ANNALES
POLONICI MATHEMATICI
LIX.3 (1994)

# Convex meromorphic mappings

by Albert E. Livingston (Newark, Del.)

**Abstract.** We study functions $f(z)$ which are meromorphic and univalent in the unit disk with a simple pole at $z = p$, $0 < p < 1$, and which map the unit disk onto a domain whose complement is either convex or is starlike with respect to a point $w_0 \neq 0$.

**1. Introduction.** Let $S(p)$, $0 < p < 1$, be the class of functions meromorphic and univalent in the unit disk $\Delta = \{z : |z| < 1\}$ with a simple pole at $z = p$ with a power series expansion $f(z) = z + b_2 z^2 + \ldots$ for $|z| < p$. The class $S(p)$ has been investigated by a number of authors. We let $C(p)$ be the subclass of $S(p)$ made up of functions $f$ such that $\overline{\mathbb{C}} \setminus f[\Delta]$ is a convex set. Royster [11] considered the class $K(p)$ consisting of members of $S(p)$ for which there exists $\delta$, $0 < \delta < 1$, so that for $\delta < |z| < 1$,

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] < 0 .$$

Obviously $K(p) \subset C(p)$. Royster also studied the class $\Sigma(p) \subset S(p)$ consisting of functions $f$ such that

$$\operatorname{Re}\left[\frac{1+pz}{1-pz} - \frac{z+p}{1+pz} - \left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0$$

for $z \in \Delta$, and proved that $K(p) = \Sigma(p)$ for $0 < p < 2 - \sqrt{3}$ but for $p > 2 - \sqrt{3}$, $K(p)$ is a proper subset of $\Sigma(p)$. Pfaltzgraff and Pinchuk [10] essentially proved that $C(p) = \Sigma(p)$ for $0 < p < 1$, by way of the Herglotz representation of functions of positive real part [12]. We will give another proof of this fact. We will also consider several coefficient problems. If $f$ is a member of $S(p)$ we will consider the two expansions

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n, \quad |z| < p , \tag{1.1}$$

1991 *Mathematics Subject Classification*: Primary 30C45.
*Key words and phrases*: convex, starlike, meromorphic.

and

$$f(z)=\sum_{n=-1}^{\infty}a_n(z-p)^n\,,\qquad |z-p|<1-p\,. \tag{1.2}$$

Goodman [2] conjectured that if $f$ is a member of $S(p)$, then

$$|b_n|\le\frac{1+p^2+\ldots+p^{2n-2}}{p^{n-1}}\,. \tag{1.3}$$

Jenkins [3] proved that (1.3) is true for any value of $n$ for which the Bieberbach conjecture holds. Since DeBrange [1] has now proven that conjecture to be valid for all $n$, it follows that (1.3) holds for all $n$. The inequality (1.3) is actually sharp in $C(p)$, since the extremal function $f(z)=-pz/(z-p)(1-pz)$ maps $\Delta$ onto the complement of the real interval $[-p/(1-p)^2,-p/(1+p)^2]$. Miller [9] proved that if $f$ is a member of $\Sigma(p)$, then

$$\left|b_2-\frac{(1+p^2+p^4)}{p(1+p^2)}\right|\le\frac{p}{1+p^2}$$

from which it follows that

$$\mathrm{Re}(b_2)\ge\frac{1+p^4}{p(1+p^2)}>1\,.$$

Miller [9] also obtained a lower bound for $\mathrm{Re}\,b_3$, which is positive for $p$ near 0, for $f$ in $C(p)=\Sigma(p)$. We will obtain the sharp inequality

$$\mathrm{Re}\,b_3\ge\frac{1-p^2+p^4}{p^2}>1$$

if $f$ is in $C(p)=\Sigma(p)$.

Concerning the expansion (1.2), the sharp estimate $|a_{-1}|\le p^2/(1-p^2)$ if $f$ is a member of $S(p)$ has been proven by Kirwan and Schober [4] and also Komatu [5]. Komatu [5] obtained the sharp bound on $|a_1|$ for $f$ in $S(p)$ and the extremal function is a member of $C(p)$. We will give another proof in $C(p)$ and also obtain the sharp bound on $|a_2|$ for $f$ in $C(p)$.

## 2. The class of $C(p)$.

In this section we will give a different necessary and sufficient condition for membership in $C(p)$ and a new proof that $C(p)=\Sigma(p)$.

THEOREM 1. *$f$ is a member of $C(p)$ if and only if for $z\in\Delta$,*

$$\mathrm{Re}\left[1+p^2-2pz+\frac{(z-p)(1-pz)f''(z)}{f'(z)}\right]<0\,. \tag{2.1}$$

Proof. If $f$ is a member of $S(p)$ let $h(z)=f((z+p)/(1+pz))$; then $h$ has a simple pole at $z=0$ and $\overline{\mathbb{C}}\setminus h[\Delta]=\overline{\mathbb{C}}\setminus f[\Delta]$. Thus, $f$ is a member of

$C(p)$ if and only if $h$ is convex with a simple pole at $z=0$. This is the case if and only if [10]

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)<0$$

for $z\in\Delta$. A straightforward computation gives

$$\operatorname{Re}\left(1+\frac{zh''(z)}{h'(z)}\right)=\operatorname{Re}Q(z)$$

where

$$Q(z)=\frac{1-pz}{1+pz}+\frac{\dfrac{(1-p^2)z}{(1+pz)^2}f''\left(\dfrac{z+p}{1+pz}\right)}{f'\left(\dfrac{z+p}{1+pz}\right)}\,.$$

But $\operatorname{Re}Q(z)<0$ for $z\in\Delta$ if and only if $\operatorname{Re}Q((z-p)/(1-pz))<0$ for $z\in\Delta$. However,

$$Q\left(\frac{z-p}{1-pz}\right)=\frac{1-p^2-2pz}{(1-p^2)}+\frac{(z-p)(1-pz)f''(z)}{(1-p^2)f'(z)}\,,$$

which gives (2.1).

