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On arithmetic progressions with equal products

by

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To Wolfgang M. Schmidt
at the occasion of his sixtieth birthday

1. Introduction. In this paper we consider the diophantine equation

$$x(x+d_1)\ldots(x+(L-1)d_1) = y(y+d_2)\ldots(y+(M-1)d_2) \tag{1}$$

in positive integers d_1, d_2, $L > 1$, $M > 1$, x, y. We assume throughout the paper that d_1 and d_2 are fixed and that L/M has a given ratio. Put $k = \gcd(L, M)$, $l = L/k$ and $m = M/k$. Hence l and m are fixed and $\gcd(l, m) = 1$.

There are several results in the literature in case $d_1 = d_2 = 1$. In 1963 Mordell [6] proved that (1) with $(L, M) = (2, 3)$ implies that $(x, y) = (2, 1)$ or $(14, 5)$. MacLeod and Barrodale [5] showed in 1970 that (1) has no solutions if $(L, M) = (2, 4)$, $(2, 6)$, $(2, 8)$, $(2, 12)$, $(4, 8)$ or $(5, 10)$ and admits only the solution $(x, y) = (8, 1)$ if $(L, M) = (3, 6)$. Two years later Boyd and Kisilevsky [1] proved that $(x, y) = (2, 1), (4, 2), (55, 19)$ are the only solutions of (1) if $(L, M) = (3, 4)$. For fixed L and $M = 2L$, MacLeod and Barrodale further showed that (1) admits only finitely many solutions. In 1990 Saradha and Shorey [8] proved that there exists only one solution with $M = 2L$, namely $(L, M, x, y) = (3, 6, 8, 1)$. In 1991 they showed in [9] that (1) has no solutions with $M = 3L$ or $M = 4L$. Recently, Mignotte and Shorey showed that this is also the case when $M = 5L$ or $M = 6L$. In general, for $m > 1$ and $M = mL$, Saradha and Shorey [10] proved that equation (1) implies that $\max(L, x, y)$ is bounded by an effectively computable number depending only on m.

In 1992 Saradha and Shorey [11] started the study of equation (1) for more general pairs (d_1, d_2). They showed that if $d_1 = d_2 = d$ and $l = 1$, $m > 1$ then there exists an effectively computable upper bound for $k = L, x$ and y which depends only on d and m. Later, Saradha and Shorey [12] showed that equation (1) implies that k is bounded by an effectively computable number depending only on d_1, d_2 and m. They further showed that x and y are bounded by such a number unless

(i) $m = k = 2$, $d_1 = 2d_2^2$, $x = y^2 + 3d_2 y$, or

(ii) d_1/d_2^m is a product of $m > 2$ distinct positive integers composed of primes not exceeding m and $m \ge \alpha(k)$ where $\alpha(k) = 14$ for $2 \le k \le 7$, $\alpha(8) = 50$ and $\alpha(k) = \exp(k \log k - 1.25475k - \log k + 1.56577)$ for $k \ge 9$.

Condition (i) is necessary. In the present paper we shall show that condition (ii) is superfluous.

The authors [13] studied equation (1) with $L = M$. Observe that in this case $d_1 = d_2$ implies $x = y$. Therefore there is no loss of generality in assuming that $d_1 < d_2$ and $\gcd(d_1, d_2, x, y) = 1$. We proved that under these assumptions L, M, x and y are bounded by an effectively computable number depending only on d_2 unless $d_1 = 1$, $d_2 = 4$, $x = L + 1$, $y = 2$. Observe that

$$(L+1)(L+2)\dots(2L) = 2 \cdot 6 \cdot \ldots \cdot (4L-2) \qquad \text{for } L = 2, 3, \dots,$$

since both sides equal $(2L)!/L!$. Hence, in case $L = M$ there is an infinite class of exceptions.

The following two theorems cover all pairs (L, M) with $L \ne M$ dealt with in the literature up to now.

THEOREM 1. *Let d_1, d_2, L, M, x and y be as in the first paragraph of this section. If*

$$L \in \{2, 4\} \quad \text{and} \quad M \text{ is odd} \tag{2}$$

then $\max(x, y) \le C_1$ *where C_1 is some effectively computable number depending only on d_1, d_2 and M.*

THEOREM 2. *Let d_1, d_2, L, M, x and y be as in the first paragraph of this section. Suppose that* $\gcd(L, M) > 1$ *and $L \ne M$. Then*

$$\max(L, M, x, y) \le C_2 \tag{3}$$

where C_2 is some effectively computable number depending only on d_1, d_2 and L/M, unless

$$(d_1, d_2, L, M, x, y) \text{ or } (d_2, d_1, M, L, y, x) \quad \text{equals} \quad (d, 2d^2, 4, 2, z, z^2 + 3dz)$$

for some positive integers d and z.

