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Minimal bi-Lipschitz embedding dimension of ultrametric spaces

by

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Abstract. We prove that an ultrametric space can be bi-Lipschitz embedded in $\mathbb{R}^n$ if its metric dimension in Assouad's sense is smaller than n. We also characterize ultrametric spaces up to bi-Lipschitz homeomorphism as dense subspaces of ultrametric inverse limits of certain inverse sequences of discrete spaces.

1. Introduction. A map $f\colon M \to M'$ of metric spaces is said to be *bi-Lipschitz* if there is a constant $L \geq 1$ such that $d(x,y)/L \leq d(f(x), f(y)) \leq Ld(x,y)$ for all $x, y \in M$; then f is also called L-bi-Lipschitz (we denote every metric by d if not otherwise specified). It is an open problem to characterize the metric spaces which can be bi-Lipschitz embedded in a (finite-dimensional) Euclidean space. If a compact metric space M can locally be bi-Lipschitz embedded in $\mathbb{R}^n$, $n \geq 2$, then M can be bi-Lipschitz embedded in $\mathbb{R}^{n(n+1)}$ by [13, Remark 4.6]; in particular, a compact Lipschitz n-manifold has a bi-Lipschitz embedding in $\mathbb{R}^{2n+1}$ by [12, Corollary 4.6]. By [4, Lemme 4.9], a metric space admits an L-bi-Lipschitz embedding in $\mathbb{R}^n$ if all of its finite subsets have this property.

These results cannot be considered satisfactory characterizations, but Assouad [2]–[4] takes a promising approach. For a metric space (M, d), he defines a bi-Lipschitz invariant called the metric dimension (see Definition 3.2) and proves that in order for (M, d) to be of finite metric dimension it is sufficient that for some $p \in (0, 1]$ and necessary that for each $p \in (0, 1)$ the metric space (M, d^p) is bi-Lipschitz embeddable in a Euclidean space. However, whether the necessity also extends to the case $p = 1$ is not known, but Assouad conjectures this is so.

In this paper we study the existence of bi-Lipschitz embeddings in Eu-

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clidean spaces for a more tractable, yet important subclass of metric spaces, that of ultrametric ones. Recall that an *ultrametric space* is a metric space M whose metric d, then also called an ultrametric, satisfies the strong triangle inequality $d(x,y) \le \max\{d(x,z), d(y,z)\}$ for all $x, y, z \in M$. Based on the fact that every positive power of an ultrametric is also an ultrametric, Assouad deduces from his result that an ultrametric space M can be bi-Lipschitz embedded in a Euclidean space if and only if M is of finite metric dimension (see Proposition 3.3). In our simple Proposition 3.1 we show that ultrametric subspaces of Euclidean spaces are finite. This implies that the Euclidean bi-Lipschitz image of an ultrametric space M of finite metric dimension cannot itself be chosen ultrametric if M is infinite. On the other hand, if M is finite and $\operatorname{card} M = n+1$, then M is evidently bi-Lipschitz homeomorphic to the ultrametric vertex set of a regular n-simplex in $\mathbb{R}^n$. (See the paragraph after the next one for a more exact result.)

In our main result, Theorem 3.8, we prove that for each $n \in \mathbb{N}$ an ultrametric space of metric dimension $< n$ can be bi-Lipschitz embedded in $\mathbb{R}^n$. Since every metric space which is bi-Lipschitz embeddable in $\mathbb{R}^n$ must be of metric dimension $\le n$, for ultrametric spaces of fractional metric dimension our result thus gives the smallest possible Euclidean bi-Lipschitz embedding dimension. However, whether this also holds for all ultrametric spaces of integral metric dimension remains open, but for each $n \in \mathbb{N}$ we construct an example of a compact ultrametric space of metric dimension n for which this is really the case, that is, which is not bi-Lipschitz embeddable in $\mathbb{R}^n$. Crucial for our proof is the characteristic property of ultrametric spaces that for every fixed radius the closed balls form a partition of the space. A variant of our method was used by the second author in [15, Theorem 2.3] to show that every compact ultrametric space is bi-Lipschitz embeddable in a Hilbert space.

For the sake of comparison, let us recall results concerning other types of embeddings of ultrametric spaces. Note first that a metrizable space admits an ultrametric if and only if its topological dimension (in the large inductive or covering sense) is at most zero [6, Theorem 4.1.24 and Problem 4.1.G]. Thus every separable ultrametric space can be topologically embedded in the Cantor set in $\mathbb{R}^1$ [6, Theorem 1.3.15]. By van Rooij [17, Corollary 2.2], every compact ultrametric space can be topologically embedded in every complete non-Archimedean (i.e., ultrametric) nontrivially valued field. The result referred to above is that an ultrametric space can be isometrically embedded in $\mathbb{R}^n$ if and only if it has at most $n+1$ points. This was proved independently by Timan [19] (at least in a special case), Lemin [11], and Aschbacher, Baldi, Baum, and Wilson [1, Theorems 1.1 and 6.7]. Further, every ultrametric space can be isometrically embedded in a Hilbert space. This was shown independently by Kelly ([8, Theorem 8.1], [9, Theorem 2], [21, Theorem 2.4]), Timan and Vestfrid [20], and Lemin [11]. Timan [19] also constructed isometric embeddings of certain countable ultrametric spaces in L_p-spaces, $p \geq 1$. Finally, by Schikhof [18, Theorem A.10] every ultrametric space can be isometrically embedded in a spherically complete (thus complete) non-Archimedean valued field.