Remark. If $f$ is a member of $C(p)$ and

$$P(z)=2pz-1-p^2-\frac{(z-p)(1-pz)f''(z)}{f'(z)}$$

then $\operatorname{Re}P(z)>0$, $z\in\Delta$, $P(p)=1-p^2$ and $P'(p)=0$.

LEMMA 1. *Let $P(z)$ satisfy $\operatorname{Re}P(z)>0$, $z\in\Delta$, and $P(0)=1$. If $0<p<1$, then for $z\in\Delta$,*

$$\operatorname{Re}\left[\frac{(z-p)(1-pz)P(z)+p}{z}-pz\right]>0\,.$$

Proof. Let $0<r<1$ and $P_r(z)=P(rz)$. Then

$$Q_r(z)=\frac{(z-p)(1-pz)P_r(z)+p}{z}-pz$$

is analytic for $|z|\le 1$. If $|z|=1$, then

$$Q_r(z)=\frac{(z-p)(1-pz)P_r(z)}{z}-p\left(z-\frac{1}{z}\right)$$

and

$$\operatorname{Re}Q_r(z)=|1-pz|^2\operatorname{Re}P_r(z)>0\,.$$

Since $Q_r(z)$ is analytic for $|z| \le 1$, $\operatorname{Re} Q_r(z) > 0$ for $z \in \Delta$. Letting $r \to 1$, we obtain for $z \in \Delta$,

$$\operatorname{Re}\left[\frac{(z-p)(1-pz)P(z)+p}{z} - pz\right] \ge 0\,.$$

But equality cannot occur in the last inequality since the quantity on the left side equals $1-p^2$ when $z = p$.

Lemma 2. *If* $\operatorname{Re} P(z) > 0$ *for* $z \in \Delta$ *and* $P(p) = 1-p^2$, *then for* $z \in \Delta$,

$$\operatorname{Re}\left[\frac{zP(z)-p+pz^2}{(z-p)(1-pz)}\right] > 0\,.$$

Proof. Let $p < r < 1$ and $\alpha = (r-1)p/(r-p^2)$ and $L_r(z) = r(z-\alpha)/(1-\overline{\alpha}z)$. It is easily verified that $L_r[\Delta] = \{z : |z| < r\}$ and $L_r(p) = p$. Let

$$Q_r(z) = \frac{zP(L_r(z))-p+pz^2}{(z-p)(1-pz)}\,.$$

$Q_r(z)$ is analytic for $|z| \le 1$ and $\operatorname{Re} P(L_r(z)) > 0$ for $|z| \le 1$. If $|z| = 1$ then

$$\begin{aligned}\operatorname{Re} Q_r(z) &= \operatorname{Re}\left[\frac{zP(L_r(z))}{(z-p)(1-pz)} + \frac{pz(z-1/z)}{(z-p)(1-pz)}\right]\\ &= \frac{1}{|1-pz|^2}\operatorname{Re} P(L_r(z)) > 0\,.\end{aligned}$$

Since $Q_r$ is analytic for $|z| \le 1$, it follows that $\operatorname{Re} Q_r(z) > 0$ for $z \in \Delta$. Letting $r \to 1$, we obtain for $z \in \Delta$,

$$\operatorname{Re}\left[\frac{zP(z)-p+pz^2}{(z-p)(1-pz)}\right] \ge 0\,.$$

But equality cannot occur in the last inequality since the expression on the left equals 1 when $z = 0$.

Theorem 2. $C(p) = \Sigma(p)$ *for* $0 < p < 1$.

Proof. Let $f$ be a member of $\Sigma(p)$ and

$$P(z) = -1 - \frac{zf''(z)}{f'(z)} + \frac{1+pz}{1-pz} - \frac{z+p}{z-p}\,.$$

Then $\operatorname{Re} P(z) > 0$, $z \in \Delta$, and $P(0) = 1$. Straightforward computations give

$$2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)} = \frac{(z-p)(1-pz)P(z)+p}{z} - pz\,.$$

Therefore, by Lemma 1,

$$\operatorname{Re}\left[2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)}\right] > 0$$

for $z \in \Delta$, and thus by Theorem 1, $f$ is a member of $C(p)$.

Conversely, suppose $f$ is a member of $C(p)$ and let

$$P(z) = 2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)}\,.$$

Then by Theorem 1, $\operatorname{Re} P(z) > 0$, $z \in \Delta$, and $P(p) = 1-p^2$. Straightforward computations give

$$-1 - \frac{zf''(z)}{f'(z)} - \frac{z+p}{z-p} + \frac{1+pz}{1-pz} = \frac{zP(z) - p + pz^2}{(z-p)(1-pz)}\,.$$

Thus, by Lemma 2,

$$\operatorname{Re}\left[-1 - \frac{zf''(z)}{f'(z)} - \frac{z+p}{z-p} + \frac{1+pz}{1-pz}\right] > 0$$

for $z \in \Delta$. Therefore $f$ is a member of $\Sigma(p)$.

