STUDIA MATHEMATICA 107 (3) (1993)

A characterization of some weighted norm inequalities for the fractional maximal function

by

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Abstract. A new characterization is given for the pairs of weight functions v, w for which the fractional maximal function is a bounded operator from $L_v^p(X)$ to $L_w^q(X)$ when $1 < p < q < \infty$ and X is a homogeneous space with a group structure. The case when X is n-dimensional Euclidean space is included.

1. Introduction. The purpose of this paper is to derive a characterization of the pairs of weight functions v, w for which the fractional maximal function is a bounded operator from $L_v^p(X)$ to $L_w^q(X)$ when $1 < p < q < \infty$ and X is a homogeneous space. The most precise results occur if X also has a group structure. In the basic case when X is n-dimensional Euclidean space $\mathbb{R}^n$, the main result to be proved gives a simple necessary and sufficient condition in order that a pair of nonnegative measurable functions $v(x), w(x)$ (hereafter called weight functions) satisfy the norm inequality

$$(1.1)\qquad \Big(\int_{\mathbb{R}^n} |M_\alpha f(x)|^q w(x)\,dx\Big)^{1/q} \le c\Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}, \qquad 1 < p < q < \infty,$$

where

$$M_\alpha f(x) = \sup_{B : x\in B} \frac{1}{|B|^{1-\alpha/n}} \int_B |f(y)|\,dy, \qquad 0 < \alpha < n,$$

B denotes a ball in $\mathbb{R}^n$, and c is a constant independent of f.

A necessary and sufficient condition for v and w to satisfy (1.1), including even the case when $q = p$, was given by E. Sawyer in [S1], but the condition considered below is simpler than the one in [S1] in the sense that it does not involve the operator M_α itself. The exact characterization is given in the following theorem.

1991 *Mathematics Subject Classification*: Primary 42B25.
Research supported in part by NSF grant DMS91-04195.

THEOREM 1. *Let $1 < p < q < \infty$ and $0 < \alpha < n$. Then (1.1) holds if and only if v, w satisfy*

$$(1.2)\quad \Big(\int_{\mathbb{R}^n} \frac{w(x)}{(|B|^{1/n} + |x - x_B|)^{(n-\alpha)q}}\,dx\Big)^{1/q}\Big(\int_B v(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \le C$$

for all balls $B \subset \mathbb{R}^n$, where x_B is the center of B, $1/p + 1/p' = 1$, and C is a positive constant which is independent of B.

Condition (1.2) is the same as that given by M. Gabidzashvili and V. Kokilashvili in [GK] in order to characterize the weights for which there is a weak-type inequality of the form

$$(1.3)\quad \sup_{t>0} t\Big(\int_{\{x\in\mathbb{R}^n: |I_\alpha f(x)|>t\}} v(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \le c\Big(\int_{\mathbb{R}^n} |f(x)|^{q'} w(x)^{-1/(q-1)}\,dx\Big)^{1/q'},$$

$1 < p < q < \infty$, where $1/q + 1/q' = 1$ and $I_\alpha f$ is the fractional integral of f defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy.$$

Note that I_α is self-adjoint, as distinguished from M_α, and that (1.3) is the weak-type inequality corresponding to the *dual* version of the inequality

$$(1.4)\quad \Big(\int_{\mathbb{R}^n} |I_\alpha f(x)|^q w(x)\,dx\Big)^{1/q} \le c\Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}.$$

As pointed out in [SW1], a corollary of the main results of [GK] and [S3] is that (1.4) holds for $1 < p < q < \infty$ if and only if both (1.2) and

$$(1.5)\quad \Big(\int_B w(x)\,dx\Big)^{1/q}\Big(\int_{\mathbb{R}^n} \frac{v(x)^{-1/(p-1)}}{(|B|^{1/n} + |x - x_B|)^{(n-\alpha)p'}}\,dx\Big)^{1/p'} \le C$$

hold. By Theorem 1 with w, v, q and p replaced respectively by $v^{-1/(p-1)}$, $w^{-1/(q-1)}$, p' and q', it follows that (1.5) characterizes the inequality "dual" to (1.1), namely, the inequality

$$(1.6)\quad \Big(\int_{\mathbb{R}^n} |M_\alpha f(x)|^{p'} v(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \le c\Big(\int_{\mathbb{R}^n} |f(x)|^{q'} w(x)^{-1/(q-1)}\,dx\Big)^{1/q'}, \quad 1 < p < q < \infty.$$

Combining results, we immediately obtain

THEOREM 2. *For $1 < p < q < \infty$, (1.4) holds if and only if both (1.1) and (1.6) hold.*

We also note, as shown in [SW1], that (1.2) is equivalent to the simpler condition

$$(1.7)\quad |B|^{\alpha/n-1}\Big(\int_B w(x)\,dx\Big)^{1/q}\Big(\int_B v(x)^{-1/(p-1)}\,dx\Big)^{1/p'} \le C$$

in case the weight σ defined by $\sigma = v^{-1/(p-1)}$ satisfies the following reverse doubling condition:

(1.8) There exist $\beta, \theta > 1$ such that $\sigma(\theta B) \ge \beta\sigma(B)$ for all balls B,

where θB denotes the ball which is concentric with B and whose radius is θ times that of B, and $\sigma(B) = \int_B \sigma(x)\,dx$. Condition (1.8) is implied by the doubling condition $\sigma(2B) \le c\sigma(B)$ for all B. It is also true that (1.5) is equivalent to (1.7) if w satisfies the reverse doubling condition. A sufficient condition for (1.1) when $p \le q$ which is closely related to (1.7) is given in [P] and can also be derived by using the methods in [SW1]. In order to give an additional idea of the relations between the conditions, we mention in passing the well-known fact that (1.7) is necessary and sufficient for the weak-type inequality

$$\sup_{t>0} t\Big(\int_{\{x\in\mathbb{R}^n: M_\alpha f(x)>t\}} w(x)\,dx\Big)^{1/q} \le c\Big(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\Big)^{1/p}$$

for any pair of weights and any $1 < p \le q < \infty$.