Section 2 deals with a related problem. Analogously to the characterization of ultrametrizable topological spaces up to homeomorphism as subspaces of countable products of discrete spaces [17, Theorem 2.1], we ultrametrize the inverse limit of every doubly infinite inverse sequence of discrete spaces satisfying a certain one-sidedness condition and show every ultrametric space to be bi-Lipschitz homeomorphic to a dense subset of such an ultrametric inverse limit. These results generalize those of the second author in [15]. We also analogously generalize results due to Lemin [10] about isometric characterization of ultrametric spaces. For compact ultrametric spaces the results of Section 2 have earlier been obtained by Michon [14].

For basic properties of ultrametric spaces we refer to [17] and [18]. We assume $0 \notin \mathbb{N}$.

2. Bi-Lipschitz embeddings in inverse limits. We first construct a class of ultrametric spaces as inverse limits.

2.1. Construction. Let X_j be a discrete topological space for each $j \in \mathbb{Z}$, let $g_j\colon X_{j+1} \to X_j$ be a map for each $j \in \mathbb{Z}$, and let Σ be the inverse limit space $\lim_{\leftarrow}(X_j, g_j)$ of the doubly infinite (i.e., with $\mathbb{Z}$ as index set) inverse sequence $(X_j, g_j)_{j\in\mathbb{Z}}$, i.e., Σ is the subspace $\{(x_j) \mid g_j(x_{j+1}) = x_j$ for each $j \in \mathbb{Z}\}$ of the product space $\prod_{j\in\mathbb{Z}} X_j$. Moreover, suppose that if $x, y \in \Sigma$, then $x_j = y_j$ for some $j \in \mathbb{Z}$, implying that $x_i = y_i$ whenever $i \leq j$. We call doubly infinite inverse sequences $(X_j, g_j)_{j\in\mathbb{Z}}$ of this kind *one-sided*.

Let $(r_j)_{j\in\mathbb{Z}}$ be a sequence of positive real numbers such that $r_{j+1} < r_j$ for each $j \in \mathbb{Z}$, that $\lim_{j\to\infty} r_j = 0$, and that $\lim_{j\to-\infty} r_j = \infty$. Then we can define a complete compatible ultrametric ϱ on Σ by setting $\varrho(x, y) = r_j$ for two distinct points x, y if j is the greatest integer with $x_j = y_j$. We call this metric ϱ on Σ a *comparison ultrametric*. If all spaces X_j are countable, Σ is separable; if they are finite, Σ is compact. If $\operatorname{card} X_1 = 1$, guaranteeing the one-sidedness property, then we may obviously reduce the index set $\mathbb{Z}$ to $\mathbb{N}$ in the definition of (Σ, ϱ) getting a canonically isometric copy of (Σ, ϱ).

The following theorem shows that the class of ultrametric spaces constructed in 2.1 is in a certain sense universal for the category of ultrametric spaces and bi-Lipschitz embeddings.

2.2. Theorem. *Let M be an ultrametric space. Then M is bi-Lipschitz embeddable as a dense subset in the inverse limit of a one-sided doubly infinite inverse sequence of discrete spaces with a comparison ultrametric.*

Proof. Let $K > 1$ be a constant and $(r_j)_{j\in\mathbb{Z}}$ a sequence such that $0 < r_{j+1} < r_j \le Kr_{j+1}$ for each $j \in \mathbb{Z}$, that $\lim_{j\to\infty} r_j = 0$, and that $\lim_{j\to-\infty} r_j = \infty$. Let $B(x,r) = \{y \in M \mid d(x,y) \le r\}$ for $x \in M$, $r > 0$. Noting that $B(x,r) = B(y,r)$ if $y \in B(x,r)$, construct partitions $X_j = \{B(x,r_j) \mid x \in M\}$ $(j \in \mathbb{Z})$ of M and maps g_j: $X_{j+1} \to X_j$, $B(x,r_{j+1}) \mapsto B(x,r_j)$ $(j \in \mathbb{Z})$. Then let $\Sigma = \lim_{\leftarrow}(X_j, g_j)$. If $(B_j)_{j\in\mathbb{Z}}$ and $(B'_j)_{j\in\mathbb{Z}}$ are points of Σ, there are $j < 0$ and $B''_j \in X_j$ such that $B_0 \cup B'_0 \subset B''_j$, which implies $B_j = B''_j = B'_j$. Thus, the inverse sequence $(X_j, g_j)_{j\in\mathbb{Z}}$ is one-sided. Now give Σ the comparison ultrametric ϱ associated with the sequence $(r_j)_{j\in\mathbb{Z}}$.

Define a map f: $M \to \Sigma$ by $f(x) = (B(x,r_j))_{j\in\mathbb{Z}}$. Consider $x, y \in M$, $x \ne y$. Choose $j \in \mathbb{Z}$ with $r_{j+1} < d(x,y) \le r_j$. Then x and y are in the same member of X_j but in distinct members of X_{j+1}. Hence, $\varrho(f(x), f(y)) = r_j$, and therefore $d(x,y) \le \varrho(f(x), f(y)) \le Kd(x,y)$. Consequently, f is a bi-Lipschitz embedding.

To see fM to be dense in Σ, note first that as $fM = \{(B_j)_{j\in\mathbb{Z}} \in \Sigma \mid \bigcap_{j\in\mathbb{Z}} B_j \ne \emptyset\}$, we have $fM = \Sigma$ if and only if M is complete. Now let $\widehat{M}$ be the completion of M; it, too, is an ultrametric space. Let $(\widehat{X}_j, \widehat{g}_j)_{j\in\mathbb{Z}}$ be the inverse sequence, $\widehat{\Sigma} = \lim_{\leftarrow}(\widehat{X}_j, \widehat{g}_j)$ the ultrametric space, and $\widehat{f}$: $\widehat{M} \to \widehat{\Sigma}$ the bi-Lipschitz homeomorphism associated with $\widehat{M}$ and $(r_j)_{j\in\mathbb{Z}}$. For each $j \in \mathbb{Z}$ we have a bijection h_j: $\widehat{X}_j \to X_j$, $B \mapsto B \cap M$. Since $h_j\widehat{g}_j = g_j h_{j+1}$, the sequence $(h_j)_{j\in\mathbb{Z}}$ induces a bijection h: $\widehat{\Sigma} \to \Sigma$, $(B_j)_{j\in\mathbb{Z}} \mapsto (B \cap M)_{j\in\mathbb{Z}}$, and h is an isometry. From $h\widehat{f}|M = f$ we then conclude that fM is dense in Σ. ∎

