we deduce from (5) with $1/q + 1/q' = 1$ that

$$SD_\sigma = T^* \in \mathfrak{L}^e_{s,\infty}(\ell_{q'}, Y) \quad \text{with } 1/s + 1/p = 1/q' + 1/r\,.$$

1)⇒3). We apply 1)⇒2) for $Y = X^*$ which is of weak type p.

3)⇒1). The proof consists of two steps:

Step I: *There is a constant $c \ge 0$ such that for all $n \in \mathbb{N}$ and $u \in \mathfrak{L}(\ell_2^n, X)$,*

$$n^{1/p'}\Gamma_n(u) \le c\min\{K(X), 1 + \ln n\}l^*(u^*) \quad \textit{with } 1/p' + 1/p = 1\,.$$

Step II: *There is a constant $c \ge 0$ such that for all $v \in \ell^*(X^*, \ell_2)$ and $n \in \mathbb{N}$,*

$$n^{1/p'}a_n(v) \le c\min\{K(X), 1 + \ln n\}l^*(v) \quad \textit{with } 1/p' + 1/p = 1\,.$$

Before proving Step I or II let us indicate how the assertion follows from Step II. If $p = 1$ there is nothing to show, since every Banach space is of type 1, hence of weak type 1. If $p > 1$ choose $1 < r < q < p$. By Step II there is a constant $c_{p,q} \geq 0$ such that for all $v \in \ell^*(X^*, \ell_2)$,

$$\sup_{n\in\mathbb{N}} n^{1/q'} a_n(v) \leq c_{p,q} \sup_{n\in\mathbb{N}} n^{1/p'}(1+\ln n)^{-1} a_n(v) \leq c_{p,q} c l^*(v) .$$

Hence X^* is of weak type q and therefore of type $r > 1$. But this means that X and X^* are K-convex (see [PS]). Again by Step II it now follows that for all $v \in \ell^*(X^*, \ell_2)$,

$$\ell^a_{p',\infty}(v) \leq cK(X) l^*(v) ,$$

which proves that X^* is of weak type p.

We will use (1) in order to pass from Step I to Step II. By the definition of the Grothendieck numbers it is sufficient to prove that for all $n \in \mathbb{N}$, $F \subset X^*$ with $\dim F = n$ and $v \in \mathfrak{L}(F, \ell_2^n)$,

$$n^{1/p'} \Gamma_n(v) \leq c \min\{K(X), 1 + \ln n\} l^*(v) .$$

For this choose a finite family $(x_k)_{k=1}^m \subset B_X$ such that for all $f \in F$,

$$\tfrac{1}{2}\|f\| \leq \sup_{k=1,\ldots,m} |\langle x_k, f\rangle| \leq \|f\| .$$

This yields a canonical injection I from F into ℓ_∞^m:

$$I : F \to \ell_\infty^m, \qquad f \mapsto (\langle x_k, f\rangle)_{k=1}^m .$$

By the extension property of ℓ^* we can find an operator $V \in \mathfrak{L}(\ell_\infty^m, \ell_2^n)$ such that

$$VI = v \quad \text{and} \quad l^*(V) \leq 2 l^*(v) .$$

Defining

$$Q : \ell_1^m \to X, \qquad (\alpha_k)_{k=1}^m \mapsto \sum_{k=1}^m a_k x_k ,$$

and

$$w := QV^* \in \mathfrak{L}(\ell_2^n, X) ,$$

one can easily check that $w^* \iota_F = v$. The symmetry of the Grothendieck numbers and Step I imply

$$\begin{aligned} n^{1/p'} \Gamma_n(v) = n^{1/p'} \Gamma_n(w^* \iota_F) &\leq n^{1/p'} \Gamma_n(w) \leq c \min\{K(X), 1 + \ln n\} l^*(w^*) \\ &\leq c \min\{K(X), 1 + \ln n\} l^*(V^{**}) \leq 2c \min\{K(X), 1 + \ln n\} l^*(v) . \end{aligned}$$