2. The proof of Theorem 1. We shall use the following lemma.

LEMMA 1. *Let $P(X)$ be an odd monic polynomial with real coefficients of degree $M > 1$ such that for some positive number v there are $(M-1)/2$ distinct real numbers β with $P(\beta) = v$, $P'(\beta) = 0$. Then*

$$P(X) = a_1 T_M(a_2 X)$$

where $T_M(X)$ *is the* M*-th Chebyshev polynomial* $\cos(M \arccos X)$ *and* a_1 *and* a_2 *are non-zero real constants.*

Proof. As P is odd, there are also $(M-1)/2$ distinct real numbers β with $P(\beta) = -v$, $P'(\beta) = 0$. Since P' has degree $M-1$, the union of the numbers β is the set of roots of P' and all the roots of P' are simple. It follows that every root of P' is a point where P has a local extremum. Since P is monic and odd, the extremum attained for the lowest value of β is a maximum v, then follows a minimum $-v$, a maximum v and so forth, alternating and ending with a minimum $-v$. Observe that there is a unique point β_0, greater than the largest value where P attains a maximum, such that $P(\beta_0) = v$, $P'(\beta_0) > 0$. Of course, $P(-\beta_0) = -v$, $P'(-\beta_0) > 0$. Define $\widetilde{P}_M(X) = \beta_0^{-M} P(\beta_0 X)$. Then $\widetilde{P}_M(X)$ is a monic polynomial of degree M with $|\widetilde{P}_M(X)| \le \beta_0^{-M} v$ for $|X| \le 1$ and it assumes the values $\beta_0^{-M} v$ and $-\beta_0^{-M} v$ each $(M+1)/2$ times in the interval $[-1,1]$ in alternating way.

Put $\widetilde{T}_M(X) = 2^{-M+1} T_M(X)$. Then $\widetilde{T}_M$ has the smallest maximum absolute value on $[-1,1]$ among all monic polynomials of degree M and every monic polynomial $\ne \pm\widetilde{T}_M$ has a higher maximum absolute value. (See e.g. [7], pp. 56–57.) Suppose $\widetilde{P}_M \ne \widetilde{T}_M$. Then $\beta_0^{-M} v > \max_{-1 \le X \le 1} |\widetilde{T}_M(X)|$, so that $\widetilde{P}_M(X) - \widetilde{T}_M(X)$ is positive at each point β with $\widetilde{P}_M(\beta) = \beta_0^{-M} v$ and negative at each point β with $\widetilde{P}_M(\beta) = -\beta_0^{-M} v$. Since both $\widetilde{P}_M$ and $\widetilde{T}_M$ are monic, it follows that $\widetilde{P}_M - \widetilde{T}_M$ is a polynomial of degree at most $M-1$ which has M sign changes in the interval $[-1,1]$, which is a contradiction. Thus $\widetilde{P}_M = \widetilde{T}_M$, which implies that

$$P(X) = 2(\beta_0/2)^M T_M(X/\beta_0). \quad \blacksquare$$

The second lemma is due to Brindza. It is proved by the method of estimating linear forms of logarithms.

LEMMA 2. *Let* $f(X) \in \mathbb{Z}[X]$, $f(X) = a_0(X-\alpha_1)^{r_1} \ldots (X-\alpha_n)^{r_n}$, *be a polynomial with distinct roots* $\alpha_1, \ldots, \alpha_n$. *Then there exists an effectively computable number* C_3 *depending only on* f *such that the equation* $z^2 = f(y)$ *in rational integers* y, z *implies* $\max(|y|,|z|) \le C_3$ *unless at most two exponents* r_j *are odd.*

Proof. See [2] or [15], Theorem 8.3.