Except for dealing with difficulties which are related to the fact that M_α is not self-adjoint, the techniques needed to derive Theorem 1 are closely related to ones in [S1], [S2] and either [GK] or [SW1]. As we shall see, the behavior of the fractional integral plays a role in the proof of Theorem 1, which is one reason why the results in [GK] and [SW1] are useful. Included in [SW1] is a method which allows an extension of the basic theorem in [GK] to situations where the Besicovitch covering lemma does not hold. Such an approach is particularly useful for deriving a version of Theorem 1 in the setting of a homogeneous space with a group structure. In order to state this more general result, we need several definitions. We consider a quasi-metric space (X, d) and a doubling measure μ on the Borel subsets of X; i.e., $d : X \times X \to [0, \infty)$ is assumed to satisfy

(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) there is a positive constant K such that $d(x, y) \le K[d(x, z) + d(z, y)]$ for all $x, y, z \in X$,

and there is a positive constant C such that

(iv) $\mu(B(x,2r)) \le C\mu(B(x,r))$ for all $x \in X$, $r > 0$, where $B(x,r) = \{y \in X : d(x,y) < r\}$ is the ball of radius r with center x.

We shall also assume that every annulus $B(x,R) \setminus B(x,r)$, $x \in X$, $0 < r < R < \infty$, is nonempty, and that X has a group structure with respect to the operation "+" such that for all $x, y, z \in X$ and all balls $B \subset X$, the following two invariance properties hold:

(v) $d(x+z, y+z) = d(x,y)$,

(vi) $\mu(-B+z) = \mu(B)$, where $-B = \{x : -x \in B\}$.

We do not assume that X is a commutative group. Instead of assuming right invariance in (v) and (vi), we could assume left invariance without affecting the validity of Theorem 3 below. It follows easily from (v) and (ii) that $d(0,-x) = d(0,x)$ for all $x \in X$, and it follows from (vi) (by taking $z = 0$) that $\mu(-B) = \mu(B)$.

For $0 < \gamma < 1$, the fractional maximal function $M_\gamma f$ of f is defined by

$$M_\gamma f(x) = \sup_{B: x \in B} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f(y)|\, d\mu(y)\,. \tag{1.9}$$

Here we have abused the notation M_γ in relation to the definition of the classical fractional maximal function given earlier; by letting $\gamma = \alpha/n$ in (1.9), we get the classical definition in case $X = \mathbb{R}^n$, $d(x,y) = |x-y|$ and $d\mu(x) = dx$.

Given a ball B, x_B again denotes the center of B, and we use the notation B_{xy} for the ball with center x and radius $d(x,y)$, i.e.,

$$B_{xy} = B(x, d(x,y))\,.$$

The following theorem is the main result of the paper.

THEOREM 3. *Let (X,d,μ) satisfy conditions* (i)–(vi) *above. Let $1 < p < q < \infty$, $0 < \gamma < 1$, and $M_\gamma f$ be defined by* (1.9). *Then the norm inequality*

$$\Big(\int_X |M_\gamma f(x)|^q w(x)\, d\mu(x)\Big)^{1/q} \le c\Big(\int_X |f(x)|^p v(x)\, d\mu(x)\Big)^{1/p} \tag{1.10}$$

holds for all f, with c independent of f, if and only if

$$\Big(\int_X \frac{w(x)}{[\mu(B)+\mu(B_{x_B x})]^{(1-\gamma)q}}\, d\mu(x)\Big)^{1/q} \times \Big(\int_B v(x)^{-1/(p-1)}\, d\mu(x)\Big)^{1/p'} \le C \tag{1.11}$$

for all balls $B \subset X$, with C independent of B.

For a homogeneous space X which has no group structure, the following result holds.

THEOREM 4. *Let (X,d,μ) satisfy only conditions* (i)–(iv) *above. Let $1 < p < q < \infty$, $0 < \gamma < 1$, and $M_\gamma f$ be defined by* (1.9). *If $v^{-1/(p-1)}\, d\mu$ satisfies the doubling condition, then the norm inequality* (1.10) *holds if and only if*

$$\mu(B)^{\gamma-1}\Big(\int_B w\, d\mu\Big)^{1/q}\Big(\int_B v^{-1/(p-1)}\, d\mu\Big)^{1/p'} \le C \tag{1.12}$$

for all balls $B \subset X$, with C independent of B.

Condition (1.12) is of course analogous to (1.7). With the additional assumption that $w\,d\mu$ is a doubling measure, Theorem 4 follows from its analogue (see Theorem 3(B) of [SW1]) for the fractional integral $I_\gamma f$ defined by

$$I_\gamma f(x) = \int_X \frac{f(y)}{\mu(B_{xy})^{1-\gamma}}\, d\mu(y)\,,$$

by using the fact that $M_\gamma f(x) \le I_\gamma f(x)$.

In Section 2, the necessity of (1.11) for the norm inequality if proved. The proof does not require conditions (v) or (vi) and so also shows the necessity of (1.12) under the hypothesis of Theorem 4. The sufficiency results are proved in Section 3. As mentioned earlier, many of the ideas used in the proof are variants of those in [SW1] and [S1]. There are, however, some differences, and since the method may be useful for still more general situations (e.g., more general maximal functions), we have tried to give most of the details for the parts of the proof which are different.