Finally, let us prove Step I. The closed graph theorem implies the existence of a constant $c \geq 0$ such that for all $T \in \mathfrak{N}_{r,w}(X, Y)$,

$$\ell^e_{s,\infty}(T) \leq c N_{r,w}(T) .$$

For $1 \leq p, q \leq \infty$ and $n \in \mathbb{N}$ we denote by $\iota^n_{p,q}$ the formal identity from ℓ_p^n to ℓ_q^n. By the definition of the Grothendieck numbers we can find a subspace $F \subset X$ with $\operatorname{codim} F \leq n$ and $R \in \mathfrak{L}(X/F, \ell_\infty^n)$ with $\|R\| \leq 1$ such that

$$\Gamma_n(u) \leq 2\Gamma_n(\iota^n_{\infty,2} R Q_F u) .$$

Now we define

$$T := R^* \iota^n_{\infty,1} R \in \mathfrak{L}(X/F, (X/F)^*) ,$$
$$v := \iota^n_{\infty,2} R Q_F \in \mathfrak{L}(X, \ell_2^n) .$$

By (2), Lemma 3.2, (4) and (3) it follows that

$$\begin{aligned} (\Gamma_n(u))^2 &\leq 4(\Gamma_n(vu))^2 = 4(v_n(vu))^2 \leq 256(e_{2n}(vu))^2 \\ &\leq 256 \cdot 144 e_{4n}(u^* v^* vu) = 36864 e_{4n}(u^* Q_F^* T Q_F u) \\ &\leq 36864 e_n(u^* Q_F^*) e_{2n}(T) e_n(Q_F u) \\ &\leq 73728 e_n((Q_F u)^*) e_{2n}(Q_F^* T Q_F) e_n(Q_F u) \\ &\leq c_0 n^{-1/2} l(Q_F u) c (2n)^{-1/s} N_{r,w}(Q_F^* T Q_F) n^{-1/2} l(Q_F u) \\ &\leq c_0 c n^{-1-1/s} [K(X/F) l^*(u^*)]^2 N_{r,w}(\iota^n_{\infty,1}) \\ &\leq c_0^2 c c_{r,w} n^{-1-1/s+1/r} [\min\{K(X), 1 + \ln n\} l^*(u^*)]^2 , \end{aligned}$$

which proves Step I. ∎

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MATHEMATISCHES SEMINAR
UNIVERSITÄT KIEL
LUDEWIG-MEYN-STR. 4
D-24098 KIEL, GERMANY
E-mail: NMS06@RZ.UNI-KIEL.D400.DE

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Factorization of Montel operators

by

S. DIEROLF (Trier) and P. DOMAŃSKI (Poznań)

Abstract. Consider the following conditions. (a) Every regular LB-space is complete; (b) if an operator T between complete LB-spaces maps bounded sets into relatively compact sets, then T factorizes through a Montel LB-space; (c) for every complete LB-space E the space $C(\beta\mathbb{N}, E)$ is bornological. We show that (a)⇒(b)⇒(c). Moreover, we show that if E is Montel, then (c) holds. An example of an LB-space E with a strictly increasing transfinite sequence of its Mackey derivatives is given.

0. Introduction. In Banach space theory there is a famous result [11] (see also [13] and [15, Theorem 6.3.4]) that every weakly compact operator between Banach spaces factorizes through a reflexive Banach space. The ideal of operators mapping bounded sets into relatively (weakly) compact sets seems to be the proper analogue in the Fréchet setting of the ideal of (weakly) compact operators in the Banach case. This leads to the following factorization problem: Does every Montel operator (i.e., an operator mapping bounded sets into relatively compact sets) between Fréchet spaces factorize through a Fréchet–Montel space? Surprisingly enough, it seems that not much is known about it as well as about the dual problem concerning factorization of all Montel maps between (complete) LB-spaces through a Montel LB-space.

The best result which could be derived from the known facts (see Corollary 3.2 below) says that if E is a quasinormable Fréchet space, F is an arbitrary Fréchet space and $T : E \to F$ is a Montel map, then T factorizes through a Fréchet–Schwartz space.

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