**3. The coefficients $a_n$.** In this section we use Theorem 1 to study the coefficients $a_1$ and $a_2$ in (1.2), if $f$ is a member of $C(p)$. We will make use of the following lemma.

LEMMA 3. *Let $P(z)$ be analytic in $\Delta$ and satisfy* $\operatorname{Re} P(z) > 0$, $z \in \Delta$, $P(p) = 1 - p^2$ *and* $P'(p) = 0$, $0 < p < 1$. *If* $P(z) = (1-p^2) + d_2(z-p)^2 + d_3(z-p)^3 + \ldots$ *for* $|z-p| < 1-p$, *then*

$$|d_2| \le \frac{2}{1-p^2}\,, \tag{3.1}$$

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \le \frac{6p}{(1-p^2)^2}\,, \qquad 2/3 \le p < 1\,, \tag{3.2}$$

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \le \frac{2(1+\frac{9}{4}p^2)}{1-p^2}\,, \qquad 0 < p \le 2/3\,. \tag{3.3}$$

*All the inequalities are sharp.*

Proof. Let

$$w(z) = \frac{P(z) - (1-p^2)}{P(z) + 1 - p^2}\,.$$

Then $w(p) = 0$ and $|w(z)| \le 1$, $z \in \Delta$. Also

$$w'(z) = \frac{2(1-p^2)P'(z)}{[P(z) + (1-p)^2]^2}$$

and hence $w'(p) = 0$. Comparing coefficients in the expansions of both sides of

$$[P(z) + (1-p^2)]w(z) = P(z) - (1-p^2)\,,$$

we obtain

$$d_2 = (1-p^2)w''(p) \tag{3.4}$$

and

$$\frac{p}{1-p^2}d_2 + d_3 = pw''(p) + (1-p)\frac{w'''(p)}{3}\,. \tag{3.5}$$

We can write

$$w(z) = \phi\biggl(\frac{z-p}{1-pz}\biggr)$$

where $\phi$ is analytic for $|z| < 1$, $\phi(0) = \phi'(0) = 0$ and $|\phi(z)| \le 1$, $z \in \Delta$. In particular, we obtain

$$w''(p) = \frac{\phi''(0)}{(1-p^2)^2}\,.$$

Since $|\phi''(0)/2| \le 1$, we have $|w''(p)| \le 2/(1-p^2)^2$. Thus from (3.4) we obtain

$$|d_2| = (1-p^2)|w''(p)| \le \frac{2}{1-p^2},$$

which is (3.1).

Next from (3.5) we obtain

$$\frac{p}{1-p^2}d_2 + d_3 = \frac{1}{(1-p^2)^2}\biggl[\frac{\phi'''(0)}{3} + 3p\phi''(0)\biggr]\,.$$

If $\phi(z) = c_2z^2 + c_3z^3 + \ldots$, $z \in \Delta$, then

$$\frac{p}{1-p^2}d_2 + d_3 = \frac{2}{(1-p^2)^2}[c_3 + 3pc_2]\,.$$

Using known inequalities for bounded functions, we obtain

$$|c_3 + 3pc_2| \le |c_3| + 3p|c_2| \le 1 - |c_2|^2 + 3p|c_2|\,.$$

Therefore

$$\left|\frac{p}{1-p^2}d_2 + d_3\right| \le \frac{2}{(1-p^2)^2}[1 + 3p|c_2| - |c_2|^2]\,. \tag{3.6}$$

Let $x = |c_2|$ and $h(x) = 1 + 3px - x^2$, $0 \le x \le 1$. Then $h'(x) = 3p - 2x$. If $p \ge 2/3$, then $h'(x) \ge 0$ for $0 \le x \le 1$ and hence

$$h(x) \le h(1) = 3p\,, \quad 2/3 \le p < 1\,. \tag{3.7}$$

If $0 < p < 2/3$, then $h(x)$ achieves its maximum at $x = 3p/2$. Hence

$$h(x) \le 1 + \tfrac{9}{4}p^2\,, \quad 0 < p \le 2/3\,. \tag{3.8}$$

Combining (3.6), (3.7) and (3.8) gives (3.2) and (3.3).

Equality is attained in (3.1) by the function

$$P(z) = \frac{1 + p^2 - 4pz + (1+p^2)z^2}{1-z^2}\,,$$

which is obtained by taking $w(z) = [(z-p)/(1-pz)]^2$. The same function gives equality in (3.2).

If $0 < p < 2/3$, let

$$\phi(z) = \frac{z^2(z + \frac{3}{2}p)}{1 + \frac{3}{2}pz}$$

and $w(z) = \phi((z-p)/(1-pz))$. The resulting function $P(z) = (1-p^2)(1+w(z))/(1-w(z))$ gives equality in (3.3).