2. Necessity. In this section, for $0 < \gamma < 1$ and a ball B with center x_B, we use the notation

$$s_B(x) = \frac{1}{[\mu(B)+\mu(B_{x_B x})]^{1-\gamma}}\,, \qquad x \in X\,. \tag{2.1}$$

We also let $\sigma = v^{-1/(p-1)}$, so that (1.11) takes the simple form

$$\Big(\int_X s_B^q w\, d\mu\Big)^{1/q}\Big(\int_B \sigma\, d\mu\Big)^{1/p'} \le C\,. \tag{2.2}$$

Assuming that $1 < p \le q < \infty$ and (1.10) holds, we fix a ball B and choose $f = \sigma\chi_B$ in (1.10), obtaining

$$\Big(\int_X M_\gamma(\sigma\chi_B)^q w\, d\mu\Big)^{1/q} \le c\Big(\int_B \sigma\, d\mu\Big)^{1/p}. \tag{2.3}$$

We will show that

$$s_B(x) \int_B \sigma\, d\mu \le c M_\gamma(\sigma\chi_B)(x)\,, \qquad x \in X\,, \tag{2.4}$$

with c independent of x and B. To prove (2.4), first note that if $y \in B$ and $x \in X$ then

$$\begin{aligned} d(y,x) &\le K[d(y,x_B)+d(x_B,x)] \quad \text{by (iii)} \\ &\le K[r(B)+d(x_B,x)], \end{aligned}$$

where $r(B)$ denotes the radius of B. Thus, B is contained in the ball B_x with center x and radius $K[r(B)+d(x_B,x)]$, and consequently,

$$(2.5) \qquad M_\gamma(\sigma\chi_B)(x) \ge \frac{1}{\mu(B_x)^{1-\gamma}} \int_{B_x} \sigma\chi_B \, d\mu = \frac{1}{\mu(B_x)^{1-\gamma}} \int_B \sigma \, d\mu .$$

Next, note that B_x is contained in the ball B^* with the same center x_B as B and radius $K(2K+1)\max\{r(B), d(x,x_B)\}$ since if $z \in B_x$ then

$$\begin{aligned} d(z,x_B) &\le K\{d(z,x)+d(x,x_B)\} \le K\{K[r(B)+d(x_B,x)]+d(x,x_B)\} \\ &\le K(2K+1)\max\{r(B), d(x,x_B)\} . \end{aligned}$$

In particular,

$$\begin{aligned} \mu(B_x) &\le \mu(B^*) \\ &\le c_K \max\{\mu(B), \mu(B_{x_B x})\} \quad \text{since } \mu \text{ is doubling} \\ &\le c s_B(x)^{-1/(1-\gamma)} . \end{aligned}$$

Combining this with (2.5) proves (2.4).

By (2.3) and (2.4),

$$\Big(\int_X s_B^q w \, d\mu\Big)^{1/q} \int_B \sigma \, d\mu \le c\Big(\int_B \sigma \, d\mu\Big)^{1/p},$$

and (2.2) follows from this in case $\int_B \sigma\,d\mu \neq 0, \infty$ by division. If $\int_B \sigma\,d\mu = 0$, (2.2) is immediate. If $\int_B \sigma\,d\mu = +\infty$, then by applying the argument above to the function $v(x)+\varepsilon$, $\varepsilon > 0$ (for which the corresponding function σ is bounded), and passing to the limit, we see that

$$\int_X s_B^q w \, d\mu = 0 .$$

Hence, $w = 0$ a.e. $(d\mu)$ and (2.2) follows.

3. Sufficiency. In this section, we prove the sufficiency parts of both Theorems 3 and 4, beginning with Theorem 3. The proof for Theorem 4 is considerably simpler and given at the end of the section.

In order to define a dyadic version of M_γ, we use the notion of grids of dyadic cubes from [SW1], Lemma (3.21). Assuming that (X, d, μ) satisfies (i)–(iv) and that annuli are not empty, we then know there exists $\lambda > 1$ (in fact, $\lambda = 8K^5$ will do) so that for any (large negative) integer m, there are

points $\{x_j^k\}$ and Borel sets $\{E_j^k\}$ in X for $k = m, m+1, \ldots$ and $j = 1, 2, \ldots$ such that

$$(3.1) \qquad \begin{aligned} &\text{(a) } B(x_j^k, \lambda^k) \subset E_j^k \subset B(x_j^k, \lambda^{k+1}), \\ &\text{(b) } X = \textstyle\bigcup_j E_j^k \text{ and the } E_j^k \text{ are pairwise disjoint in } j \text{ for each } k, \\ &\text{(c) if } k < l \text{ then either } E_j^k \cap E_i^l = \emptyset \text{ or } E_j^k \subset E_i^l . \end{aligned}$$

We let $\mathcal{D} = \mathcal{D}_m = \{E_j^k : j \ge 1,\ k \ge m\}$ and refer to the sets E_j^k as dyadic cubes and denote them by Q. If $Q = E_j^k$, we let $s(Q) = \lambda^k$ and refer to $s(Q)$ as the sidelength of Q. Define

$$M_{\gamma,m} f(x) = \sup_{\substack{B: x \in B \\ r(B) \ge \lambda^m}} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f| \, d\mu$$

and also the dyadic versions

$$M_{\gamma,m}^{\mathrm{dy}} f(x) = \sup_{Q \in \mathcal{D}: x \in Q} \frac{1}{\mu(Q)^{1-\gamma}} \int_Q |f| \, d\mu ,$$

$$M_{\gamma,m,z}^{\mathrm{dy}} f(x) = \sup_{Q \in \mathcal{D}: x \in Q+z} \frac{1}{\mu(Q+z)^{1-\gamma}} \int_{Q+z} |f| \, d\mu, \qquad z \in X .$$

In the following lemma, which is analogous to Lemma (4.7) of [SW1] (and in the spirit of results in [FS] and [S1]), we also need to assume that X satisfies (v) and (vi). We use the standard notation

$$\|f\|_{L^p_{v\,d\mu}} = \Big(\int_X |f(x)|^p v(x)\, dx\Big)^{1/p},$$

$1 \le p < \infty$, etc.