2.3. Remarks. 1) In the proof, M is totally bounded (respectively, separable) if and only if the spaces X_j are finite (respectively, countable). If M is bounded (and nonempty), by choosing $r_1 \ge \operatorname{diam} M$ we have $X_j = \{M\}$ for all $j \le 1$, and hence we could replace the index set $\mathbb{Z}$ by $\mathbb{N}$ in the definition of (Σ, ϱ). Theorem 2.2 generalizes [15, Proposition 2.1], in which the ultrametric space M is assumed to be bounded and complete and in whose proof r_1 is chosen to be $\ge \operatorname{diam} M$. For compact ultrametric spaces, Theorem 2.2 also follows from results of Michon [14].

2) A consequence of Theorem 2.2, which is also easy to establish directly [17, Exercise 2.F], is that every ultrametric is bi-Lipschitz equivalent to an ultrametric whose positive values form a discrete set. As not all ultrametrics satisfy the latter condition, in Theorem 2.2 bi-Lipschitz embeddings cannot be replaced by isometric embeddings.

3) Analogously to the above characterization of ultrametric spaces up to bi-Lipschitz homeomorphism, there is the following characterization of ultrametric spaces up to isometry as dense subsets of ultrametric inverse

limits of certain countable inverse systems of discrete spaces. For bounded complete spaces this characterization has been given by Lemin [10, 6.4], although not as explicitly as here, and for compact spaces by Michon [14].

First, fix a countable dense subset A of $(0, \infty)$ for an index set. Now, if $\mathbf{S}$ is an inverse system of discrete topological spaces X_r $(r \in A)$ and maps $g_r^s\colon X_s \to X_r$ $(r,\, s \in A,\ s \geq r)$, if $\Sigma = \lim_{\leftarrow} \mathbf{S}$ is the inverse limit of $\mathbf{S}$, and if $\mathbf{S}$ is one-sided in the sense that for all $x, y \in \Sigma$ there is $r \in A$ with $x_r = y_r$, then we can define a complete compatible ultrametric ϱ on Σ by $\varrho(x, y) = \sup\{1/r \mid r \in A,\ x_r \neq y_r\}$. Conversely, if M is an ultrametric space, X_r the partition $\{B(x, 1/r) \mid x \in M\}$ of M $(r \in A)$, and $g_r^s\colon X_s \to X_r$ the map $B(x, 1/s) \mapsto B(x, 1/r)$ $(r,\, s \in A,\ s \geq r)$, then the inverse system $\mathbf{S} = (X_r, g_r^s)_A$ is one-sided, and if $\Sigma = \lim_{\leftarrow} \mathbf{S}$ is ultrametrized as above, then the map $f\colon M \to \Sigma$, $x \mapsto (B(x, 1/r))_{r \in A}$, is an isometry onto a dense subset, and $fM = \Sigma$ if and only if M is complete.

3. Bi-Lipschitz embeddings in Euclidean spaces. Our first result is a weaker form of a known one, but we present it as our proof is so simple.

3.1. Proposition. *No infinite ultrametric space is isometric to a subset of $\mathbb{R}^n$ for any $n \in \mathbb{N}$.*

Proof. Suppose the contrary, and assume that $n \in \mathbb{N}$ is the smallest number for which $\mathbb{R}^n$ contains an infinite ultrametric subspace M. Since also $\overline{M}$ is ultrametric, we may assume M to be closed. For distinct points $x, y \in M$ let $T(x, y) \subset \mathbb{R}^n$ denote the perpendicular bisector of the segment $[x, y]$. Then $z \in M$ and $|x - y| < |x - z|$ imply $z \in T(x, y)$. In fact, $|x - z| \leq \max\{|x - y|, |y - z|\} = |y - z|$ and $|y - z| \leq \max\{|x - y|, |x - z|\} = |x - z|$, and so $|x - z| = |y - z|$.

We first show M to be discrete. If not, M has a cluster point x. Let $A = M \setminus \{x\}$. Choose a sequence (y_j) in A converging to x. Let $F_k = \bigcap_{j>k} T(x, y_j)$ for $k \in \mathbb{N}$. Then $F_1 \subset F_2 \subset \ldots$ are proper affine subspaces of $\mathbb{R}^n$ (possibly empty), which implies that such is also their union F. If $z \in A$, choose k with $|x - y_j| < |x - z|$ for $j > k$; then $z \in F_k$. Thus, $A \subset F$. Consequently, $M \subset F$, a contradiction.

It follows that we can write $M = \{x_j \mid j \in \mathbb{N}\}$ with $x_1 \neq x_2$ and $x_j \to \infty$. Choose k with $|x_1 - x_2| < |x_1 - x_j|$ for $j > k$, and let $A = \{x_j \mid j > k\}$. Then $A \subset T(x_1, x_2)$. Since A is an infinite ultrametric set, this is a contradiction. ■

The following definition for the metric dimension is equivalent to that of Assouad [3], [4].

3.2. Definition. Let M be a metric space. Suppose that $s \geq 0$ and $C \geq 0$ are numbers such that $\operatorname{card} Y \leq C(b/a)^s$ whenever $a > 0$ and $b \geq a$ are numbers and $Y \subset M$ a set with $a \leq d(x, y) \leq b$ for $x, y \in Y$ and $x \neq y$.

