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ON THE CLASS OF FUNCTIONS HAVING INFINITE LIMIT ON A GIVEN SET

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Introduction. Given a topological space X and a real function f on X define

$$L_f(X) = \{x \in X : \lim_{t \to x} f(t) = +\infty\}.$$

According to [1] for a linear set A there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $A = L_f(\mathbb{R})$ if and only if A is a countable G_δ-set. Our purpose is to prove a similar result in a more general setting and to investigate the cardinality and topological properties of the class of functions $f : X \to \mathbb{R}$ for which $L_f(X)$ equals a given non-empty, countable G_δ-set.

We will need some auxiliary notions and notations. Denote by $\overline{E}$ and E^{c}, respectively, the closure and the set of all condensation points of a subset E of a topological space, and by $\operatorname{card} E$ its cardinality. Denote by $\mathcal{F}$ the space $\mathbb{R}^X$.

A topological space X is called a *Fréchet space* if for every $E \subset X$ and every $x \in \overline{E}$ there exists a sequence in E converging to x (cf. [2]). Every first-countable space is a Fréchet space ([2], p. 78), but there exists a Fréchet space that is not first-countable ([2], p. 79).

A topological space X is said to be *hereditarily Lindelöf* if for each $E \subset X$ every open cover of E has a countable refinement. A well-known property of these spaces is as follows ([4], p. 57):

LEMMA 1. *If X is a hereditarily Lindelöf space, then $E \setminus E^{\mathrm{c}}$ is countable for each $E \subset X$.*

Main results. Using Lemma 1 it can be shown similarly to [1] that for a Hausdorff, hereditarily Lindelöf space X having no isolated points, $L_f(X)$ is a countable G_δ-set for every $f \in \mathcal{F}$. We will be interested in the reverse problem, namely to find, for every non-empty, countable G_δ-set $A \subset X$, a function $f \in \mathcal{F}$ for which $L_f(X) = A$.

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In what follows X will be a Fréchet, Hausdorff, hereditarily Lindelöf space such that $X = X^{\mathrm{c}}$. Let A be a given non-empty, countable G_δ-subset of X. Define

$$\mathcal{S} = \{f \in \mathcal{F} : L_f(X) = A\}.$$

THEOREM 1. *The set $\mathcal{S}$ is non-empty.*

Proof. Let $A = \{a_1, a_2, \ldots\} \subset X, A = \bigcap_{n=1}^{\infty} G_n$ where $G_1 = X$, G_n is open in X, $G_{n+1} \subsetneq G_n$ $(n \in \mathbb{N})$. We can assume that $F_n = G_n \setminus G_{n+1}$ is uncountable for each $n \in \mathbb{N}$. Put $H = \bigcup_{n=1}^{\infty} (F_n \cap F_n^{\mathrm{c}})$.

According to Lemma 1 the set $B = (\bigcup_{n=1}^{\infty} F_n) \setminus H$ is countable, since $B \subset \bigcup_{n=1}^{\infty} (F_n \setminus F_n^{\mathrm{c}})$. Write $B = \{b_1, b_2, \ldots\}$. Observe that $A \cap \bigcup_{n=1}^{\infty} F_n = \emptyset$, so

$$X \setminus H = \Big(A \cup \bigcup_{n=1}^{\infty} F_n\Big) \setminus H = A \cup B.$$

Therefore $B \subset \overline{H}$ (since $B \subset X^{\mathrm{c}}$), hence for each $k \in \mathbb{N}$ there exists a sequence $c_i^{(k)} \in H$ $(i \in \mathbb{N})$ converging to b_k. Set $C = \bigcup_{i,k \in \mathbb{N}} \{c_i^{(k)}\}$. Define a function $f \in \mathcal{F}$ as follows:

$$\begin{aligned} f(a_k) &= k \quad \text{for all } k \in \mathbb{N}, \\ f(c_i^{(k)}) &= k \quad \text{for all } i, k \in \mathbb{N}, \\ f(x) &= n \quad \text{for all } x \in F_n \setminus C, \ n \in \mathbb{N}. \end{aligned}$$

We will prove that $L_f(X) = A$.

First choose $x \in X \setminus A$. Then either $x \in B$ or $x \in H$. If $x \in B$ then $x = b_k$ for some $k \in \mathbb{N}$, and consequently $x \notin L_f(X)$ since $\lim_{i \to \infty} f(c_i^{(k)}) = k$. If $x \in H$ then $x \in F_m^{\mathrm{c}}$ for some $m \in \mathbb{N}$, so there exists a directed set Σ and a net $\{x_\sigma : \sigma \in \Sigma\}$ in $F_m \setminus C$ converging to x. Thus again $x \notin L_f(X)$ since $\lim_\sigma f(x_\sigma) = m$.

Finally, suppose $x \in A$. Take an arbitrary $n \in \mathbb{N}$. Then $x \in G_n$. The space X is Hausdorff, so there is a neighbourhood S_1 of x which contains no member of the sequence $\{c_i^{(k)}\}_{i=1}^{\infty}$ for all $1 \le k \le n$ (notice that $c_i^{(k)} \to b_k \notin A$ as $i \to \infty$). Further, there exists a neighbourhood S_2 of x containing none of $a_1, \ldots, a_n$ except possibly x. It is now not hard to see that $f(t) \ge n$ for each $t \in G_n \cap S_1 \cap S_2$, $t \ne x$, whence $x \in L_f(X)$. ∎

Remark 1. If X is a Hausdorff, second-countable, Baire space with no isolated points (in particular, if X is a separable, complete metric space with no isolated points) then Theorem 1 holds. Indeed, in this case every non-empty open subset of X is uncountable (see [3], Proposition 1.29) and thus $X = X^{\mathrm{c}}$; further, second-countable spaces are Fréchet and hereditarily Lindelöf.