Lemma (3.2). *Let X satisfy all of* (i)–(vi), $1 \le q \le \infty$ *and* $w(x) \ge 0$. *Then*

$$\|M_{\gamma,m} f\|_{L^q_{w\,d\mu}} \le c \sup_{z \in X} \|M_{\gamma,m,z}^{\mathrm{dy}} f\|_{L^q_{w\,d\mu}}$$

with c independent of m and f.

Proof. For $N \in \mathbb{Z}$, $N > m$, let $B_N = B(0, \lambda^N)$ where 0 denotes the identity element of X. We claim that

$$(3.3) \qquad \begin{aligned} &\sup_{\substack{B: x \in B \\ \lambda^m \le r(B) < \lambda^N}} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f| \, d\mu \\ &\le c \frac{1}{\mu(B_{N+3})} \int_{B_{N+3}} M_{\gamma,m,z}^{\mathrm{dy}} f(x) \, d\mu(z) \qquad \text{if } x \in B_N . \end{aligned}$$

The conclusion of the lemma follows easily from (3.3) by applying Minkowski's integral inequality and then letting $N \to \infty$.

To prove (3.3), fix $x \in B_N$ and choose a ball B with $x \in B$ and $\lambda^m \le r(B) < \lambda^N$ for which the left-hand side of (3.3) is less than $2\int_B |f|\,d\mu \times \mu(B)^{\gamma-1}$. Pick k with $m+1 \le k \le N$ such that $\lambda^{k-1} \le r(B) < \lambda^k$. Let Ω be all those $z \in B_{N+3}$ for which there exists $Q \in \mathcal{D}$ with $s(Q) = \lambda^{k+1}$ and $B \subset Q + z$. For such z and Q, we have $Q = E_i^{k+1}$ for some i and

$$\begin{aligned} Q + z &\subset B(x_i^{k+1}, \lambda^{k+2}) + z && \text{by (3.1)(a)} \\ &= B(x_i^{k+1} + z, \lambda^{k+2}) && \text{by (v)}. \end{aligned}$$

Thus $B \subset B(x_i^{k+1} + z, \lambda^{k+2})$, and since $r(B) \ge \lambda^{k-1}$, it follows by doubling that $\mu(B(x_i^{k+1}+z, \lambda^{k+2})) \le c\mu(B)$, and therefore $\mu(Q+z) \le c\mu(B)$. Hence,

$$\begin{aligned} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f|\,d\mu &\le \frac{c}{\mu(Q+z)^{1-\gamma}} \int_{Q+z} |f|\,d\mu \\ &\le cM^{\mathrm{dy}}_{\gamma,m,z} f(x) \quad \text{since } x \in B \subset Q + z \text{ and } Q \in \mathcal{D}. \end{aligned}$$

This shows that the expression on the left in (3.3) is at most $cM^{\mathrm{dy}}_{\gamma,m,z} f(x)$ if $z \in \Omega$. Consequently, (3.3) follows by integration with respect to $d\mu(z)$ over Ω if we show that $\mu(\Omega) \ge c\mu(B_{N+3})$, $c > 0$.

The proof that $\mu(\Omega) \ge c\mu(B_{N+3})$ is essentially identical to the proof of the corresponding estimate in [SW1], Lemma (4.7). For completeness, we repeat the argument. We first show that for any j and any $z \in -B(x_j^{k+1}, \lambda^k) + x$, it is true that $B \subset E_j^{k+1} + z$. Since $B(x_j^{k+1}, \lambda^{k+1}) \subset E_j^{k+1}$ by (3.1)(a), it is enough to show that if $y \in B$ then $y - z \in B(x_j^{k+1}, \lambda^{k+1})$. Note that $x - z \in B(x_j^{k+1}, \lambda^k)$, so that

$$\begin{aligned} d(y-z, x_j^{k+1}) &\le K[d(y-z, x-z) + d(x-z, x_j^{k+1})] \\ &\le K[d(y,x) + \lambda^k] \quad \text{by (v)} \\ &\le K[K\{d(y, x_B) + d(x_B, x)\} + \lambda^k] \\ &\le K[K\{\lambda^k + \lambda^k\} + \lambda^k] \quad \text{since } x, y \in B \text{ and } r(B) < \lambda^k \\ &= K(2K+1)\lambda^k < \lambda^{k+1}, \end{aligned}$$

as desired, since $K(2K+1) < \lambda$ $(= 8K^5)$.

Now let

$$\Gamma = \{j : E_j^{k+1} \cap B(x, \lambda^{N+2}) \ne \emptyset\}.$$

We claim that $-B(x_j^{k+1}, \lambda^k) + x \subset B_{N+3}$ if $j \in \Gamma$. Let $w \in B(x_j^{k+1}, \lambda^k)$. We must show that $d(-w + x, 0) < \lambda^{N+3}$. Since $j \in \Gamma$, there exists $u \in E_j^{k+1} \cap B(x, \lambda^{N+2})$. By (3.1)(a), $E_j^{k+1} \subset B(x_j^{k+1}, \lambda^{k+2})$, so that $u \in B(x_j^{k+1}, \lambda^{k+2})$ too. Therefore, $d(u, x_j^{k+1}) < \lambda^{k+2}$, $d(u, x) < \lambda^{N+2}$ and $d(w, x_j^{k+1}) < \lambda^k$. Also, $d(x, 0) < \lambda^N$ since $x \in B_N$. Hence, by repeated (3 times) application of the triangle inequality (iii) and since $k \le N$, we obtain $d(w, 0) < 4K^3\lambda^{N+2}$. Thus,

$$\begin{aligned} d(-w + x, 0) &\le K[d(-w + x, x) + d(x, 0)] \\ &\le K[d(-w, 0) + \lambda^N] \quad \text{by (v) and since } x \in B_N \\ &= K[d(w, 0) + \lambda^N] \\ &\le K[4K^3\lambda^{N+2} + \lambda^N] < \lambda^{N+3} \end{aligned}$$

since $K[4K^3 + 1] < \lambda$. This verifies the claim above.