Then M is called (C,s)*-homogeneous*. We say that M is *s-homogeneous* if M is (C,s)-homogeneous for some C. The infimum (in $[0,\infty]$) of the numbers s (if any) for which M is s-homogeneous is called the *metric dimension* of M and denoted by $\operatorname{Dim} M$ or also by $\operatorname{Dim}(M,d)$.

We need the following basic properties of these concepts (cf. [3, Proposition 2] or [4, 2.2]). An L-bi-Lipschitz image of a (C,s)-homogeneous space is $(L^{2s}C,s)$-homogeneous. Thus, Dim is a bi-Lipschitz invariant. Consider $A\subset M$. If M is (C,s)-homogeneous, so is A, and conversely whenever A is dense. Thus, $\operatorname{Dim} A\le \operatorname{Dim} M$, with equality if A is dense. The space $\mathbb{R}^n$ is n-homogeneous and $\operatorname{Dim}\mathbb{R}^n=\operatorname{Dim}[0,1]^n=n$ for all $n\ge 0$. If $p\in(0,1)$, then $\operatorname{Dim}(M,d^p)=(1/p)\operatorname{Dim}(M,d)$. If $M=\bigcup_{i=1}^k A_i$, then $\operatorname{Dim} M=\max_{1\le i\le k}\operatorname{Dim} A_i$.

We also mention the following fact [3, Remarque 2] yielding a simple characterization for being of finite metric dimension. A metric space is (C,s)-homogeneous for some (C,s) if and only if there is $q\in\mathbb{N}$ such that for each $r>0$, each closed ball of radius r can be covered by (in the ultrametric case: is the union of) k closed balls of radius $r/2$ with $k\le q$. Here (C,s) and q can be chosen to depend only on each other.

Before proving our main result, Theorem 3.8, we first present and discuss a weaker form of it due to Assouad [3, Remarque 2 and Proposition 3(g)].

3.3. Proposition. *Let M be an ultrametric space. Then there is a bi-Lipschitz embedding of M in $\mathbb{R}^n$ for some $n\in\mathbb{N}$ if and only if* $\operatorname{Dim} M<\infty$.

For the "if"-part note that the metric d of M can be written as $d=(d^2)^{1/2}$ with d^2, too, being an ultrametric and that $\operatorname{Dim}(M,d^2)=\frac{1}{2}\operatorname{Dim}(M,d)<\infty$; hence, the assertion follows from Assouad's result ([2, Proposition 1.30], [4, Proposition 2.6]) mentioned in the second paragraph of the introduction. Both versions of Assouad's proof show that in the "if"-part n and an upper bound for the bi-Lipschitz constant of the embedding can be chosen to depend only on a pair (C,s) of numbers for which M is (C,s)-homogeneous.

3.4. Remark. The sufficiency part of Proposition 3.3 is not valid even for compact spaces if the metric dimension Dim is replaced either by the Hausdorff dimension $\dim_{\mathrm{H}}$ (cf. [7, (2.11)]) or by the upper fractal dimension $\overline{\dim}_{\mathrm{f}}$ in the sense of [15, p. 558] (called the upper box-counting dimension in [7, (3.5)]). This amounts to the fact that $\operatorname{Dim} M=\infty$ implies neither $\dim_{\mathrm{H}} M=\infty$ nor $\overline{\dim}_{\mathrm{f}} M=\infty$ as we now establish.

Recall that for a metric space M we can define $\overline{\dim}_{\mathrm{f}} M\in[0,\infty]$ as follows. If $M=\emptyset$, set $\overline{\dim}_{\mathrm{f}} M=0$; if M is not totally bounded, set $\overline{\dim}_{\mathrm{f}} M=\infty$; otherwise, letting $N_r(M)\in\mathbb{N}$ for $r>0$ be the smallest number of open r-balls needed to cover M, set $\overline{\dim}_{\mathrm{f}} M=\limsup_{r\to 0}\log N_r(M)/\log(1/r)$.

We have $\dim_H M \le \operatorname{Dim} M$ [3, Proposition 2] and $\dim_H M \le \overline{\dim}_f M$ (cf. [7, (3.17)]) for all metric spaces M. By Proposition 3.5 below, $\overline{\dim}_f M \le \operatorname{Dim} M$ if M is bounded.

Now in [13, 4.12] it was shown that the compact ultrametric space $M = \mathbb{N} \cup \{\infty\}$ with the metric defined by $d(i,j) = 1/\log(i+1)$ if $i < j$ has no bi-Lipschitz embedding into any Euclidean space although $\dim_H M = 0$ as M is countable. In fact, $\overline{\dim}_f M = \infty$, and therefore there is not even any embedding $f\colon M \to \mathbb{R}^n$, $n \in \mathbb{N}$, with $f^{-1}|fM$ Hölder continuous; cf. Example 3.6. More strongly, [15, Example 3.2] gives a compact ultrametric space M homeomorphic to the Cantor set such that $\overline{\dim}_f M < \infty$ and such that M is not bi-Lipschitz embeddable in $\mathbb{R}^n$ for any $n \in \mathbb{N}$ (see Example 3.6 for a countable example). This was established by showing $\dim_m M = \infty$ for the metric dimension $\dim_m$ in the sense of [15, Definition 3.1]. The dimensions $\dim_m$ and Dim are easily seen to coincide for totally bounded metric spaces. Proposition 3.3 is not, however, valid for $\dim_m$ either, because for arbitrary metric spaces M we only have the inequality $\dim_m M \le \operatorname{Dim} M$, and this assumes the form $0 < \infty$ if M is an infinite ultrametric space with $d(x,y) = 1$ for $x \ne y$.

