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ON A GLOBALIZATION PROPERTY

Abstract. Let (X,τ) be a topological space. Let Φ be a class of real-valued functions defined on X. A function $\phi \in \Phi$ is called a *local Φ-subgradient* of a function $f : X \to \mathbb{R}$ at a point x_0 if there is a neighbourhood U of x_0 such that $f(x) - f(x_0) \geq \phi(x) - \phi(x_0)$ for all $x \in U$. A function $\phi \in \Phi$ is called a *global Φ-subgradient* of f at x_0 if the inequality holds for all $x \in X$. The following properties of the class Φ are investigated:

(a) when the existence of a local Φ-subgradient of a function f at each point implies the existence of a global Φ-subgradient of f at each point (globalization property),

(b) when each local Φ-subgradient can be extended to a global Φ-subgradient (strong globalization property).

Let (X,τ) be a topological space. Let f be a real-valued function defined on X.

Let Φ be a class of real-valued functions defined on X. We say that the function f is *Φ-convex* if it can be represented as a supremum of functions belonging to Φ.

A function $\phi \in \Phi$ is called a *local Φ-subgradient* of the function f at a point x_0 if there is a neighbourhood U of x_0 such that for all $x \in U$,

$$f(x) - f(x_0) \geq \phi(x) - \phi(x_0)\,. \tag{1}$$

A function $\phi \in \Phi$ is called a *global Φ-subgradient* (briefly, *Φ-subgradient*) of f at x_0 if (1) holds for all $x \in X$.

It is easy to show that the fact that f has a local Φ-subgradient at each point does not imply that f has a Φ-subgradient at each point, nor even that f is Φ-convex.

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Example. Let $X = \mathbb{R}$. Let Φ denote the class of all quadratic functions. Let $f(x) = x^3$. Then f is not bounded from below by any function $\phi \in \Phi$. On the other hand, it has a local Φ-subgradient at each point.

It is interesting, however, that there are classes Φ such that the existence of a local Φ-subgradient of a function f at each point $x_0 \in X$ implies the existence of a global Φ-subgradient of f at each point. We then say that Φ has the *globalization property*. If each local Φ-subgradient can be extended to a global one we say that Φ has the *strong globalization property*.

If the existence of a local Φ-subgradient of a bounded function f at each point $x_0 \in X$ implies the existence of a global Φ-subgradient of f at each point we say that Φ has the *bounded globalization property*.

Let $A \subset X$. We say that the set A has the *Φ-globalization property* (*strong Φ-globalization property*, *bounded Φ-globalization property*) if the family Φ restricted to A has the globalization property (resp. strong globalization property, bounded globalization property).

In particular, if X is a linear topological space and Φ is the class of continuous linear functionals on X then a set A with the Φ-globalization property will be said to have the (*strong, bounded*) *linear globalization property* or briefly the (*strong*, *bounded*) *globalization property.*

Proposition 1. *Let (X, τ) be a linear topological space. Then X has the strong linear globalization property.*

Proof. We begin with the one-dimensional space $X = \mathbb{R}$. Recall that a function f defined on the real line is convex if and only if

$$\limsup_{t\to t_0+0} \frac{f(t)-f(t_0)}{t-t_0} \geq \liminf_{t\to t_0-0} \frac{f(t)-f(t_0)}{t-t_0}\,. \tag{2}$$

The existence of a local linear subgradient of f at each point implies (2). Thus f is convex. For arbitrary dimension we simply observe that the restriction of f to any one-dimensional subspace is convex. This implies that f is convex. Therefore each local linear subgradient is a (global) linear subgradient. ■

The same considerations give

Proposition 2. *A convex set in a linear topological space has the strong linear globalization property.*

It is interesting to know which families of linear functionals have the bounded globalization property.

Proposition 3. *Let X be the unit sphere in a Banach space $(Y, \|\cdot\|)$, and $X = \{x \in Y : \|x\| = 1\}$. Let Φ be the family of continuous linear functionals restricted to X. Then Φ has the bounded globalization property.*

P r o o f. Let f be a bounded function defined on X and having a local Φ-subgradient at each point $x_0 \in X$. Let $a \in \mathbb{R}$ be chosen so that $f_1(x) = f(x) - a \geq 0$ for all $x \in X$. We show that f_1 has a Φ-subgradient at each point of X, which automatically implies that so does f. We extend f_1 to the whole space Y putting

$$f_2(x) = \begin{cases} \|x\| f_1(x/\|x\|) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

It is easy to see that f_2 has local Φ-subgradient 0 at 0, because $\inf_{x \in X}(f_2(x) - \inf\{\phi(x) : \phi \in \Phi\}) \geq 0$. Take any point $x_0 \neq 0$. The function $f_1(x/\|x\|)$ has a local Φ-subgradient ϕ_1 at $x_0/\|x_0\|$. Observe that $f_1(x_0/\|x_0\|) - \phi_1(x_0/\|x_0\|) \geq 0$.