It follows from the estimates in the previous two paragraphs and from the definition of Ω that Ω contains $-B(x_j^{k+1}, \lambda^k) + x$ if $j \in \Gamma$. Since the sets E_j^{k+1} are pairwise disjoint in j, it follows from (3.1)(a) that the sets $-B(x_j^{k+1}, \lambda^k) + x$ are also pairwise disjoint in j. Hence,

$$\begin{aligned} \mu(\Omega) &\ge \sum_{j\in\Gamma} \mu(-B(x_j^{k+1}, \lambda^k) + x) \\ &= \sum_{j\in\Gamma} \mu(B(x_j^{k+1}, \lambda^k)) && \text{by (vi)} \\ &\ge c \sum_{j\in\Gamma} \mu(E_j^{k+1}) && \text{by (3.1)(a) and doubling} \\ &\ge c\mu(B(x, \lambda^{N+2})) && \text{by definition of } \Gamma \text{ and (3.1)(b)} \\ &\ge c\mu(B(x, \lambda^{N+3})) && \text{by doubling} \\ &= c\mu(B(0, \lambda^{N+3}) + x) && \text{by (v)} \\ &= c\mu(B(0, \lambda^{N+3})) = c\mu(B_{N+3}) && \text{by (vi)} \end{aligned}$$

since $B(0, \lambda^{N+3}) = -B(0, \lambda^{N+3})$. This completes the proof of Lemma (3.2).

Lemma (3.2) reduces the proof of (1.10) to the proof of the corresponding inequality for $M^{\mathrm{dy}}_{\gamma,m,z} f$ with a constant which is independent of m and z. By observing at each step that the constants are independent of z, we will take $z = 0$ for simplicity. Replacing f by $f\sigma$, we then want to show that

$$\|M^{\mathrm{dy}}_{\gamma,m}(f\sigma)\|_{L^q_{w\,d\mu}} \le c\|f\|_{L^p_{\sigma\,d\mu}} \tag{3.4}$$

with c independent of m and f, provided that $1 < p < q < \infty$ and (1.11) holds. Following the ideas in [S1], let

$$X_k = \{x \in X : M^{\mathrm{dy}}_{\gamma,m}(f\sigma)(x) > 2^k\},$$

so that

$$\int_X [M^{\mathrm{dy}}_{\gamma,m}(f\sigma)]^q w\,d\mu \approx \sum_{k=-\infty}^{\infty} 2^{kq} \int_{X_k \setminus X_{k+1}} w\,d\mu. \tag{3.5}$$

We may assume without loss of generality that $f \ge 0$. Let $\{Q_j^k\}_j$ be the maximal (with respect to inclusion) dyadic cubes in $\mathcal{D}$ with

$$\frac{1}{\mu(Q_j^k)^{1-\gamma}} \int_{Q_j^k} f\sigma\, d\mu > 2^k .$$

Then the Q_j^k are pairwise disjoint in j by their maximality and (3.1)(c), and $X_k = \bigcup_j Q_j^k$. Hence (3.5) is at most a constant times

$$\text{(3.6)}\quad \sum_{k,j}\left(\frac{1}{\mu(Q_j^k)^{1-\gamma}}\int_{Q_j^k} f\sigma\, d\mu\right)^q \int_{Q_j^k\setminus X_{k+1}} w\, d\mu = \sum_{k,j} b_{jk}\left(\frac{1}{a_{jk}}\int_{Q_j^k} f\sigma\, d\mu\right)^q$$

where

$$\text{(3.7)}\qquad a_{jk} = \int_{Q_j^k} \sigma\, d\mu, \qquad b_{jk} = \left[\frac{a_{jk}}{\mu(Q_j^k)^{1-\gamma}}\right]^q \int_{Q_j^k\setminus X_{k+1}} w\, d\mu .$$

LEMMA (3.8). *Let $\{Q_i\}_{i\in I}$ be a collection of dyadic cubes in the sense of* (3.1), *and let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be sequences of positive numbers which satisfy for given $1 < p \le q < \infty$ the conditions*

$$\int_{Q_i} \sigma\, d\mu \le c a_i \quad \text{and} \quad \sum_{i: Q_i\subset Q_{i_0}} b_i \le c a_{i_0}^{q/p},$$

with c independent of i and i_0. Then

$$\left[\sum_i b_i\left(\frac{1}{a_i}\int_{Q_i} f\sigma\, d\mu\right)^q\right]^{1/q} \le c\left(\int_X f^p\sigma\, d\mu\right)^{1/p}.$$

The lemma is proved in the usual Euclidean case when $p = q$ in [SW2] and when $a_i = b_i$ and $p \le q$ in [SW1]. The proof for the general case is similar and is omitted. Later in the section, for the proof of Theorem 4, we will need a slightly different version which is valid for cubes with less structure than (3.1) but for a_i, b_i with more structure.

The desired estimate (3.4) follows by applying Lemma (3.8) to (3.6), provided we show that

$$\text{(3.9)}\qquad \Big(\sum_{\substack{j,k\\ Q_j^k\subset Q_s^t}} b_{jk}\Big)^{1/q} \le c\Big(\int_{Q_s^t} \sigma\, d\mu\Big)^{1/p}$$

for each Q_s^t, with c independent of s, t. Up to this point in the proof, we have essentially followed [S1]. We now estimate the left side of (3.9) by duality.