3.5. Proposition. *If M is a bounded metric space, then $\overline{\dim}_f M \le \operatorname{Dim} M$.*

Proof. If M is nonempty and totally bounded, we can replace $N_r(M)$ in the definition of $\overline{\dim}_f M$ by $N'_r(M) = \max\{\operatorname{card} Y \mid Y \subset M,\ d(x,y) \ge r$ if $x, y \in Y,\ x \ne y\}$ as $N_r(M) \le N'_r(M) \le N_{r/2}(M)$.

We may assume that $b = \operatorname{diam} M > 0$ and $\operatorname{Dim} M < \infty$. Suppose M to be (C,s)-homogeneous. Let $r \in (0,b]$. If $Y \subset M$ and $d(x,y) \ge r$ for $x, y \in Y$, $x \ne y$, then $\operatorname{card} Y \le C(b/r)^s$. Thus, M is totally bounded and $N'_r(M) \le C(b/r)^s$. This implies $\overline{\dim}_f M \le s$. Hence, $\overline{\dim}_f M \le \operatorname{Dim} M$. ■

3.6. Example. It is an open problem to characterize the metric spaces, or even the compact ultrametric ones, which are bi-Hölder embeddable in $\mathbb{R}^n$ for some $n \in \mathbb{N}$ (cf. [5], [15]). We show that such spaces need not have finite metric dimension and thus need not be bi-Lipschitz embeddable in $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Recall that a map $f\colon M \to N$ of metric spaces is called *Hölder of exponent* $\alpha > 0$ if $d(f(x), f(y)) \le c\,d(x,y)^\alpha$ for all $x, y \in M$ with some $c > 0$. If, in addition, the inverse of f on fM exists and is Hölder of exponent $\beta > 0$, we call f *bi-Hölder*. Note that if M is compact and $f\colon M \to \mathbb{R}^n$ an embedding with f^{-1} Hölder of exponent α, then $\overline{\dim}_f M \le (1/\alpha)\,\overline{\dim}_f fM \le n/\alpha < \infty$. However, it is not known whether M can conversely be bi-Hölder embedded in a Euclidean space if $\overline{\dim}_f M < \infty$. In our example this is not possible with an embedding f for which f is Hölder of exponent γ and f^{-1} Hölder of exponent $1/\gamma$ for some $\gamma > 0$ as this implies $\operatorname{Dim} fM = (1/\gamma)\operatorname{Dim} M$.

Our example is a modification of an example in (a preliminary version of) [5]. Thus, let $M = \mathbb{N} \cup \{\infty\}$, let $\alpha > 0$, and endow M with the compact ultrametric defined by $d(i,j) = i^{-\alpha}$ if $i < j$. Then $\overline{\dim}_{\mathrm{f}}\, M = 1/\alpha$. Define an embedding $f\colon M \to \mathbb{R}^1$ by $f(i) = i^{-\alpha}$ for $i < \infty$ and $f(\infty) = 0$. Let $\beta = \alpha/(\alpha+1)$. Then there is $c > 0$ such that $|f(i) - f(j)| \le d(i,j) \le c|f(i) - f(j)|^{\beta}$ for all $i, j \in M$, and so f is bi-Hölder. The first inequality is trivial, and the second inequality follows from the fact that if $i, j \in M$, $i < j$, and $g(x) = (1+x)^{-\alpha}$, then

$$\begin{aligned} d(i,j)|f(i) - f(j)|^{-\beta} &\le i^{-\alpha}|i^{-\alpha} - (i+1)^{-\alpha}|^{-\beta} \\ &= i^{-\alpha(1-\beta)}(1 - (1+i^{-1})^{-\alpha})^{-\beta} \\ &= ((g(0) - g(i^{-1}))/i^{-1})^{-\beta} = h(i) \le h(1) \end{aligned}$$

as $g'(x) < 0 < g''(x)$ for $x \ge 0$. Finally, to see that $\operatorname{Dim} M = \infty$, let $s > 0$, $i \in \mathbb{N}$, and $Y = \{i, i+1, \ldots, 2i+1\}$. Then note that $(2i)^{-\alpha} \le d(j,k) \le i^{-\alpha}$ if $j, k \in Y$, $j \ne k$, and that $\operatorname{card} Y/(i^{-\alpha}/(2i)^{-\alpha})^s = (i+2)2^{-\alpha s} \to \infty$ as $i \to \infty$.

3.7. Construction. We now turn to our main result. We first construct for each $n \in \mathbb{N}$ a family $\{(U_k^n, \varrho) \mid k \in \mathbb{N}\}$ of ultrametric spaces, which will be shown to be universal for the category of ultrametric spaces M with $\operatorname{Dim} M < n$ and of bi-Lipschitz embeddings. Thus, fix $n, k \in \mathbb{N}$. Let $L = 2k+1$ and $A = \{0, 1, \ldots, k\}^n$. Let $\varphi_a\colon \mathbb{R}^n \to \mathbb{R}^n$ for $a \in A$ be the similarity map $x \mapsto L^{-1}(x + 2a)$. Let $\mathcal{F}$ be the family of axis-parallel closed cubes $L^i \varphi_{a_1} \ldots \varphi_{a_j}[0,1]^n$ in $[0,\infty)^n$ with $i \in \mathbb{N} \cup \{0\}$, $j \in \mathbb{N}$, and $a_1, \ldots, a_j \in A$. Then, letting $r_j = L^{-j}$ for $j \in \mathbb{Z}$ and endowing $\mathbb{R}^n$ with the box norm $\|x\| = \max\{|x_1|, \ldots, |x_n|\}$, it is clear that $\mathcal{F}$ has the following properties:

1) if $\mathcal{F}_j = \{F \in \mathcal{F} \mid \operatorname{diam} F = r_j\}$ for $j \in \mathbb{Z}$, then $\mathcal{F} = \bigcup_{j \in \mathbb{Z}} \mathcal{F}_j$;
2) $\operatorname{dist}(F, F') \ge r_j$ if $j \in \mathbb{Z}$ and $F, F' \in \mathcal{F}_j$ with $F \ne F'$;
3) if $j \in \mathbb{Z}$, then each $F \in \mathcal{F}_{j+1}$ is contained in a unique cube $h(F) \in \mathcal{F}_j$;
4) if $j \in \mathbb{Z}$, then each $F \in \mathcal{F}_j$ contains exactly $(k+1)^n$ cubes of $\mathcal{F}_{j+1}$; and
5) $F_j = [0, r_j]^n \in \mathcal{F}_j$ for each $j \in \mathbb{Z}$.