THEOREM 3. *Let $f$ be a member of $C(p)$ and have the expansion* (1.2). *Then*

$$|a_1| \le \frac{p^2}{(1-p^2)^3}\,, \tag{3.9}$$

$$|a_2| \le \frac{(4+9p^2)|a_{-1}|}{12(1-p^2)^3}\,, \qquad 0 < p \le 2/3\,, \tag{3.10}$$

$$|a_2| \le \frac{p}{(1-p^2)^3}|a_{-1}| \le \frac{p^3}{(1-p^2)^4}\,, \qquad 2/3 \le p \le 1\,. \tag{3.11}$$

*All the inequalities are sharp.*

Remark. Making use of the area theorem, Komatu [5] proved inequality (3.9) for the larger class $S(p)$.

Proof of Theorem 3. Let

$$P(z) = 2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)}\,.$$

Then $P(z)$ satisfies the hypotheses of Lemma 3. Comparing coefficients on both sides of the equation

$$[2p(z-p) - (1-p^2)]f'(z) - (z-p)[(1-p^2) - p(z-p)]f''(z) = P(z)f'(z)$$

we obtain

$$2a_1(1-p^2) = a_{-1}d_2 \tag{3.12}$$

and

$$6(1-p^2)a_2 = 2pa_1 + a_{-1}d_3\,. \tag{3.13}$$

Combining (3.1) and (3.12) gives

$$|a_1| \le \frac{|a_{-1}|}{(1-p^2)^2}\,.$$

However, $|a_{-1}| \le p^2/(1-p^2)$ (cf. [4], [5]), giving (3.9).

Combining (3.12) and (3.13) gives

$$a_2 = \frac{1}{6(1-p^2)}\left[\frac{p}{1-p^2}d_2 + d_3\right]a_{-1}\,. \tag{3.14}$$

If $0 < p \le 2/3$, then (3.3) and (3.14) gives (3.10). If $2/3 \le p < 1$, then (3.2) combined with (3.14) gives (3.11).

Equality is attained in (3.9) and (3.11) by $f(z) = -pz/((z-p)(1-pz))$. If $0 < p \le 2/3$, equality is attained in (3.10) by the function $f$ which satisfies

$$2pz - 1 - p^2 - \frac{(z-p)(1-pz)f''(z)}{f'(z)} = P(z)$$

where $P(z)$ is the function satisfying the hypotheses of Lemma 3 and giving equality in (3.3). Since $\operatorname{Re} P(z) = 0$ on $|z| = 1$ with finitely many exceptions and since

$$1 + \frac{zf''(z)}{f'(z)} = \frac{z}{(z-p)(1-pz)}\left[p\left(z - \frac{1}{z}\right) - P(z)\right],$$

it follows that on $|z| = 1$,

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1}{|1-pz|^2}\operatorname{Re} P(z) = 0$$

with finitely many exceptions.

Laborious computations give

$$P(z) = \frac{(1-p^2)(2+2p-p^2)}{2-2p-p^2}\cdot\frac{(1+z)(z-e^{i\gamma})(z-e^{-i\gamma})}{(1-z)(z-e^{i\beta})(z-e^{-i\beta})}$$

where $e^{i\gamma} \ne e^{i\beta}$, and $e^{i\beta}$ is not real for $0 < p < 2/3$. Thus $\operatorname{Re}(1 + zf''(z)/f'(z)) = 0$ on $|z| = 1$ with 3 exceptional points. It follows that for the extremal function in the case $0 < p < 2/3$, $\overline{\mathbb{C}} \setminus f[\Delta]$ is the interior of a triangle.

Remark. In the case $0 < p < 2/3$ of Theorem 3, using the inequality $|a_{-1}| \le p^2/(1-p^2)$ in (3.10) does not result in a sharp inequality.

THEOREM 4. *If $f$ is a member of $C(p)$ with expansion* (1.2), *then*

$$\left|p + \frac{a_0(1-p^2)}{a_{-1}}\right| \le \frac{1+p^2}{p},$$

*and the inequality is sharp.*

Proof. Let

$$h(z) = \frac{-a_{-1}}{(1-p^2)f\left(\dfrac{p-z}{1-pz}\right)},$$

then $h$ is a member of $S(p)$ and for $|z-p| < 1-p$,

$$h(z) = z + \left(p + \frac{(1-p^2)a_0}{a_{-1}}\right)z^2 + \ldots$$

Using (1.3) when $n = 2$, we get

$$\left|p + \frac{(1-p^2)a_0}{a_{-1}}\right| \le \frac{1+p^2}{p}.$$

Equality is attained by $f(z) = -pz/((z-p)(1-pz))$.

**4. The coefficients $b_n$.** Let $f$ be a member of $C(p)$ and have the expansion (1.1) for $|z| < p$. As remarked in the introduction, sharp upper bounds on $|b_n|$ are known for all $n$ and a sharp lower bound on $\mathrm{Re}(b_2)$ follows from results in [9]. In this section we will obtain a sharp lower bound on $\mathrm{Re}(b_3)$ which suggests a conjecture concerning $\mathrm{Re}(b_n)$ for all $n$.