Proof of Theorem 1. Consider equation (1) with $L = 2$. Since $x(x+d_1) = (x + \frac{1}{2}d_1)^2 - \frac{1}{4}d_1^2$, this implies

$$z^2 = 4y(y+d_2)\ldots(y+(M-1)d_2) + d_1^2 \tag{4}$$

where $z = 2x + d_1$. Now consider equation (1) with $L = 4$. Then

$$(x^2 + 3d_1 x + d_1^2)^2 - d_1^4 = y(y+d_2)\ldots(y+(M-1)d_2)$$

whence

$$z^2 = y(y+d_2)\dots(y+(M-1)d_2)+d_1^4 \tag{5}$$

where $z = x^2 + 3d_1x + d_1^2$. We conclude that all cases of Theorem 1 can be reduced to an equation

$$z^2 = \delta y(y+d_2)\dots(y+(M-1)d_2)+c^2$$

where c is some positive rational integer and $\delta = 1$ or 4.

We deal first with the cases with $M = 3$. In these cases equation (1) is reduced to the elliptic equation

$$z^2 = \delta y(y+d_2)(y+2d_2)+c^2. \tag{6}$$

Put $f(Y) = \delta Y(Y+d_2)(Y+2d_2)+c^2$. According to Lemma 2 we have $\max(|y|,|z|) \le C_3$ where C_3 depends only on d_2 and c unless f has a double root α. In the latter case we have

$$0 = \delta^{-1}f'(\alpha) = 3\alpha^2 + 6d_2\alpha + 2d_2^2,$$

which implies $\alpha = -d_2 \pm \frac{d_2}{3}\sqrt{3}$. Hence

$$0 = f(\alpha) = \delta(\alpha^3 + 3d_2\alpha^2 + 2d_2^2\alpha) + c^2 = \mp\tfrac{2}{9}\delta d_2^3\sqrt{3} + c^2.$$

Since δ, c and d_2 are rational, this implies $d_2 = 0$, which is a contradiction. So we obtain $\max(x,y) \le C_4$ where C_4 is some computable number depending only on d_1 and d_2.

We are left with the cases $L \in \{2,4\}$ and M is odd, $M \ge 5$. Here we consider the hyperelliptic equation $z^2 = f(y)$ where

$$f(Y) := \delta Y(Y+d_2)\dots(Y+(M-1)d_2)+c^2 \tag{7}$$

and c is some positive rational integer. According to Lemma 2 we have $\max(y,z) \le C_3$ where C_3 depends only on c, d_2 and M, unless f has exactly one root of odd order. In the latter case we may assume without loss of generality that

$$f(Y) = \delta(Y-\alpha_1)^{r_1}(Y-\alpha_2)^{r_2}\dots(Y-\alpha_n)^{r_n}$$

with $\alpha_1,\dots,\alpha_n$ distinct roots, r_1 odd and $r_2,\dots,r_n$ even. Since $f(Y)-c^2$ has M distinct real roots, the roots of f' are real and simple by Rolle's theorem. Thus $r_1 = 1$, $r_2 = \dots = r_n = 2$. Therefore $M = 2n-1$ and f has $(M-1)/2$ double roots $\alpha_2,\dots,\alpha_n$.

Consider the polynomial

$$g(Y) := \left(Y - \frac{M-1}{2}\right)\left(Y - \frac{M-3}{2}\right)\dots\left(Y + \frac{M-3}{2}\right)\left(Y + \frac{M-1}{2}\right).$$

Observe that g is an odd function and that

$$f(Y) = \delta d_2^M g\left(\frac{Y}{d_2} + \frac{M-1}{2}\right) + c^2.$$

Since f has $(M-1)/2$ double roots, the polynomial g has $(M-1)/2$ distinct real values β with $g(\beta) = -\delta c^2 d_2^{-M}$, $g'(\beta) = 0$. Since g is odd, Lemma 1 gives that the function g is of the form $g(Y) = a_1 T_M(a_2 Y)$ where T_M is the Mth Chebyshev polynomial and a_1 and a_2 are non-zero real constants. This implies that

$$f(Y) = \delta a_1 d_2^M T_M\left(\frac{a_2}{d_2}Y + \frac{a_2(M-1)}{2}\right) + c^2.$$

We infer from (7) that

$$Y(Y+d_2)\dots(Y+(M-1)d_2) = a_1 d_2^M T_M\left(\frac{a_2}{d_2}Y + \frac{a_2(M-1)}{2}\right).$$

However, it follows from the definition of Chebyshev polynomials that for $M > 3$ the roots of T_M are not equidistant, which yields a contradiction. We conclude that $\max(x,y) \le C_5$ where C_5 is some number depending only on d_1, d_2 and M. ∎

3. Lemmas for the proof of Theorem 2. Using the notation of the first paragraph of the introduction we rewrite (1) as follows:

$$x(x+d_1)\dots(x+(lk-1)d_1) = y(y+d_2)\dots(y+(mk-1)d_2). \tag{8}$$

Without loss of generality we shall assume that $l < m$. We shall frequently use the Vinogradov symbol $\ll$ and then tacitly assume that the implied constants are effectively computable positive numbers depending only on d_1, d_2, l and m. Note that l and m are completely determined by L/M.