The third property defines a function $h\colon \mathcal{F} \to \mathcal{F}$. Let $U = U_k^n = \bigcap_{j \in \mathbb{Z}} \bigcup \mathcal{F}_j \subset \mathbb{R}^n$. Then U is closed and perfect. Define an ultrametric ϱ on U by setting $\varrho(x,y) = r_j$ for two distinct points $x, y \in U$ if j is the greatest integer for which x and y are in the same member of $\mathcal{F}_j$. Then $\|x - y\| \le \varrho(x,y) \le L\|x - y\|$ and thus $|x-y|/\sqrt{n} \le \varrho(x,y) \le L|x-y|$. Hence, ϱ is bi-Lipschitz compatible.

We study U further, but the following properties of U are not really needed later.

We first show $(U, \|\cdot\|)$ to be (C_0, t)-homogeneous with $C_0 = (k+1)^{2n}$ and $t = n\log(k+1)/\log(2k+1) < n$. Thus, let $0 < a \le b$, let $Y \subset U$, and let $a \le \|x-y\| \le b$ if $x, y \in Y$ and $x \ne y$. Choose $i, j \in \mathbb{Z}$ with $r_{i+1} < a \le r_i$ and $r_{j+1} \le b < r_j$; then $i \ge j$. Now each cube in $\mathcal{F}_{i+1}$ contains at most one point of Y, and Y is contained in one cube of $\mathcal{F}_j$. Hence $\operatorname{card} Y \le (k+1)^{n(i+1-j)} = C_0(r_{j+1}/r_i)^t \le C_0(b/a)^t$ as needed.

It is also easily seen that $\operatorname{Dim}(U, \|\cdot\|) \ge t$. Alternatively, this follows from the equality $\dim_{\mathrm{H}} U = t$ below. Hence, $\operatorname{Dim} U = t$. Note that $t \to n$ as $k \to \infty$. Obviously U is the union of countably many disjoint isometric copies of the compact set $U_0 = U \cap F_0$, which is homeomorphic to the middle third Cantor set and for $n = k = 1$ even coincides with it. Since $\{\varphi_a U_0 \mid a \in A\}$ is a partition of U_0, in the terminology of [7, p. 113], U_0 is a self-similar fractal invariant for the similarities φ_a with ratios L^{-1}. Since $\operatorname{card} A \cdot L^{-t} = 1$, from [7, Theorem 9.3] we then conclude that $\dim_{\mathrm{H}} U_0 = \overline{\dim}_{\mathrm{f}}\, U_0 = t$ and that the t-dimensional Hausdorff measure of U_0 is positive and finite. Consequently, $\dim_{\mathrm{H}} U = t$.

3.8. THEOREM. *Let M be an ultrametric space and $n \in \mathbb{N}$. If* $\operatorname{Dim} M < n$, *then M is bi-Lipschitz embeddable in $\mathbb{R}^n$. Conversely, if M is bi-Lipschitz embeddable in $\mathbb{R}^n$, then* $\operatorname{Dim} M \le n$; *in fact, M is n-homogeneous.*

Proof. It suffices to prove the first part. Here it is reasonable not to use terminology developed in Section 2 although the structure of ultrametric spaces related there to inverse limits is pertinent also now. Choose $s < n$ and $C > 0$ such that M is (C, s)-homogeneous. Let $k \in \mathbb{N}$ be the smallest number with $C(2k+1)^s \le (k+1)^n$. In $\mathbb{R}^n$ we use the constructions of 3.7 with this value of k. For $j \in \mathbb{Z}$, let $\mathcal{B}_j$ be the partition of M by closed balls of radius r_j. Let $\mathcal{B}$ be the sum (i.e., disjoint union) of the family $\{\mathcal{B}_j \mid j \in \mathbb{Z}\}$. Define a function g: $\mathcal{B} \to \mathcal{B}$ such that if $j \in \mathbb{Z}$ and $B \in \mathcal{B}_{j+1}$, then $g(B)$ is the unique ball in $\mathcal{B}_j$ containing B. If $B \in \mathcal{B}_j$ and $B', B'' \in \mathcal{B}_{j+1}$ with $B' \ne B''$, then $\operatorname{diam} B \le r_j$ and $\operatorname{dist}(B', B'') > r_{j+1}$, which implies $\operatorname{card} g^{-1}(B) \le C(r_j/r_{j+1})^s = C(2k+1)^s$. Thus, $\operatorname{card} g^{-1}(B) \le (k+1)^n = \operatorname{card} h^{-1}(F)$ for all $B \in \mathcal{B}$ and $F \in \mathcal{F}$.