THEOREM 5. *Let $f$ be a member of $C(p)$ with expansion* (1.1). *Then*

$$\mathrm{Re}\, b_2 \ge \frac{1+p^4}{p(1+p^2)} > 1 \tag{4.1}$$

*and*

$$\mathrm{Re}\, b_3 \ge \frac{1-p^2+p^4}{p^2} = \frac{1+p^6}{p^2(1+p^2)} > 1\,. \tag{4.2}$$

*Both inequalities are sharp, each being attained by the function*

$$f(z) = \frac{p(1+p^2)z - 2p^2z^2}{(1-p^2)(p-z)(1-pz)}\,.$$

P r o o f. Let

$$P(z) = 2pz - 1 - p^2 - \frac{(-p+(1+p^2)z-pz^2)f''(z)}{f'(z)}\,,$$

then $\mathrm{Re}\, P(z) > 0$, $z \in \Delta$, $P(p) = 1-p^2$ and $P'(p) = 0$. Let $P(z) = c_0 + c_1 z + c_2 z^2 + \ldots$ Comparing coefficients on both sides of the equation

$$P(z)f'(z) = [2pz - (1+p^2)]f'(z) - [-p+(1+p^2)z-pz^2]f''(z)\,,$$

we obtain

$$c_0 = 2pb_2 - (1+p^2) \tag{4.3}$$

and

$$2c_0b_2 + c_1 = 2p - 4(1+p^2)b_2 + 6pb_3\,. \tag{4.4}$$

Using (4.3) and (4.4) we obtain

$$b_2 = \frac{c_0 + (1+p^2)}{2p} \tag{4.5}$$

and

$$6p^2b_3 = c_0^2 + 3(1+p^2)c_0 + pc_1 + 2(1+p^2+p^4)\,. \tag{4.6}$$

Let $w(z) = [P(z)-(1-p^2)]/[P(z)+(1-p^2)]$, then $|w(z)| < 1$ for $z \in \Delta$ and $w(p) = w'(p) = 0$. Thus we can write

$$w(z) = \left(\frac{z-p}{1-pz}\right)^2 \phi(z)$$

where $|\phi(z)| < 1$ for $z \in \Delta$. We have

$$P(z) = \frac{(1-p^2)(1+w(z))}{1-w(z)}.$$

Thus

$$c_0 = P(0) = \frac{(1-p^2)(1+w(0))}{1-w(0)} = \frac{(1-p^2)(1+p^2\phi(0))}{1-p^2\phi(0)}. \tag{4.7}$$

It follows that

$$\operatorname{Re} c_0 \geq (1-p^2)\frac{1-p^2|\phi(0)|}{1+p^2|\phi(0)|} \geq \frac{(1-p^2)^2}{1+p^2}.$$

Using this inequality in conjunction with (4.5) gives (4.1), which has also been proven by Miller [9].

Next, we have

$$\begin{aligned} c_1 = P'(0) &= \frac{2(1-p^2)w'(0)}{(1-w(0))^2} \\ &= \frac{2(1-p^2)[-2p(1-p^2)\phi(0) + p^2\phi'(0)]}{(1-p^2\phi(0))^2}. \end{aligned}$$

Combining (4.6), (4.7) and (4.8), we eventually obtain

$$\begin{aligned} 6p^2 b_3 = (1-p^2)\left[(1-p^2) + \frac{2p^3\phi'(0)}{(1-p^2\phi(0))^2} + 3(1+p^2)\frac{1+p^2\phi(0)}{1-p^2\phi(0)}\right] \\ + 2(1+p^2+p^4). \end{aligned} \tag{4.9}$$

Now let

$$Q(z) = \frac{1+p^2\phi(z)}{1-p^2\phi(z)}.$$

Then $\operatorname{Re} Q(z) \geq (1-p^2)/(1+p^2) > 0$ for $z \in \Delta$ and (4.9) can be written as

$$\begin{aligned} 6p^2 b_3 = (1-p^2)[(1-p^2) + pQ'(0) + 3(1+p^2)Q(0)] \\ + 2(1+p^2+p^4). \end{aligned} \tag{4.10}$$

Let $T(z) = Q(z) - (1-p^2)/(1+p^2)$. Since $\operatorname{Re} T(z) > 0$ for $z \in \Delta$, it is known that

$$|T'(0)| \leq 2 \operatorname{Re} T(0).$$

Thus

$$|Q'(0)| \leq 2 \operatorname{Re}\left[Q(0) - \frac{1-p^2}{1+p^2}\right].$$

Hence

$$2 \operatorname{Re} Q(0) \geq |Q'(0)| + 2\frac{1-p^2}{1+p^2}.$$

Using the last inequality with (4.10) we obtain

$$\begin{aligned}6p^2 \operatorname{Re} b_3 &\ge (1-p^2)\left[(1-p^2) - p|Q'(0)| + \frac{3}{2}(1+p^2)|Q'(0)| + 3(1-p^2)\right]\\ &\quad + 2(1+p^2+p^4)\\ &= (1-p^2)\left[(1-p^2) + \frac{3+3p^2-2p}{2}|Q'(0)| + 3(1-p^2)\right]\\ &\quad + 2(1+p^2+p^4)\\ &\ge (1-p^2)[4(1-p^2)] + 2(1+p^2+p^4)\\ &= 6(1-p^2+p^4)\,,\end{aligned}$$

which gives (4.2).

An examination of the proof indicates that equality holds in (4.1) and (4.2) if and only if $\phi(z) \equiv -1$. This leads to the extremal function stated in the theorem.

Remark. It seems reasonable to expect that the extremal function for Theorem 5 is extremal for all $n$. That is, we expect that if $f$ is a member of $C(p)$, then $\operatorname{Re}(b_n) \ge (1+p^{2n})/(p^{n-1}(1+p^2))$ for all $n$.