Lemma 3. *$k \ll \log(x+1)$.*

Proof. We assume that k is larger than some suitable number depending only on d_1, d_2, l and m. We express this by saying that k is taken sufficiently large. Let p be the smallest prime number which does not divide $d_1 d_2$. Then

$$\operatorname{ord}_p(y(y+d_2)\dots(y+(mk-1)d_2)) \ge \left[\frac{mk}{p}\right] + \left[\frac{mk}{p^2}\right] + \dots$$

On the other hand,

$$\operatorname{ord}_p(x(x+d_1)\dots(x+(lk-1)d_1)) \le \max_{i=0,\dots,lk-1} \operatorname{ord}_p(x+id_1) + \left[\frac{lk}{p}\right] + \left[\frac{lk}{p^2}\right] + \dots$$

It now follows from (8) and $l < m$ that, for k sufficiently large,

$$k \ll \left[\frac{mk}{p}\right] - \left[\frac{lk}{p}\right] \le \max_{i=0,\dots,lk-1} \operatorname{ord}_p(x+id_1) \ll \log(x+kd_1).$$

Hence $k \ll \log(x+1)$. ∎

Lemma 4. *$x^l - y^m \ll y^{m-1}\log(y+1)$, $y^m - x^l \ll x^{l-1}\log(x+1)$.*

Proof. We have, by (8), $y^{km} \le (x + kld_1)^{kl}$. Hence, by Lemma 3, $y^{m/l} - x \le kld_1 \ll \log(x+1)$. This implies $y^m - x^l \ll x^{l-1}\log(x+1)$. For the proof of the first inequality we derive from (8) and Lemma 3 as above that $x^{l/m} - y \ll \log(x+1)$, which implies that $\log(x+1) \ll \log(y+1)$ and the first inequality follows. ∎

The following inequalities follow immediately from Lemmas 3 and 4:

$$k \ll \log(y+1), \tag{9}$$

$$x^l \ll y^m, \quad y^m \ll x^l, \quad |x^l - y^m| \ll \min(x^l, y^m)\frac{\log y}{y}. \tag{10}$$

Because of Lemma 3, (9) and (10) we may assume that x and y are larger than some suitable number depending only on d_1, d_2, l and m. We express this by saying that x (or y) is taken sufficiently large. From now onwards, we shall assume without reference that x and y are sufficiently large.

We adopt the notation of [10] in slightly modified form. We define positive integers $A_j(\nu,k)$ for $\nu \in \{l,m\}$ by

$$z(z+1)\ldots(z+\nu k-1) = \sum_{j=0}^{\nu k-1} A_j(\nu,k) z^{\nu k-j}.$$

Further we determine rational numbers $B_j(\nu,k)$ and $H_j(\nu,k)$ such that

$$(z^\nu + B_1(\nu,k)z^{\nu-1} + \ldots + B_\nu(\nu,k))^k = \sum_{j=0}^{\nu k} H_j(\nu,k) z^{\nu k-j}$$

satisfies

$$H_j(\nu,k) = A_j(\nu,k) \quad \text{for } 0 \le j \le \nu.$$

We introduce the notation

$$G_j(\nu,k) = A_j(\nu,k) - H_j(\nu,k) \quad \text{for } 0 < j \le \nu k.$$

LEMMA 5. *There exist effectively computable absolute constants c_1 and c_2 such that*

(a) $A_j(\nu,k) \le (\nu k)^{2j}$ *for* $0 \le j < \nu k$,

(b) $B_j(\nu,k) \le c_1^{j^{3/2}}(\nu k)^{2j}$ *for* $1 \le j \le \nu$,

(c) $|G_j(\nu,k)| \le c_2^{j\sqrt{\nu}}(\nu k)^{2j}$ *for* $0 < j \le \nu k$,

(d) $k^{2j-1}B_j(\nu,k) \in \mathbb{Z}$ *for* $1 \le j \le \nu$,

(e) $k^{2j-1}G_j(\nu,k) \in \mathbb{Z}$ *for* $0 < j \le \nu k$.