We construct an injection α: $\mathcal{B} \to \mathcal{F}$ such that $\alpha\mathcal{B}_j \subset \mathcal{F}_j$ for each $j \in \mathbb{Z}$ and such that $h\alpha = \alpha g$. Fix a point $x_0 \in M$ (assuming $M \ne \emptyset$). For $j \in \mathbb{Z}$ let $B_j \in \mathcal{B}_j$ be the ball with $x_0 \in B_j$. Construct inductively subfamilies $\mathcal{B}_0^* \subset \mathcal{B}_1^* \subset \ldots$ of $\mathcal{B}$ by letting $\mathcal{B}_0^* = \{B_j \mid j \in \mathbb{Z}\}$ and $\mathcal{B}_{i+1}^* = \bigcup\{g^{-1}(B) \mid B \in \mathcal{B}_i^*\}$ for $i \ge 0$. Then $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathcal{B}_i^*$. Now define α on $\mathcal{B}_1^*$ by letting $\alpha|g^{-1}(B_j)$ for each $j \in \mathbb{Z}$ be an arbitrary injection $g^{-1}(B_j) \to h^{-1}(F_j)$ which maps B_{j+1} to F_{j+1}. Suppose inductively that $i \ge 1$ and α is defined on $\mathcal{B}_i^*$. Then define α on $\mathcal{B}_{i+1}^* \setminus \mathcal{B}_i^*$ by letting $\alpha|g^{-1}(B)$ for each $B \in \mathcal{B}_i^* \setminus \mathcal{B}_{i-1}^*$ be an arbitrary injection $g^{-1}(B) \to h^{-1}(\alpha(B))$. This procedure yields the desired function α.

Now we can define a function $f\colon M \to U$ (with $f(x_0) = 0$) by setting $\{f(x)\} = \bigcap\{\alpha(B) \mid x \in B \in \mathcal{B}\}$ for each $x \in M$. Consider $x, y \in M$ with $x \neq y$. Choose $j \in \mathbb{Z}$ with $r_{j+1} < d(x,y) \le r_j$. Then x and y are in the same member of $\mathcal{B}_j$ but in distinct members of $\mathcal{B}_{j+1}$. Hence, $f(x)$ and $f(y)$ are in the same member of $\mathcal{F}_j$ but in distinct members of $\mathcal{F}_{j+1}$ implying $\varrho(f(x), f(y)) = r_j$. It follows that $1 \le \varrho(f(x), f(y))/d(x,y) \le r_j/r_{j+1} = L$ and thus that $1/L \le |f(x) - f(y)|/d(x,y) \le L\sqrt{n}$. Hence, $f\colon M \to \mathbb{R}^n$ is L_1-bi-Lipschitz with $L_1 = L\sqrt{n}$ depending only on (C, s, n). ∎

3.9. Corollary. *Every ultrametric space of finite metric dimension admits a bi-Hölder embedding in* $\mathbb{R}^1$.

Proof. If $\operatorname{Dim} M < s < \infty$, then $M' = (M, d^s)$ is an ultrametric space with $\operatorname{Dim} M' = (1/s) \operatorname{Dim} M < 1$; now compose the identity map $M \to M'$ with a bi-Lipschitz embedding $M' \to \mathbb{R}^1$ provided by Theorem 3.8. ∎

We say that an embedding between metric spaces is LIP if it is locally bi-Lipschitz.

3.10. Lemma. *Let $n \in \mathbb{N}$, $n \ge 2$. If M is a locally compact separable metric space which locally can be bi-Lipschitz embedded in $\mathbb{R}^n$, then there is a closed* LIP *embedding $f\colon M \to \mathbb{R}^{n(n+1)}$. This also holds if "locally compact" and "closed" are omitted.*

Proof. See [13, Remark 4.6] for the first part. For the second part, let M^* be the completion of M. If $x \in M$, choose an open neighborhood U_x of x in M^* with a bounded bi-Lipschitz embedding $f_x\colon V_x = U_x \cap M \to \mathbb{R}^n$. Then f_x extends to a bi-Lipschitz homeomorphism $f_x^*\colon \operatorname{cl}_{M^*} V_x \to \operatorname{cl} f_x V_x$. Since $U_x \subset \operatorname{cl}_{M^*} V_x$, it follows that U_x is locally compact. Hence, if $M' = \bigcup_{x\in M} U_x$, then by the first part there is an LIP embedding $f'\colon M' \to \mathbb{R}^{n(n+1)}$. Now $f = f'|M$ is the desired embedding. ∎

By Lemma 3.10 we get the following consequence of Theorem 3.8.

3.11. Corollary. *Let $n \in \mathbb{N}$, $n \ge 2$. If M is a separable ultrametric space whose every point has a neighborhood of metric dimension $< n$, then there is an* LIP *embedding $f\colon M \to \mathbb{R}^{n(n+1)}$. If M is locally compact, f can be chosen closed.*

3.12. Example. Let $n \in \mathbb{N}$. Then there is a compact ultrametric space M with $\operatorname{Dim} M = n$ but such that M is not n-homogeneous and consequently has no bi-Lipschitz embedding in $\mathbb{R}^n$. In fact, let $A = \{2^k \mid k \in \mathbb{N}\}$, let $N_j = 4^n$ if $j \in A$ and $N_j = 2^n$ if $j \in \mathbb{N} \setminus A$, let $(X_j, g_j)_{j\in\mathbb{N}}$ be an inverse sequence of finite discrete topological spaces and maps such that $\operatorname{card} X_1 = 1$ and $\operatorname{card} g_j^{-1}(x) = N_j$ for all $j \in \mathbb{N}$ and $x \in X_j$, let $M = \lim_{\leftarrow}(X_j, g_j)$, and endow M with the comparison ultrametric associated with the sequence $(2^{-j})_{j\in\mathbb{N}}$ (see 2.1). Then for each $s > n$ there is $C_s > 0$ such that

$\prod_{i=0}^{p} N_{j+i} \leq C_s(2^{-j}/2^{-j-p})^s$ for all $j \geq 1$ and $p \geq 0$; this follows from the estimate $\operatorname{card}(A \cap [j, j+p]) \leq \log_2(1+p) + 1$. However, as A is infinite, the same is not true of $s = n$. It can now be shown that M is (C_s, s)-homogeneous for each $s > n$ but not n-homogeneous. Then $\operatorname{Dim} M = n$.