Let ϕ_2 denote a functional of norm one supporting X at $x_0/\|x_0\|$, i.e. such that $\phi_2(x_0/\|x_0\|) = 1$. Observe that the functional $\phi(x) = \phi_1(x) + b\phi_2(x)$ is a local linear subgradient of f_1 at $x_0/\|x_0\|$ for all $b \geq 0$. If $b = f_1(x_0/\|x_0\|) - \phi_1(x_0/\|x_0\|)$ then $\phi(x_0/\|x_0\|) = f_1(x_0/\|x_0\|)$ and $f_1(x) \geq \phi(x)$ in some neighbourhood V of $x_0/\|x_0\|$ on X. Then by the homogeneity of f_2 and ϕ, $f_2(x_0) = \phi(x_0)$ and $f_2(x) \geq \phi(x)$ in a neighbourhood U of x_0. Thus ϕ is a local linear subgradient of f_2 at x_0.

By Proposition 1 each local linear subgradient is also a global linear subgradient. Observe that its restriction to X gives a Φ-subgradient on X. ∎

COROLLARY 1. *Let f be a periodic function with period 2π. If at each point t there is a local subgradient of f of the form $a_t \sin t + b_t \cos t$, where all a_t, b_t are bounded as functions of t, then at each t there is a global subgradient of this form.*

P r o o f. We simply rewrite Proposition 3 in polar coordinates. ∎

COROLLARY 2. *Let $f(t,s)$ be a function with period 2π with respect to t, $-\pi/2 \leq s < \pi/2$. If at each point (t,s) there is a local subgradient of f of the form $a_{(t,s)} \sin s + b_{(t,s)} \cos s \sin t + c_{(t,s)} \cos s \cos t$, where all $a_{(t,s)}, b_{(t,s)}, c_{(t,s)}$ are bounded as functions of (t,s), then at each (t,s) there is a global subgradient of this form.*

P r o o f. We simply rewrite Proposition 3 in spherical coordinates. ∎

PROBLEM 1. Does the family Φ in Proposition 3 have the globalization property?

Of course if an open set $X \subset \mathbb{R}^2$ is not connected Proposition 3 does not hold. Indeed, if $X = X_1 \cup X_2$, where X_1, X_2 are disjoint and open in X, then the function

$$f(x) = \begin{cases} 1 & \text{for } x \in X_1, \\ 0 & \text{for } x \in X_2 \end{cases}$$

has local Φ-subgradient 0 at each point of X. But it is not a Φ-subgradient of f at x_0 for $x_0 \in X_1$.

For connected sets the situation is more complicated. Of course in the case of the space $\mathbb{R}^1$ each connected set is automatically convex and Proposition 3 holds. For $\mathbb{R}^2$ we have the following

PROPOSITION 4. *Let X be a simply connected non-convex open set in $\mathbb{R}^2$. Let Φ be the restrictions of linear functionals to X. Then Φ does not have the globalization property.*

Proof. Since X is not convex and simply connected there is a half-plane $H = \{(x, y) : ax + by < 1\}$ such that the intersection of H and X is not connected. We denote two components of $X \cap H$ by X_1 and X_2. Consider the intersection of the line $L = \{(x, y) : ax + by = 1\}$ with X. We can find $(x_1, y_1) \in \overline{X}_1 \cap L$ which is a boundary point of $\operatorname{conv}(X_1 \cap L)$ and an interior point of $\operatorname{conv}(X \cap L)$ on L.

Let $L_1 = \{(x, y) : a_1x + b_1y = 1\}$ be a line containing (x_1, y_1) and such that there is $(x_2, y_2) \in X_1$ with $a_1x_2 + b_1y_2 > 1$. The existence of such a line is easy to show. Let $f(x, y) = \max[0, a_1x + b_1y - 1]$. It is easy to see that f has a local Φ-subgradient at each point of X. On the other hand, f has no Φ-subgradient at any point of X such that $a_1x + b_1y > 1$, in particular at (x_2, y_2). ∎

As a consequence of Proposition 2 we obtain

PROPOSITION 5. *Let Y be a convex subset of a linear topological space. Let Ψ be a class of linear functionals restricted to Y. Let X be a topological space and let h be a homeomorphism of X onto Y. Define $\Phi = \{\phi : \phi(x) = \psi(h(x)),\ \psi \in \Psi\}$. Then Φ has the globalization property.*

Proof. Let f be a real-valued function on X. Suppose that ϕ is a local Φ-subgradient of f at $x_0 \in X$. Since h is a homeomorphism the image of an open set is open. Thus $\psi(y) = \phi(h^{-1}(y))$ is a local Ψ-subgradient of $f(h^{-1}(y))$ at $y_0 = h(x_0)$. Since this holds for all x_0 and h maps X onto Y, the function $f(h^{-1}(y))$ has a local Ψ-subgradient at each point of Y. Then it has a Ψ-subgradient, call it again ψ, at each $y_0 \in Y$, i.e.

$$f(h^{-1}(y)) - f(h^{-1}(y_0)) \ge \psi(y) - \psi(y_0) \quad \text{for all } y \in Y\,.$$

Thus

$$f(x) - f(x_0) \ge \psi(h(x)) - \psi(h(x_0)) \quad \text{for all } x \in X$$

and the function $\phi(x) = \psi(h(x)) \in \Phi$ is a Φ-subgradient of f. ∎

PROBLEM 2. Is it essential in Proposition 5 that the mapping h is one-to-one?

PROBLEM 3. Suppose that a family Φ of functions defined on a topological space X has the globalization property. Are there a convex set Y in a topological space and a homeomorphism h of X onto Y such that the functions $\phi(h^{-1}(y))$ are linear?

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