Proof. (a) $A_j(\nu,k) \le \binom{\nu k-1}{j}(\nu k)^j < (\nu k)^{2j}$.

(b) As the proof of [10], Lemma 1.

(c) As the proof of [10], Lemma 2.

(d), (e) As the proof of [10], Lemma 3. ∎

Put

$$(11)\qquad \begin{aligned} L_j &= B_j(l,k)d_1^j \qquad \text{for } 1 \le j \le l,\\ M_j &= B_j(m,k)d_2^j \qquad \text{for } 1 \le j \le m, \end{aligned}$$

$$(12)\qquad \begin{aligned} L_j^* &= G_j(l,k)d_1^j \qquad \text{for } 1 \le j \le lk,\\ M_j^* &= G_j(m,k)d_2^j \qquad \text{for } 1 \le j \le mk. \end{aligned}$$

Then

$$(13)\qquad \begin{aligned} &x(x+d_1)\ldots(x+(lk-1)d_1)\\ &\quad = (x^l + L_1x^{l-1} + \ldots + L_l)^k + L_{l+1}^* x^{kl-l-1} + L_{l+2}^* x^{kl-l-2} + \ldots \end{aligned}$$

and

$$(14)\qquad \begin{aligned} &y(y+d_2)\ldots(y+(mk-1)d_2)\\ &\quad = (y^m + M_1y^{m-1} + \ldots + M_m)^k\\ &\qquad\qquad + M_{m+1}^* y^{km-m-1} + M_{m+2}^* y^{km-m-2} + \ldots \end{aligned}$$

Lemma 6.

$$(15)\qquad x^l + L_1x^{l-1} + \ldots + L_l = y^m + M_1y^{m-1} + \ldots + M_m.$$

Proof. By (13), (14), (8), (12) and Lemma 5(c),

$$\begin{aligned} D :&= |(x^l + L_1x^{l-1} + \ldots + L_l)^k - (y^m + M_1y^{m-1} + \ldots + M_m)^k|\\ &= \Big|\sum_{i=l+1}^{lk} L_i^* x^{lk-i} - \sum_{j=m+1}^{mk} M_j^* y^{mk-j}\Big|\\ &\le \sum_{i=l+1}^{lk} c_2^{i\sqrt{l}}(lkd_1)^{2i}x^{lk-i} + \sum_{j=m+1}^{mk} c_2^{j\sqrt{m}}(mkd_2)^{2j}y^{mk-j}. \end{aligned}$$

By Lemma 3 and (9), we obtain

$$\begin{aligned} D &\le \frac{(lkd_1)^{2l+2}c_2^{(l+1)\sqrt{l}}x^{lk-l-1}}{1 - c_2^{\sqrt{l}}(lkd_1)^2/x} + \frac{(mkd_2)^{2m+2}c_2^{(m+1)\sqrt{m}}y^{mk-m-1}}{1 - c_2^{\sqrt{m}}(mkd_2)^2/y}\\ &\ll \left(\frac{k^2}{x}\right)^{l+1} x^{lk} + \left(\frac{k^2}{y}\right)^{m+1} y^{mk}. \end{aligned}$$

On the other hand, we have $L_1 > 0$, $M_1 > 0$ and

$$\begin{aligned} D &= |(x^l + L_1x^{l-1} + \ldots + L_l)^k - (y^m + M_1y^{m-1} + \ldots + M_m)^k| \\ &\geq |(x^l + L_1x^{l-1} + \ldots + L_l) - (y^m + M_1y^{m-1} + \ldots + M_m)| \\ &\quad \times \min(x^{l(k-1)}, y^{m(k-1)}). \end{aligned}$$

Suppose (15) does not hold. Then, by (11), Lemma 5(d) and $l < m$, it follows that $D \geq w^{k-1}/k^{2m-1}$ with $w = \min(x^l, y^m)$. On combining the lower and upper bound for D we obtain, using the fact that $l < m$ and $x \gg y$ by Lemma 4,

$$w^{k-1} \leq k^{2m-1}D \ll \frac{k^{4m+1}}{y} \max(x^{lk-l}, y^{mk-m}).$$

Hence, by (9) and (10),

$$w^{k-1} \ll \frac{k^{4m+1}}{y}(w + |x^l - y^m|)^{k-1} \ll \frac{(\log y)^{4m+1}}{y} w^{k-1}\left(1 + \frac{\log y}{y}\right)^{k-1}.$$