**5. Starlike functions.** Miller [7]–[9] considered functions $f$ of $S(p)$ for which there exists $\varrho$, $0<\varrho<1$, so that $\operatorname{Re}[zf'(z)/(f(z)-w_0)] < 0$ for $\varrho < |z| < 1$ and a fixed $w_0 \in \mathbb{C}, w_0 \ne 0$. These functions map $\Delta$ onto the complement of a set which is starlike with respect to $w_0$. This class of functions is a subclass of the class $\Sigma^*(p, w_0)$ defined as the class of functions $f$ in $S(p)$ such that for $z \in \Delta$,

$$\operatorname{Re}\left[\frac{pz}{1-pz} - \frac{p}{z-p} - \frac{zf'(z)}{(f(z)-w_0)}\right] > 0\,.$$

Actually, the two classes are the same if $0 < p < \sqrt{3-2\sqrt{2}}$ (cf. [9]). But for $p \ge \sqrt{3-2\sqrt{2}}$ and proper choice of $w_0$ the first class is a proper subset of the second. We will prove that $\Sigma^*(p, w_0)$ is the class of all functions $f$ in $S(p)$ such that $\overline{\mathbb{C}} \setminus f[\Delta]$ is starlike with respect to $w_0$, which we denote by $\Sigma^{\mathrm{s}}(p, w_0)$.

THEOREM 6. *$f$ is a member of $\Sigma^{\mathrm{s}}(p, w_0)$ if and only if, for $z \in \Delta$,*

$$\operatorname{Re}\left[\frac{(z-p)(1-pz)f'(z)}{f(z)-w_0}\right] < 0\,.$$

Proof. Suppose $f$ is a member of $S(p)$ and let

$$g(z) = f\left(\frac{z+p}{1+pz}\right).$$

$f$ is a member of $\Sigma^{\mathrm{s}}(p,w_0)$ if and only if $\overline{\mathbb{C}}\setminus g(\Delta)$ is starlike with respect to $w_0$. This is the case if and only if $F(z)=g(z)-w_0$ maps $\Delta$ onto the complement of a set which is starlike with respect to the origin. Since $F$ has its pole at the origin, $\overline{\mathbb{C}}\setminus F[\Delta]$ is starlike with respect to the origin if and only if $\operatorname{Re}[zF'(z)/F(z)]<0$ for $z\in\Delta$. The last inequality is true if and only if

$$\operatorname{Re}\left[\left(\frac{z-p}{1-pz}\right)F'\left(\frac{z-p}{1-pz}\right)\Big/F\left(\frac{z-p}{1-pz}\right)\right]<0$$

for $z\in\Delta$. A straightforward computation gives

$$\left(\frac{z-p}{1-pz}\right)F'\left(\frac{z-p}{1-pz}\right)\Big/F\left(\frac{z-p}{1-pz}\right)=\frac{(z-p)(1-pz)f'(z)}{(1-p^2)(f(z)-w_0)}$$

and the theorem follows.

Theorem 7. *$\Sigma^{\mathrm{s}}(p,w_0)=\Sigma^*(p,w_0)$ for all $p$, $0<p<1$, and all $w_0\neq 0$.*

Proof. Let $f$ be a member of $\Sigma^*(p,w_0)$ and

$$P(z)=\frac{pz}{1-pz}-\frac{p}{z-p}-\frac{zf'(z)}{f(z)-w_0},$$

then $\operatorname{Re}P(z)>0$ for $z\in\Delta$ and $P(0)=1$. From this we obtain

$$\frac{(z-p)(1-pz)f'(z)}{f(z)-w_0}=-\frac{(z-p)(1-pz)P(z)+p(1-z^2)}{z}. \tag{5.1}$$

Let $0<r<1$ and

$$Q_r(z)=\frac{(z-p)(1-pz)P(rz)+p(1-z^2)}{z},$$

then $Q_r(z)$ is analytic for $|z|\le 1$, and

$$\operatorname{Re}Q_r(z)=|1-pz|^2\operatorname{Re}P(rz)>0$$

for $|z|=1$. Thus $\operatorname{Re}Q_r(z)>0$ for $z\in\Delta$. If we let $r\to 1$, we obtain

$$\operatorname{Re}\left[\frac{(z-p)(1-pz)P(z)+p(1-z^2)}{z}\right]\ge 0. \tag{5.2}$$

However, the expression on the left side of (5.2) is strictly positive for $z=p$. Thus equality cannot occur in (5.2). Hence from (5.1),

$$\operatorname{Re}\left[\frac{(z-p)(1-pz)f'(z)}{f(z)-w_0}\right]<0$$

for $z\in\Delta$. Thus by Theorem 6, $f$ is a member of $\Sigma^{\mathrm{s}}(p,w_0)$.

Conversely, suppose $f$ is a member of $\Sigma^{\mathrm{s}}(p,w_0)$ and let

$$P(z)=-\frac{(z-p)(1-pz)f'(z)}{f(z)-w_0},$$

then $\operatorname{Re} P(z) > 0$ for $z \in \Delta$ and $P(p) = 1 - p^2$. We obtain

$$\frac{pz}{1-pz} - \frac{p}{(z-p)} - \frac{zf'(z)}{f(z)-w_0} = \frac{zP(z) - p(1-z^2)}{(z-p)(1-pz)}\,. \tag{5.3}$$

By Lemma 2 the real part of the expression on the right side of (5.3) is strictly positive for $z$ in $\Delta$. Thus $f$ is a member of $\Sigma^*(p, w_0)$.