3.13. Example. If in Example 3.12 we choose $N_j = 2^n$ for each $j \in \mathbb{N}$, then M is a $(2^n, n)$-homogeneous compact ultrametric space with $\operatorname{Dim} M = n$. We conjecture that M cannot be bi-Lipschitz embedded in $\mathbb{R}^n$ and now establish this conjecture for $n = 1$. In fact, we show that if $n = 1$, there is no embedding $f\colon M \to \mathbb{R}^1$ with $f^{-1}\colon fM \to M$ a Lipschitz map.

Thus, suppose there is such an embedding f. Then we may assume that $|f(x)-f(y)| \geq d(x,y)$ for all $x, y \in M$. For $k \in \mathbb{N}$, let $I_k = \{1, \ldots, 2^k\}$, let A_k be the (nonempty) set of sequences $x = (x_i)_{i \in I_k}$ in M with $d(x_i, x_j) \geq 2^{-k}$ if $i \neq j$, and let l_k be the minimum of the finitely many numbers $l(x) = \sum_{i=2}^{2^k} d(x_{i-1}, x_i)$ with $x \in A_k$. Choosing $x \in A_k$ with $f(x_1) < \ldots < f(x_{2^k})$ we get

$$\operatorname{diam} fM \geq \sum_{i=2}^{2^k} |f(x_{i-1}) - f(x_i)| \geq l(x) \geq l_k\,.$$

Thus, it suffices to show $l_k = \frac{1}{2}k$ as then fM cannot be bounded. Obviously, each $x \in A_k$ has a permutation $y \in A_k$ such that if $1 \leq j \leq k$, then $d(y_{i-1}, y_i) = 2^{-j}$ for 2^{j-1} of the indexes i, which implies $l(y) = \frac{1}{2}k$. It is now enough to prove by induction that $l_k \geq \frac{1}{2}k$.

This being obvious for $k = 1$, let $k > 1$ be such that $l_{k-1} \geq \frac{1}{2}(k-1)$. Consider $x \in A_k$. There is an increasing injection $\varphi\colon I_{k-1} \to I_k$ with $y = (x_{\varphi(i)})_{i \in I_{k-1}} \in A_{k-1}$. If $2 \leq i \leq 2^{k-1}$, then $d(y_{i-1}, y_i) \leq \max\{d(x_{j-1}, x_j) \mid \varphi(i-1) < j \leq \varphi(i)\}$, and consequently,

$$\begin{aligned}\sum_{i=2}^{2^{k-1}} \sum_{j=\varphi(i-1)+1}^{\varphi(i)} d(x_{j-1}, x_j) &\geq \sum_{i=2}^{2^{k-1}} (d(y_{i-1}, y_i) + (\varphi(i) - \varphi(i-1) - 1)2^{-k}) \\ &= l(y) + (\varphi(2^{k-1}) - \varphi(1) - 2^{k-1} + 1)2^{-k}\,.\end{aligned}$$

Moreover,

$$\sum_{j=2}^{\varphi(1)} d(x_{j-1}, x_j) + \sum_{j=\varphi(2^{k-1})+1}^{2^k} d(x_{j-1}, x_j) \geq (\varphi(1) - 1 + 2^k - \varphi(2^{k-1}))2^{-k}.$$

Hence, $l(x) \geq l(y) + 2^{k-1}2^{-k} \geq l_{k-1} + \frac{1}{2} \geq \frac{1}{2}k$. It follows that $l_k \geq \frac{1}{2}k$.

3.14. Conjecture. If $n \in \mathbb{N}$ and M is an n-homogeneous ultrametric space with $\operatorname{Dim} M = n$, there is no bi-Lipschitz embedding $f\colon M \to \mathbb{R}^n$.

The conjecture is equivalent to the claim that an ultrametric space M can be bi-Lipschitz embedded in $\mathbb{R}^n$ only if $\operatorname{Dim} M < n$. In the conjecture we may clearly assume M to be complete. In seeking more examples (for $n \geq 2$) to support or counterexamples to refute the conjecture, the results of the second author [16] about compact metric spaces bi-Lipschitz homeomorphic to an ultrametric space might be useful. For instance, we must disregard the counterexample candidate $M = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subset \mathbb{R}^1$ with $\operatorname{Dim} M = 1$ (cf. Example 3.6) as M has no bi-Lipschitz compatible ultrametric by [16, Proposition 2.14]. Let us mention that, on the other hand, M gives an example of a compact space for which $\operatorname{Dim} M$ is finite but different from $\overline{\dim}_{\mathrm{f}} M = \frac{1}{2}$ [7, Example 3.5].

To establish Conjecture 3.14 for infinite compact spaces we may assume them to have only one cluster point:

3.15. Proposition. *Each infinite compact metric space M contains a countable closed subset A with a unique cluster point for which* $\operatorname{Dim} A = \operatorname{Dim} M$.

Proof. We may assume $s = \operatorname{Dim} M > 0$. By the last of the basic properties after Definition 3.2, there is a point $x_0 \in M$ such that $\operatorname{Dim} U = s$ for each neighborhood U of x_0. Choose a sequence $s_j \in (0, s)$ with $s_j \to s$. Then for each $j \in \mathbb{N}$ there are numbers $0 < a_j \leq b_j$ and a finite set $Y_j \subset B(x_0, 1/j)$ such that $a_j \leq d(x, y) \leq b_j$ whenever $x, y \in Y_j$, $x \neq y$, and such that $\operatorname{card} Y_j > j(b_j/a_j)^{s_j}$. It is easy to see that $A = \{x_0\} \cup \bigcup_{j \in \mathbb{N}} Y_j$ is the desired set. ■

Added in proof (September 1993)**.** Assouad's conjecture in the Introduction does not hold; this has been shown by Stephen Semmes in his manuscript *Bilipschitz embeddings of metric spaces into Euclidean spaces*. Conjecture 3.14 has been confirmed by Kerkko Luosto.

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