This implies, by (9),

$$\log y \ll \log\log y + k \log\left(1 + \frac{\log y}{y}\right) \ll \log\log y + \frac{(\log y)^2}{y}.$$

This proves that $y \ll 1$, which is a contradiction. ∎

LEMMA 7. *We have*

$$L_i^* = 0 \quad \textit{for } l < i < 2l$$

and

$$M_j^* = 0 \quad \textit{for } m < j < 2m. \tag{16}$$

Proof. Define I and J by

$$L_{l+1}^* = \ldots = L_{I-1}^* = 0, \quad L_I^* \neq 0, \qquad M_{m+1}^* = \ldots = M_{J-1}^* = 0, \quad M_J^* \neq 0.$$

By (15), (8), (13) and (14), we obtain $\sum_{i=I}^{lk} L_i^* x^{lk-i} = \sum_{j=J}^{mk} M_j^* y^{mk-j}$. Therefore, by (10), it suffices to show that either $A_{l+1} = \ldots = A_{2l-1} = 0$ or $B_{m+1} = \ldots = B_{2m-1} = 0$. Suppose that this assertion is false. Then we can take $l < I < 2l$ and $m < J < 2m$. Observe that $mI = lJ$ and $\gcd(l,m) = 1$ imply $l \,|\, I$, a contradiction. We prove the lemma when $mI < lJ$ and the proof for the case $mI > lJ$ is similar. We have

$$mI \leq lJ - 1. \tag{17}$$

Hence, by (12), Lemma 5(c), Lemma 3 and (9),

$$\begin{aligned}|L_I^* x^{lk-I}| &\le \sum_{i=I+1}^{lk} |L_i^*| x^{lk-i} + \sum_{j=J}^{mk} |M_j^*| y^{mk-j} \\ &\le \sum_{i=I+1}^{lk} c_2^{i\sqrt{l}} (lkd_1)^{2i} x^{lk-i} + \sum_{j=J}^{mk} c_2^{j\sqrt{m}} (mkd_2)^{2j} y^{mk-j} \\ &\le 2c_2^{(I+1)\sqrt{l}} \frac{(lkd_1)^{2I+2}}{x^{I+1}} x^{lk} + 2c_2^{J\sqrt{m}} \frac{(mkd_2)^{2J}}{y^J} y^{mk}.\end{aligned}$$

By (12), Lemma 5(e) and $L_I^* \neq 0$, we have $|L_I^*| \ge k^{-2I}$. Hence, by $I < 2l$, Lemma 3 and (10),

$$\begin{aligned}(18)\quad 1 &\le 2c_2^{(I+1)\sqrt{l}} k^{2I} \frac{(lkd_1)^{2I+2}}{x} + 2c_2^{J\sqrt{m}} k^{2I} \frac{(mkd_2)^{2J}}{y^{J-mI/l}} \left(\frac{y^m}{x^l}\right)^{k-I/l} \\ &\le 2\frac{(c_2^{\sqrt{l}} lkd_1)^{8l}}{x} + 2\frac{c_2^{J\sqrt{m}} k^{4l} (mkd_2)^{2J}}{y^{J-mI/l}} \left(1 + \frac{|y^m - x^l|}{x^l}\right)^k \\ &\le \frac{1}{2} + 2\frac{c_2^{J\sqrt{m}} k^{4l} (mkd_2)^{2J}}{y^{J-mI/l}} \left(1 + \frac{c_3 \log y}{y}\right)^k\end{aligned}$$

for some number c_3 depending only on d_1, d_2, l and m. Since $J < 2m$, (18) implies, by (17),

$$(19)\qquad y^{1/l} \le c_2^{J\sqrt{m}} (mkd_2)^{10m} \left(1 + \frac{c_3 \log y}{y}\right)^k.$$

By (19) and (9)

$$(20)\qquad \log y \ll \log k + k \log\left(1 + \frac{c_3 \log y}{y}\right) \ll \log\log y + \frac{(\log y)^2}{y}.$$

It is clear from (20) that $y \ll 1$, a contradiction. ∎

The next lemma is due to R. Balasubramanian.

Lemma 8. *Let $m > 2$. There exists an effectively computable number C_6 depending only on m and an integer q with $m < q < 2m$ such that*

$$(21)\qquad G_q(m,k) \neq 0 \quad \textit{for } k \ge C_6.$$

Proof. See [10], Lemma 7.