Miller [9] has given some estimates of coefficients in the expansion (1.1) if $f$ is a member of $\Sigma^*(p, w_0) = \Sigma^{\mathrm{s}}(p, w_0)$. We will next give sharp bounds on a few coefficients in the expansion (1.2).

THEOREM 8. *If $f(z)$ is a member of $\Sigma^*(p, w_0)$ and has expansion* (1.2) *for $|z-p| < 1-p$ then*

$$|a_0 - w_0| \le \frac{2+p}{1-p^2}|a_{-1}| \tag{5.4}$$

*and*

$$|a_1| \le \frac{|a_{-1}|}{(1-p^2)^2} \tag{5.5}$$

*Both inequalities are sharp.*

Proof. We first prove inequality (5.5). Let

$$P(z) = \frac{-(z-p)(1-pz)f'(z)}{f(z)-w_0}\,,$$

then $\operatorname{Re} P(z) > 0$ for $z \in \Delta$ and $P(p) = 1 - p^2$. Let

$$P(z) = (1-p^2) + \sum_{n=1}^{\infty} c_n (z-p)^n$$

for $|z-p| < 1-p$. Comparing coefficients on both sides of the equation

$$(f(z) - w_0)P(z) = -(z-p)(1-pz)f'(z)\,,$$

we obtain

$$a_{-1}c_1 + (1-p^2)(a_0 - w_0) = -\,pa_{-1}\,, \tag{5.6}$$

$$a_{-1}c_2 + (a_0 - w_0)c_1 + (1-p^2)a_1 = -\,a_1(1-p^2)\,. \tag{5.7}$$

Combining (5.6) and (5.7), we eventually obtain

$$-2(1-p^2)^2 a_1 = a_{-1}[(1-p^2)c_2 - pc_1 - c_1^2]\,. \tag{5.8}$$

We now claim that $|(1-p^2)c_2 - pc_1 - c_1^2| \le 2$. To prove this, let

$$Q(z) = \frac{1}{1-p^2} P\left(\frac{z+p}{1+pz}\right),$$

then $\operatorname{Re} P(z) > 0$ for $z \in \Delta$ and $Q(0) = 1$. Thus [11] there exists $m(t)$ increasing on $[0, 2\pi]$ with $\int_0^{2\pi} dm(t) = 1$, such that

$$\frac{1}{1-p^2} P\left(\frac{z+p}{1+pz}\right) = \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z}\, dm(t)\,.$$

Thus

$$P(z) = (1-p^2) \int_0^{2\pi} \frac{(1-pz)+e^{it}(z-p)}{(1-pz)-e^{it}(z-p)}\, dm(t)\,.$$

Expanding the integrand in powers of $z-p$ and integrating we obtain

$$c_1 = 2\int_0^{2\pi} e^{it}\, dm(t)$$

and

$$c_2 = \frac{2}{1-p^2}\int_0^{2\pi} (e^{2it} + pe^{it})\, dm(t) = \frac{2}{1-p^2}\int_0^{2\pi} e^{2it}\, dm(t) + \frac{pc_1}{1-p^2}\,.$$

Thus

$$(1-p^2)c_2 - pc_1 - c_1^2 = 2\int_0^{2\pi} e^{2it}\, dm(t) - c_1^2\,. \tag{5.9}$$

Now let

$$T(z) = \int_0^{2\pi} \frac{1+e^{it}z}{1-e^{it}z}\, dm(t)\,.$$

Then $\operatorname{Re} T(z) > 0$ for $z \in \Delta$ and $T(0) = 1$. If

$$T(z) = 1 + p_1 z + p_2 z^2 + \ldots, \qquad z \in \Delta\,,$$

then

$$p_1 = 2\int_0^{2\pi} e^{it}\, dm(t) = c_1 \quad \text{and} \quad p_2 = 2\int_0^{2\pi} e^{2it}\, dm(t)\,.$$

Thus from (5.9),

$$(1-p^2)c_2 - pc_1 - c_1^2 = p_2 - p_1^2\,.$$

But it is known [6] that $|p_2 - p_1^2| \le 2$. Thus

$$|(1-p^2)c_2 - pc_1 - c_1^2| \le 2\,.$$

Therefore from (5.8) we obtain

$$2(1-p^2)^2|a_1| \le 2|a_{-1}|\,,$$

which is (5.5).

Next, from (5.6),

$$|a_0 - w_0| = \frac{|a_{-1}||c_1 + p|}{1 - p^2} = \frac{|a_{-1}||p_1 + p|}{1 - p^2} \le \frac{|a_{-1}|(2 + p)}{1 - p^2}\,.$$

To see sharpness, consider

$$f(z) = w_0 + pw_0 \frac{(1 - z)^2}{(z - p)(1 - pz)}\,.$$

Since

$$\frac{(z - p)(1 - pz)f'(z)}{f(z) - w_0} = -(1 - p^2)\frac{1 + z}{1 - z}\,,$$

$f(z)$ is a member of $\Sigma^{\rm s}(p, w_0)$. Moreover, $\overline{\mathbb{C}}\backslash f[\Delta]$ is the line segment $\xi = tw_0$, $(1 + p^2)/(1 + p)^2 \le t \le (1 + p^2)/(1 - p)^2$. Also, for $|z - p| < 1 - p$,

$$\begin{aligned} f(z) = {} & \frac{pw_0(1 - p)}{(1 + p)(z - p)} \\ & + \left[w_0 + \frac{p(p - 2 + p^2)}{(1 + p)(1 - p^2)}w_0\right] + \frac{pw_0}{(1 - p)(1 + p)^3}(z - p) + \ldots, \end{aligned}$$

from which we can see that equality is attained in (5.4) and (5.5).