Write $Q_s^t = Q_0$ and

$$F(x) = \sum_{\substack{j,k\\ Q_j^k\subset Q_0}} \frac{a_{jk}}{\mu(Q_j^k)^{1-\gamma}}\chi_{Q_j^k\setminus X_{k+1}},$$

and note that the sets $Q_j^k\setminus X_{k+1}$ are pairwise disjoint in both j and k. Thus the left side of (3.9) equals

$$\begin{aligned}
\Big(\int_{Q_0} F^q w\, d\mu\Big)^{1/q} &= \sup_{g\ge 0: \|g\|_L\le 1} \int_{Q_0} Fg\, d\mu \quad \text{where } L = L^{q'}_{w^{1-q'}d\mu} \\
&= \sup_{\cdots} \int_{Q_0}\Big[\sum_{\substack{j,k\\ Q_j^k\subset Q_0}} \frac{1}{\mu(Q_j^k)^{1-\gamma}}\chi_{Q_j^k}(y)\int_{Q_j^k\setminus X_{k+1}} g\, d\mu\Big]\sigma(y)\, d\mu(y) \\
&\qquad\qquad \text{by definition of } a_{jk} \\
&= \sup_{\cdots} \int_{Q_0}\Big[\int \widetilde{k}(x,y) g(x)\, d\mu(x)\Big]\sigma(y)\, d\mu(y)
\end{aligned}$$

where

$$\widetilde{k}(x,y) = \sum_{\substack{j,k\\ Q_j^k\subset Q_0}} \frac{1}{\mu(Q_j^k)^{1-\gamma}}\chi_{Q_j^k}(y)\chi_{Q_j^k\setminus X_{k+1}}(x).$$

Since the sets $Q_j^k\setminus X_{k+1}$ are pairwise disjoint in j, k, given x, y there is at most one term of the sum with $\chi_{Q_j^k\setminus X_{k+1}}(x) \ne 0$, and y must belong to the same Q_j^k. If $Q_j^k = E_i^l$, then $x, y \in E_i^l$ implies that $d(x,y) \le c\lambda^l$, so that $B_{xy} = B(x, d(x,y)) \subset B(x, c\lambda^l)$. Thus,

$$\begin{aligned}
\mu(B_{xy}) &\le c\mu(B(x,\lambda^l)) \quad \text{by doubling} \\
&\le c\mu(Q_j^k)
\end{aligned}$$

by doubling again since $Q_j^k = E_i^l$, (3.1)(a) holds and $x \in Q_j^k$. It follows that

$$\widetilde{k}(x,y) \le c\frac{1}{\mu(B_{xy})^{1-\gamma}}.$$

Now define

$$\text{(3.10)}\qquad k(x,y) = \frac{1}{\mu(B_{xy})^{1-\gamma}},$$

and note that $k(x,y) \approx k(y,x)$ by doubling. We have shown that $\widetilde{k} \le ck$

and that the expression on the left in (3.9) is bounded by a constant times

$$\sup_{g\ge 0:\|g\|_L\le 1}\int_{Q_0}(Tg)\sigma\,d\mu\,, \tag{3.11}$$

where T is the integral operator

$$Tg(x)=\int_X k(x,y)g(y)\,d\mu(y) \tag{3.12}$$

and L is as before. (The support of g could be restricted above to Q_0, but this is not necessary for our purposes.) To estimate the size of Tg, we will use the following lemma for integral operators. We use the notation $|E|_{wd\mu}$ for the $wd\mu$ measure of a set E, i.e.,

$$|E|_{wd\mu}=\int_E w\,d\mu\,.$$

Also, if $k(x,y)$ is the kernel of an integral operator (3.12) with $k(x,y)\ge 0$, and if B is a ball in X, we write

$$\varphi(B)=\sup_{\substack{x,y\in B\\ d(x,y)\ge cr(B)}} k(x,y) \tag{3.13}$$

where c is a suitably small positive constant depending only on the constant K in (iii).

LEMMA (3.14). *Let (X,d,μ) satisfy all of (i)–(vi), let $1<p<q<\infty$ and let T be an integral operator of type (3.12) with $k(x,y)\ge 0$. Assume also that $k(x,y)\approx k(y,x)$ and that there are constants $C_1,C_2>1$ and $\varepsilon>0$ so that*

$$k(x,y)\le C_1k(x,y')\quad \textit{if}\quad d(x,y')\le C_2d(x,y) \tag{3.15}$$

and

$$\frac{\varphi(B)}{\varphi(B')}\le c\left[\frac{r(B')}{r(B)}\right]^{\varepsilon}\quad \textit{for all balls } B',B \textit{ with } B'\subset 2KB\,. \tag{3.16}$$

If w,v satisfy

$$\Big(\int_B w\,d\mu\Big)^{1/q}\Big(\int_X \min\{\varphi(B),k(x_B,x)\}^{p'}\sigma(x)\,d\mu(x)\Big)^{1/p'}\le c \tag{3.17}$$

for all balls B, where x_B is the center of B, then

$$\sup_{t>0}t|\{x\in X:|Tf(x)|>t\}|_{wd\mu}^{1/q}\le c\|f\|_{L^p_{vd\mu}}\,. \tag{3.18}$$

This is shown (but not stated as a theorem) in [SW1]: see (4.14) and (4.15). In fact, the version proved there does not require $k(x,y)\approx k(y,x)$. In the case of the classical fractional integral on $\mathbb{R}^n$, the result was first given in [GK].