Lemma 9. *Suppose that $f(X)$ and $g(Y)$ are polynomials of positive degree with rational numbers as coefficients. Assume that the degrees of f and g are relatively prime. Then $f(X) - g(Y)$ is irreducible over the rationals.*

Proof. This is due to Ehrenfeucht [4]. Cf. [14], p. 94 and [3].

4. Proof of Theorem 2. It suffices to show that equation (8) with $k > 1$ and $l < m$ implies that $\max(k, x, y) \le C_2$. By the result of Saradha and Shorey [10] mentioned in the introduction we may assume $m > 2$. By Lemma 3 and (10) we may take x and y sufficiently large so that (16) holds. Then, by (12), Lemma 8 and $m > 2$, we have $k \ll 1$. We may therefore assume that k is fixed. Hence $L_1, \ldots, L_l$ and $M_1, \ldots, M_m$ are fixed. If

$$X(X + d_1)\ldots(X + (lk-1)d_1) - Y(Y + d_2)\ldots(Y + (mk-1)d_2) \tag{22}$$

and

$$(X^l + L_1X^{l-1} + \ldots + L_l) - (Y^m + M_1Y^{m-1} + \ldots + M_m) \tag{23}$$

have no non-constant common factor, then the resultant of both polynomials with respect to X is a non-zero polynomial in Y which is a linear combination of the polynomials (22) and (23). Since every sufficiently large solution (x, y) of (8) is also a solution of (15), it follows that y is a zero of the resultant. This implies that y is bounded. This contradicts our assumption that y is sufficiently large. If (22) and (23) have a non-constant common factor, then we see from Lemma 9 that (22) has to be divisible by (23).

Put $g(Y) = Y^m + M_1Y^{m-1} + \ldots + M_m$. By taking $Y = 0, -d_2, -2d_2, \ldots, -(mk-1)d_2$ we find that each polynomial

$$f_j(X) := X^l + L_1X^{l-1} + \ldots + L_l - g(-jd_2) \quad (j = 0, 1, \ldots, mk-1)$$

is a divisor of

$$X(X + d_1)\ldots(X + (lk-1)d_1).$$

However, g can assume each value at most m times. So in $0, -d_2, \ldots, -(mk-1)d_2$ the polynomial g attains at least k distinct values. To each value corresponds a polynomial $f_j(X)$ of degree l. Since $X(X + d_1)\ldots\ldots(X + (lk-1)d_1)$ is a polynomial of degree lk and any two distinct polynomials f_j are coprime (their difference is constant), there are at most k distinct polynomials f_j. Thus the polynomials $\{f_j\}_{j=0}^{mk-1}$ split into k classes of size m such that within a class the polynomials are identical and any two polynomials from different classes are distinct.

First we consider the case where m is odd. We have shown that at the points $\{-jd_2 \mid 0 \le j < mk\}$ the monic polynomial g of odd degree $m > 2$ attains exactly k distinct values and each value precisely m times. Denote these values by $v_1 < v_2 < \ldots < v_k$ and the points where g attains the value v_i by $-j_{i,1}d_2 < -j_{i,2}d_2 < \ldots < -j_{i,m}d_2$ $(i = 1, \ldots, k)$. By Rolle's theorem each interval $(-j_{i,h}d_2, -j_{i,h+1}d_2)$ contains a zero of g'. Since g' has only $m-1$ zeros, $z_1, z_2, \ldots, z_{m-1}$ say, these zeros are distinct and simple and

$$-j_{i,1}d_2 < z_1 < -j_{i,2}d_2 < z_2 < \ldots < -j_{i,m-1}d_2 < z_{m-1} < -j_{i,m}d_2 \quad (i = 1, \ldots, k).$$

The polynomial g is increasing on $(-\infty, z_1)$, decreasing on (z_1, z_2), increasing on $(z_2, z_3), \dots$, increasing on (z_{m-1}, ∞). It follows that the set $\{-j_{i,1}d_2 \mid 1 \le i \le k\}$ consists of the k extreme negative points $\{-jd_2 \mid (m-1)k \le j \le mk-1\}$ and, more precisely, $j_{1,1} = mk-1$, $j_{2,1} = mk-2, \dots, j_{k,1} = mk-k$. Further, we have the following scheme in case m is odd ($\downarrow$ indicates g is increasing, $\uparrow$ indicates g is decreasing):