THEOREM 9. *With the notation of Theorem* 8,

$$|a_{-1}| \le \frac{p(1 - p)}{1 + p}|w_0|$$

*and the inequality is sharp.*

Proof. With $P(z)$ as in the proof of Theorem 8,

$$\frac{d}{dz}\log(z - p)(f(z) - w_0) = \frac{(1 - pz) - P(z)}{(z - p)(1 - pz)}\,.$$

Integrating, we obtain

$$f(z) - w_0 = \frac{pw_0}{z - p}\exp\int_0^z \frac{(1 - p\xi) - P(\xi)}{(\xi - p)(1 - p\xi)}\,d\xi\,.$$

Thus

$$a_{-1} = \lim_{z\to p}(z - p)(f(z) - w_0) = pw_0\exp\int_0^p \frac{(1 - p\xi) - P(\xi)}{(\xi - p)(1 - p\xi)}\,d\xi$$

and

$$|a_{-1}| = p|w_0|\exp\int_0^p \frac{(1 - p\xi) - \operatorname{Re}P(\xi)}{(\xi - p)(1 - p\xi)}\,d\xi\,. \tag{5.10}$$

We can write

$$P(z) = (1 - p^2)Q\left(\frac{z - p}{1 - pz}\right)$$

where $\operatorname{Re} Q(z) > 0$ for $z \in \Delta$ and $Q(0) = 1$. Using the well-known inequality $\operatorname{Re} Q(z) \geq (1 - |z|)/(1 + |z|)$, we obtain for $\xi$ real and $0 \leq \xi \leq p$,

$$\operatorname{Re} P(\xi) \geq (1-p^2)\frac{1 - \left|\dfrac{\xi - p}{1 - p\xi}\right|}{1 + \left|\dfrac{\xi - p}{1 - p\xi}\right|} \tag{5.11}$$

$$= (1-p^2)\frac{1 - \dfrac{p-\xi}{1-p\xi}}{1 + \dfrac{p-\xi}{1-p\xi}} = (1-p)^2\frac{1+\xi}{1-\xi}.$$

Combining (5.10) and (5.11) gives

$$|a_{-1}| \leq p|w_0| \exp \int_0^p \frac{(1-p\xi) - (1-p)^2\dfrac{1+\xi}{1-\xi}}{(\xi - p)(1-p\xi)}\,d\xi$$

$$= p|w_0| \exp \int_0^p \frac{p\xi + (p-2)}{(1-\xi)(1-p\xi)}\,d\xi$$

$$= p|w_0| \exp \int_0^p \left(\frac{-2}{1-\xi} + \frac{p}{1-p\xi}\right) d\xi = p|w_0|\left(\frac{1-p}{1+p}\right),$$

which is the inequality to be proven. Equality is attained by the function given in Theorem 8.

COROLLARY. *With notation of Theorem* 8,

$$|a_0 - w_0| \leq \frac{p(p+2)}{(1+p)^2}|w_0|$$

*and*

$$|a_1| \leq \frac{p}{(1-p)(1+p)^3}|w_0|.$$

*Both inequalities are sharp.*

### References


[1] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. 154 (1985), 137–152.

[2] A. W. Goodman, *Functions typically-real and meromorphic in the unit circle*, Trans. Amer. Math. Soc. 8 (1950), 92–105.

[3] J. A. Jenkins, *On a conjecture of Goodman concerning meromorphic univalent functions*, Michigan Math. J. 9 (1962), 25–27.

[4] W. E. Kirwan and G. Schober, *Extremal problems for meromorphic univalent functions*, J. Analyse Math. 30 (1976), 330–348.

[5] Y. Komatu, *Note on the theory of conformal representation by meromorphic functions I, II*, Proc. Japan Acad. 21 (1945), 269–284.
[6] A. E. Livingston, *The coefficients of multivalent close to convex functions*, Proc. Amer. Math. Soc. 21 (1969), 545–552.
[7] J. Miller, *Convex meromorphic mappings and related functions*, ibid. 25 (1970), 220–228.
[8] —, *Starlike meromorphic functions*, ibid. 31 (1972), 446–452.
[9] —, *Convex and starlike meromorphic functions*, ibid. 80 (1980), 607–613.
[10] J. Pfaltzgraff and B. Pinchuk, *A variational method for classes of meromorphic functions*, J. Analyse Math. 24 (1971), 101–150.
[11] W. C. Royster, *Convex meromorphic functions*, in Mathematical Essays Dedicated to A. J. MacIntyre, Ohio Univ. Press, Athens, Ohio, 1970, 331–339.
[12] G. Schober, *Univalent Functions—Selected Topics*, Lecture Notes in Math. 478, Springer, Berlin, 1975.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
501 EWING HALL
NEWARK, DELAWARE 19716
U.S.A.

*Reçu par la Rédaction le 4.10.1993*