In our case, the kernel k defined by (3.10) satisfies (3.15) since μ is doubling. Let us assume for the moment that (3.16) is also valid, noting that

$$\varphi(B)\approx\frac{1}{\inf_{z\in B}\mu(B(z,r(B)))^{1-\gamma}}\approx\frac{1}{\mu(B)^{1-\gamma}} \tag{3.19}$$

in our case by doubling. Now write (3.11) as

$$\sup_{g\ge 0:\|g\|_L\le 1}\int_0^\infty|\{x\in Q_0:Tg(x)>t\}|_{\sigma d\mu}\,dt\,, \tag{3.20}$$

with $L=L^{q'}_{w^{1-q'}d\mu}$, and apply Lemma (3.14) with p,q,w and v replaced respectively by q',p',σ, and $w^{1-q'}$. Since $p<q$ implies that $q'<p'$, it follows that the integral in (3.20) is at most

$$\int_0^\infty \min\{|Q_0|_{\sigma d\mu},(ct^{-1}\|g\|_L)^{p'}\}\,dt$$
$$\le\int_0^\infty\min\{|Q_0|_{\sigma d\mu},c/t^{p'}\}\,dt=c|Q_0|^{1/p}_{\sigma d\mu}\,,$$

which is the desired estimate (3.9), provided that the analogue of (3.17) is valid. This analogue is

$$\Big(\int_B\sigma\,d\mu\Big)^{1/p'}\Big(\int_X\min\Big\{\frac{1}{\mu(B)^{1-\gamma}},\frac{1}{\mu(B_{x_Bx})^{1-\gamma}}\Big\}^q w(x)\,d\mu(x)\Big)^{1/q}\le C$$

by (3.10) and (3.19), and is valid since it is the same as hypothesis (1.11) of Theorem 3. This completes the proof of Theorem 3 except for verifying (3.16).

To verify (3.16), note by (3.19) and doubling that (3.16) is the same as

$$\mu(B(x,r'))\le c\Big(\frac{r'}{r}\Big)^{\varepsilon}\mu(B(x,r)),\qquad r'<r\,,$$

for (a different) $\varepsilon>0$. This condition is in turn easily seen to be equivalent to the following sort of reverse doubling condition (cf. (1.8)):

(3.21) There exist $\alpha,\beta>1$ such that $\mu(B(x,\alpha r))\ge\beta\mu(B(x,r))$ for all $x\in X$, $r>0$.

The classical reverse doubling condition is (3.21) with $\alpha=2$. Condition (3.21) follows from the doubling property of μ in any homogeneous space X (i.e., (v) and (vi) are not needed), as we now show. Given $x\in X$ and $s>0$, pick y with $s\le d(x,y)\le 2s$ (we assume as always that annuli are not empty). Then, for any z,

$$s\le d(x,y)\le K[d(x,z)+d(z,y)]\,,$$

so that if $z \in B(y, \varepsilon s)$ for $0 < \varepsilon < K^{-1}$ then $(K^{-1} - \varepsilon)s < d(x, z)$. Letting $\delta = K^{-1} - \varepsilon$, we see that $\delta > 0$ and $B(y, \varepsilon s) \cap B(x, \delta s) = \emptyset$. Also, for such z,

$$d(x, z) \le K[d(x, y) + d(y, z)] < K[2s + \varepsilon s] = K(2 + \varepsilon)s,$$

so that $B(y, \varepsilon s) \subset B(x, \theta s)$, $\theta = K(2 + \varepsilon)$. Since $\delta < \theta$, it follows that both $B(y, \varepsilon s)$ and $B(x, \delta s)$ are contained in $B(x, \theta s)$. Since they are disjoint and μ is a measure,

$$\mu(B(x, \theta s)) \ge \mu(B(y, \varepsilon s)) + \mu(B(x, \delta s)). \tag{3.22}$$

Next note that $B(x, \theta s)$ is contained in a fixed enlargement of $B(y, \varepsilon s)$ since if $z \in B(x, \theta s)$ then

$$d(y, z) \le K[d(y, x) + d(x, z)] < K[2s + \theta s] = \frac{K(2 + \theta)}{\varepsilon} \varepsilon s.$$

Thus, by doubling, $\mu(B(y, \varepsilon s)) \ge \eta \mu(B(x, \theta s))$ for some η depending only on μ and K, $0 < \eta < 1$. By (3.22),

$$\mu(B(x, \theta s)) \ge \eta \mu(B(x, \theta s)) + \mu(B(x, \delta s)),$$

so that $\mu(B(x, \theta s)) \ge (1 - \eta)^{-1} \mu(B(x, \delta s))$. This proves (3.21) as can be seen by letting $\beta = (1 - \eta)^{-1}$, replacing δs by r and writing $\theta s = \theta \delta^{-1} r = \alpha r$, where $\alpha = \theta \delta^{-1}$ by definition. This completes the proof of Theorem 3.

The proof of the sufficiency part of Theorem 4 is much simpler. We use the cruder version of "dyadic" balls mentioned in [SW1], i.e., for $\lambda = K + 2K^2$ and each $k \in \mathbb{Z}$, there is a sequence $\{B_j^k\}_j$ of balls of radius λ^k such that if $\widehat{B}_j^k$ denotes the ball with radius λ^{k-1} and the same center as B_j^k, then

(3.23)
(a) every ball in X of radius λ^{k-1} is contained in at least one B_j^k,
(b) $\sum_j \chi_{B_j^k} \le M$ for all k with M independent of k,
(c) $\widehat{B}_i^k \cap \widehat{B}_j^k = \emptyset$ if $i \ne j$, $k \in \mathbb{Z}$.

We call the balls B_j^k *dyadic balls*, and if B is a dyadic ball, say $B = B_j^k$, we write $\widehat{B} = \widehat{B}_j^k$. We also write $r(B)$ for the radius of B. A useful property of dyadic balls is as follows.

(3.24) If $\{B_\alpha\}$ is a collection of dyadic balls and $\{B_j\}$ is a subcollection of maximal (with respect to inclusion) balls, then the balls $\{\widehat{B}_j\}$ are pairwise disjoint.