$$
\begin{array}{rccccccc}
v_1 = & g(-(mk-1)d_2) & = \dots \overset{\rightarrow}{=} & g(-(3k-1)d_2) = & g(-kd_2) & \overset{\rightarrow}{=} & g(-(k-1)d_2) \\
 & \downarrow & & \downarrow & \uparrow & & \downarrow \\
v_2 = & g(-(mk-2)d_2) & = \dots = & g(-(3k-2)d_2) = & g(-(k+1)d_2) & = & g(-(k-2)d_2) \\
 & \downarrow & & \downarrow & \uparrow & & \downarrow \\
 & \vdots & & \vdots & \vdots & & \vdots \\
 & \downarrow & & \downarrow & \uparrow & & \downarrow \\
v_{k-1} = & g(-(mk-k+1)d_2) & = \dots = & g(-(2k+1)d_2) = & g(-(2k-2)d_2) & = & g(-d_2) \\
 & \downarrow & & \downarrow & \uparrow & & \downarrow \\
v_k = & g(-(mk-k)d_2) & \overset{\rightarrow}{=} \dots = & g(-2kd_2) \overset{\rightarrow}{=} & g(-(2k-1)d_2) & = & g(0)
\end{array}
$$

Hence we have

$$
\begin{aligned}
g(Y) &= Y(Y+(2k-1)d_2)(Y+2kd_2)(Y+(4k-1)d_2)\dots \\
&\qquad\qquad \dots (Y+(mk-k)d_2) + v_k \\
&= (Y+d_2)(Y+(2k-2)d_2)(Y+(2k+1)d_2)(Y+(4k-2)d_2)\dots \\
&\qquad\qquad \dots (Y+(mk-k+1)d_2) + v_{k-1}.
\end{aligned}
$$

By putting $Y=0$ and $Y=-(2k-1)d_2$ in the above equality we have

$$
\begin{aligned}
v_k - v_{k-1} &= d_2^m \cdot 1 \cdot (2k-2)(2k+1)(4k-2)\dots(mk-k+1) \\
&= d_2^m \cdot (-2k+2)(-1)(2)(2k-1)\dots(mk-3k+2),
\end{aligned}
$$

which is impossible since $m \ge 3$.

If m is even, a similar reasoning yields the following scheme:

$$
\begin{array}{rccccccc}
v_1 = & g(-(mk-k)d_2) & \overset{\rightarrow}{=} \dots \overset{\rightarrow}{=} & g(-(3k-1)d_2) = & g(-kd_2) & \overset{\rightarrow}{=} & g(-(k-1)d_2) \\
 & \uparrow & & \downarrow & \uparrow & & \downarrow \\
v_2 = & g(-(mk-k+1)d_2) & = \dots = & g(-(3k-2)d_2) = & g(-(k+1)d_2) & = & g(-(k-2)d_2) \\
 & \uparrow & & \downarrow & \uparrow & & \downarrow \\
 & \vdots & & \vdots & \vdots & & \vdots \\
 & \uparrow & & \downarrow & \uparrow & & \downarrow \\
v_{k-1} = & g(-(mk-2)d_2) & = \dots = & g(-(2k+1)d_2) = & g(-(2k-2)d_2) & = & g(-d_2) \\
 & \uparrow & & \downarrow & \uparrow & & \downarrow \\
v_k = & g(-(mk-1)d_2) & = \dots = & g(-2kd_2) \overset{\rightarrow}{=} & g(-(2k-1)d_2) & = & g(0)
\end{array}
$$

Hence we have

$$
\begin{aligned}
g(Y) &= Y(Y+(2k-1)d_2)(Y+2kd_2)(Y+(4k-1)d_2)\dots \\
&\qquad\qquad \dots (Y+(mk-1)d_2) + v_k \\
&= (Y+d_2)(Y+(2k-2)d_2)(Y+(2k+1)d_2)(Y+(4k-2)d_2)\dots \\
&\qquad\qquad \dots (Y+(mk-2)d_2) + v_{k-1}.
\end{aligned}
$$

By putting $Y=0$ and $Y=-(2k-1)d_2$ in the above equality we have

$$\begin{aligned} v_k - v_{k-1} &= d_2^m \cdot 1 \cdot (2k-2)(2k+1)(4k-2)\dots(mk-2) \\ &= d_2^m \cdot (-2k+2)(-1)(2)(2k-1)\dots(mk-2k-1), \end{aligned}$$

which is impossible since $m \geq 4$. In fact, the above argument is also valid when $m=2$ and $k \geq 3$. ∎

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