THEOREM 2. *We have* $\operatorname{card} \mathcal{S} = \operatorname{card}(\mathcal{F} \setminus \mathcal{S}) = 2^{\operatorname{card} X}$.

Proof. Let $f \in \mathcal{S}$ (see Theorem 1). Using the notation of Theorem 1 put $\alpha_n = \operatorname{card}(X \setminus G_n)$ ($n \in \mathbb{N}$) and $\alpha = \operatorname{card}(X \setminus A) = \operatorname{card} X$ (X is uncountable). Then $\{\alpha_n\}_{n=1}^{\infty}$ is a non-decreasing sequence of infinite ordinals converging to α (in the order topology; see [4]). Fix $n \in \mathbb{N}$. For every $M \subset X \setminus G_n$ define the function $f_M = \max\{1, f\} \cdot \chi_{X\setminus M}$, where $\chi_{X\setminus M}$ is the characteristic function of $X \setminus M$.

It is not hard to see that $f_M \neq f_N$ and $f_M, f_N \in \mathcal{S}$ for any different subsets M, N of the closed set $X \setminus G_n$. Thus $\operatorname{card}\mathcal{S} \geq 2^{\alpha_n}$. Since α is a limit ordinal we have $\operatorname{card}\mathcal{S} \geq \sup\{2^{\alpha_n} : n \in \mathbb{N}\} = 2^{\alpha} = 2^{\operatorname{card} X}$. On the other hand, making allowance for the uncountability of X we get $\operatorname{card}\mathcal{S} \leq \operatorname{card}\mathcal{F} = (\operatorname{card}\mathbb{R})^{\operatorname{card} X} = 2^{\operatorname{card} X}$.

To show that $\operatorname{card}(\mathcal{F} \setminus \mathcal{S}) = 2^{\operatorname{card} X}$ it suffices to notice that $\chi_B \in \mathcal{F} \setminus \mathcal{S}$ for any $B \subset X$. Hence $2^{\operatorname{card} X} \leq \operatorname{card}(\mathcal{F} \setminus \mathcal{S}) \leq \operatorname{card}\mathcal{F} = 2^{\operatorname{card} X}$. ∎

To be able to investigate $\mathcal{S}$ from the topological point of view introduce the sup-metric d on $\mathcal{F}$:

$$d(f,g) = \min\{1, \sup_{x\in\mathbb{R}} |f(x) - g(x)|\}, \qquad \text{where } f, g \in \mathcal{F}.$$

It is known that $(\mathcal{F}, d)$ is a complete metric space.

THEOREM 3. *The class $\mathcal{S}$ is simultaneously open and closed in $\mathcal{F}$.*

Proof. If $f, g \in \mathcal{F}$ and $d(f,g) < 1$ then $L_f(X) = L_g(X)$. So if $f \in \mathcal{S}$ (resp. $f \in \mathcal{F} \setminus \mathcal{S}$) then the open 1-ball around f is in $\mathcal{S}$ (resp. in $\mathcal{F} \setminus \mathcal{S}$). ∎

THEOREM 4. *Both $\mathcal{S}$ and $\mathcal{F} \setminus \mathcal{S}$ are of second category in $\mathcal{F}$.*

Proof. According to Theorems 2 and 3, $\mathcal{S}$ and $\mathcal{F} \setminus \mathcal{S}$ are non-empty open sets, and consequently they are of second category in the complete metric space $(\mathcal{F}, d)$. ∎

Remark 2. In the light of Theorems 2 and 4 it is worth noticing that neither $\mathcal{S}$ nor $\mathcal{F} \setminus \mathcal{S}$ is dense in $\mathcal{F}$. Actually, if, say, $\mathcal{S}$ were dense in $\mathcal{F}$ then in view of Theorem 3 it would be a residual set in $\mathcal{F}$ and hence $\mathcal{F} \setminus \mathcal{S}$ of first category in $\mathcal{F}$.

THEOREM 5. *We have $\mathcal{S} \subset \mathcal{S}^{\mathrm{c}}$ and $\mathcal{F} \setminus \mathcal{S} \subset (\mathcal{F} \setminus \mathcal{S})^{\mathrm{c}}$.*

Proof. Let $f \in \mathcal{F}$ and $0 < \varepsilon < 1$. For $0 < \eta < \varepsilon$ define $f_\eta(x) = f(x) + \eta$ ($x \in X$). Then $d(f, f_\eta) = \eta < \varepsilon$ for all $\eta \in (0, \varepsilon)$; furthermore, $f_\eta \in \mathcal{S}$ if and only if $f \in \mathcal{S}$ ($0 < \eta < \varepsilon$). ∎

Remark 3. It is easy to see that the set

$$\mathcal{S}' = \{f \in \mathcal{F} : \lim_{t\to x} f(t) = -\infty \text{ if and only if } x \in A\}$$

also has the properties established in Theorems 1–5 for $\mathcal{S}$.

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