These properties are taken from (3.5) and (3.6) of [SW1]. Now define the corresponding dyadic maximal function

$$m_\gamma^{\mathrm{dy}} f(x) = \sup_{\substack{\text{dyadic } B \\ x \in B}} \frac{1}{\mu(B)^{1-\gamma}} \int_B |f| \, d\mu.$$

We claim that

$$M_\gamma f(x) \le c m_\gamma^{\mathrm{dy}} f(x), \quad x \in X, \tag{3.25}$$

with c independent of x. Fix x and B with $x \in B$, and choose k such that $\lambda^{k-1} < r(B) \le \lambda^k$. Then $B = B(x_B, r(B)) \subset B(x_B, \lambda^k) \subset B_j^{k+1}$ for some j by (3.23)(a). Since μ is doubling and B, B_j^{k+1} have comparable radii, it follows that $\mu(B_j^{k+1}) \le c\mu(B)$, and consequently

$$\frac{1}{\mu(B)^{1-\gamma}} \int_B |f| \, d\mu \le c \frac{1}{\mu(B_j^{k+1})^{1-\gamma}} \int_{B_j^{k+1}} |f| \, d\mu \le c m_\gamma^{\mathrm{dy}}(f)(x),$$

and (3.25) follows. We remark that (3.25) could also be accomplished by defining $m_\gamma^{\mathrm{dy}} f$ with respect to suitable enlargements of the sets in the grid used in the proof of Theorem 3.

The simple estimate (3.25) (in place of Lemma (3.2)) is what allows us to prove (1.10) without assuming that X satisfies properties (v) and (vi). In fact, by (3.25), it suffices to prove that

$$\|m_\gamma^{\mathrm{dy}}(f\sigma)\|_{L^q_{w\,d\mu}} \le c\|f\|_{L^p_{\sigma\,d\mu}},$$

$f \ge 0$. Now letting

$$X_k = \{x \in X : m_\gamma^{\mathrm{dy}}(f\sigma) > 2^k\},$$

we see as before that if $f \ge 0$ then

$$\int_X [m_\gamma^{\mathrm{dy}}(f\sigma)]^q w \, d\mu \le \sum_{k,j} b_{jk} \left(\frac{1}{a_{jk}} \int_{Q_j^k} f\sigma \, d\mu \right)^q, \tag{3.26}$$

where $\{Q_j^k\}_j$ are maximal dyadic balls satisfying

$$\frac{1}{\mu(Q_j^k)^{1-\gamma}} \int_{Q_j^k} f\sigma \, d\mu > 2^k,$$

and

$$a_{jk} = \int_{Q_j^k} \sigma \, d\mu, \qquad b_{jk} = \left[\frac{a_{jk}}{\mu(Q_j^k)^{1-\gamma}} \right]^q \int_{Q_j^k} w \, d\mu.$$

The main difference between the present situation and the earlier one is that now the Q_j^k belong to the family satisfying (3.23) and (3.24) rather than (3.1). Also, for simplicity, in the definition of b_{jk} we now use the integral of w over all of Q_j^k rather than over $Q_j^k \setminus X_{k+1}$. It follows immediately from hypothesis (1.12) that

$$b_{jk} \le c \Big(\int_{Q_j^k} \sigma \, d\mu \Big)^{q/p} = c a_{jk}^{q/p}.$$

Thus, (3.26) is bounded by a constant times

$$\sum_{j,k} a_{jk}^{q/p}\left(\frac{1}{a_{jk}}\int_{Q_j^k} f\sigma\,d\mu\right)^q.$$

Since $\sigma d\mu$ is a doubling measure, we may apply Lemma (3.15) of [SW1] directly to see that the last sum is at most $c\|f\|^q_{L^p_{\sigma d\mu}}$, which proves Theorem 4, provided that we verify the condition

$$\sum_{Q_j^k\subset Q_s^t} a_{jk}^{q/p} \le c a_{st}^{q/p}.$$

This condition is proved exactly as (3.20) in [SW1], using $q>p$ and the fact that $\sigma d\mu$ is a doubling measure, and in fact does not require the maximality of the Q_j^k.

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Received January 2, 1992
Revised version April 20, 1993 (2883)

STUDIA MATHEMATICA 107 (3) (1993)

Pseudotopologies with applications to one-parameter groups, von Neumann algebras, and Lie algebra representations

by

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Abstract. For any pair E,F of pseudotopological vector spaces, we endow the space $L(E,F)$ of all continuous linear operators from E into F with a pseudotopology such that, if G is a pseudotopological space, then the mapping $L(E,F)\times L(F,G)\ni(f,g)\to gf\in L(E,G)$ is continuous. We use this pseudotopology to establish a result about differentiability of certain operator-valued functions related with strongly continuous one-parameter semigroups in Banach spaces, to characterize von Neumann algebras, and to establish a result about integration of Lie algebra representations.

0. Introduction. If E is a Banach space and $L(E)$ is the space of all continuous linear operators in E, then, if $L(E)$ is endowed with the standard norm topology, then the composition of operators in $L(E)$ is continuous. When $L(E)$ is equipped with either the strong operator topology or weak operator topology, the composition of operators in $L(E)$ fails to be continuous unless E is finite-dimensional. If F is a Fréchet space with a topology that cannot be determined by a single norm, then, as proved by Bastiani [B] and Keller [Ke], there is no reasonable topology on $L(F)$ under which the composition of operators in $L(F)$ is continuous. In this paper, for any pair E,F of pseudotopological vector spaces, we endow the space $L(E,F)$ of all continuous linear operators from E into F with a pseudotopology such that, if G is a pseudotopological space, then the mapping $L(E,F)\times L(F,G)\ni(f,g)\to gf\in L(E,G)$ is continuous. We use this pseudotopology to establish a result about differentiability of certain operator-valued functions related to strongly continuous one-parameter semigroups in Banach spaces, to characterize von Neumann

1991 *Mathematics Subject Classification*: Primary 46A99; Secondary 47D05, 46L10, 22E60, 17B15.

Key words and phrases: pseudotopology, continuity, composition of operators, differentiability, one-parameter semigroup, von Neumann algebra, integration, Lie algebra representation.

Partially supported by KBN grant 2-1167-91-01.